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Equivalent Characterization of Centralizers on $\mathcal{B}(H)$

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Abstract Let H be a Hilbert space with $\dim H \geq 2$ and $Z \in \mathcal{B}(H)$ be an arbitrary but fixed operator. In this paper we show that an additive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies $\Phi(AB) = \Phi(A)B = A\Phi(B)$ for any $A, B \in \mathcal{B}(H)$ with $AB = Z$ if and only if $\Phi(AB) = \Phi(A)B = A\Phi(B)$, $\forall A, B \in \mathcal{B}(H)$, that is, Φ is a centralizer. Similar results are obtained for Hilbert space nest algebras. In addition, we show that $\Phi(A^2) = A\Phi(A) = \Phi(A)A$ for any $A \in \mathcal{B}(H)$ with $A^2 = 0$ if and only if $\Phi(A) = A\Phi(I) = \Phi(I)A$, $\forall A \in \mathcal{B}(H)$, and generalize main results in *Linear Algebra and its Application*, **450**, 243–249 (2014) to infinite dimensional case. New equivalent characterization of centralizers on $\mathcal{B}(H)$ is obtained.

Keywords Centralizers, full-centralized points, von Neumann algebras, nest algebras

MR(2010) Subject Classification 47B49, 47C15

1 Introduction

Let $\mathcal{R}$ be a ring with the unit I and $\Phi : \mathcal{R} \to \mathcal{R}$ be an additive map. Φ is called a left centralizer if $\Phi(AB) = \Phi(A)B$, $\forall A, B \in \mathcal{R}$, a right centralizer if $\Phi(AB) = A\Phi(B)$, $\forall A, B \in \mathcal{R}$, and a centralizer if it is both a left and right centralizer, that is, $\Phi(AB) = \Phi(A)B = A\Phi(B)$, $\forall A, B \in \mathcal{R}$. As well known that right (left) centralizers and centralizers are very important in both theory and applications, and were studied intensively (see [1, 4, 7, 8] and references therein). For example, centralizers were studied in the general framework of semiprime rings by Vukmann and Kosi-Ulbl [8]. Recently some mathematicians characterized left and right centralizers by local actions. In [1], Bresar characterized an additive map behaving like a right (left) centralizer when acting on zero-product elements, that is, an additive map $\Phi : \mathcal{R} \to \mathcal{R}$ satisfying

$$\Phi(AB) = A\Phi(B)\ (\Phi(A)B) \quad \text{for any } A, B \in \mathcal{R} \text{ with } AB = 0. \tag{1.1}$$

They proved that an additive map from a prime ring into itself satisfying (1.1) if and only if $\Phi(AB) = A\Phi(B)$ $(\Phi(A)B)$, $\forall A, B \in \mathcal{R}$, that is, Φ is a right (left) centralizer. In [2], Chen studied a linear map $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ $(n \neq 3)$ satisfying

$$\Phi(A^2) = A\Phi(A) \quad \text{for any } A \in M_n(\mathbb{F}) \text{ with } A^2 = 0. \tag{1.2}$$

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It is obvious that each right centralizer satisfies (1.2), but the reverse is not true. In fact, they proved that such map has the form $\Phi(A) = SA + \mathrm{trace}(A)T, \forall A \in M_n(\mathbb{F})$. For centralizers, Qi [7] showed that an additive map Φ from a prime ring $\mathcal{R}$ into itself satisfies

$$\Phi(AB) = A\Phi(B) = \Phi(A)B \quad \text{for any } A, B \in \mathcal{R} \text{ with } AB = P,\ P^2 = P,\ P \neq 0, I$$

if and only if $\Phi(AB) = A\Phi(B) = \Phi(A)B$, $\forall A, B \in \mathcal{R}$, that is, Φ is a centralizer. Motivated by results in [1, 2, 7], more generally we say an additive map $\Phi : \mathcal{R} \to \mathcal{R}$ is centralized at a fixed point $Z \in \mathcal{R}$ if

$$\Phi(AB) = A\Phi(B) = \Phi(A)B \quad \text{for any } A, B \in \mathcal{R} \text{ with } AB = Z, \tag{1.3}$$

and such point is called a centralized point of $\mathcal{R}$. It is obvious that an additive map $\Phi : \mathcal{R} \to \mathcal{R}$ is a centralizer if and only if it is centralized at each $Z \in \mathcal{R}$. It is natural and interesting to ask the question whether or not an additive map is a centralizer if it is centralized at one given point. We say that Z is an additive full-centralized point of a ring $\mathcal{R}$ if every additive map from $\mathcal{R}$ into itself that is centralized at Z is in fact a centralizer. Main results in [1, 7] show that 0 and every nontrivial idempotent are full-centralized points of a prime ring. In this paper, we will show that every operator is a full-centralized point of $\mathcal{B}(H)$ and a Hilbert space nest algebra Alg$\mathcal{N}$.

Let H be a Hilbert space over the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ with $\dim H \geq 2$ and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. A nest $\mathcal{N}$ in $\mathcal{B}(H)$ is a totally ordered family of orthogonal projections in $\mathcal{B}(H)$ such that for every family $\{P_\lambda\}$ of elements of $\mathcal{N}$, both $\vee P_\lambda$ and $\wedge P_\lambda$ belong to $\mathcal{N}$, and which includes 0 and I. The nest subalgebra of $\mathcal{B}(H)$ associated to a nest $\mathcal{N}$, denoted by Alg$\mathcal{N}$, is the set of all elements $A \in \mathcal{B}(H)$ satisfying $PAP = AP$ for each $P \in \mathcal{N}$.

2 Centralizers on $\mathcal{B}(H)$

In this section, first we characterize left or right centralizers on $\mathcal{B}(H)$ by additive maps behaving like left or right centralizers at an injective or a dense range operator product elements. As its application, we show that every operator is a full-centralized point of $\mathcal{B}(H)$ and Alg$\mathcal{N}$.

To prove our main result, Lemmas 2.1–2.3 are necessary. In the proof of Lemmas 2.1–2.3, the statement that *any operator in $\mathcal{B}(H)$ is a sum of two invertible operators* will be used without mentioning.

Lemma 2.1 *Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be an additive map. Then*

(1) *For an operator $Z \in \mathcal{B}(H)$ with dense range, $\Phi(AB) = \Phi(A)B$ for any $A, B \in \mathcal{B}(H)$ with $AB = Z$ if and only if $\Phi(AB) = \Phi(A)B$, $\forall A, B \in \mathcal{B}(H)$, that is, Φ is a left centralizer.*

(2) *For an injective operator $Z \in \mathcal{B}(H)$, $\Phi(AB) = A\Phi(B)$ for any $A, B \in \mathcal{B}(H)$ with $AB = Z$ if and only if $\Phi(AB) = A\Phi(B)$, $\forall A, B \in \mathcal{B}(H)$, that is, Φ is a right centralizer.*

Proof We only check (1), (2) is similar. The "if" part of (1) is obvious, we only check its "only if" part. For any invertible operator $A \in \mathcal{B}(H)$, it follows from $Z = IZ$ and $Z = A^{-1}AZ$ that $\Phi(Z) = \Phi(I)Z$, $\Phi(Z) = \Phi(A^{-1})AZ$ and $\Phi(I)Z = \Phi(A^{-1})AZ$. Now the fact that Z is an operator with dense range implies $\Phi(A^{-1})A = \Phi(I)$, and hence $\Phi(A^{-1}) = \Phi(I)A^{-1}$. Therefore, $\Phi(A) = \Phi(I)A$, $\forall A \in \mathcal{B}(H)$ and (1) is true. The proof is complete.

From now on, fix a nontrivial idempotent $P \in \mathcal{B}(H)$, and set $P_1 = P, P_2 = I - P$ and $\mathcal{A}_{ij} = P_i\mathcal{B}(H)P_j$, $1 \leq i,j \leq 2$. Then $\mathcal{B}(H) = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$, which is the Pierce decomposition of $\mathcal{B}(H)$. For any $A \in \mathcal{B}(H)$, $A_{ij} = P_iAP_j$, $1 \leq i,j \leq 2$.

Lemma 2.2 *For $Z \in \mathcal{B}(H)$ such that $Z_{21} = 0$, if an additive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies*

$$\Phi(AB) = \Phi(A)B \quad \text{for any } A, B \in \mathcal{B}(H) \text{ with } AB = Z, \tag{2.1}$$

then for any $A_{11} \in \mathcal{A}_{11},\ A_{12} \in \mathcal{A}_{12},\ A_{22} \in \mathcal{A}_{22}$,

$$\Phi(A_{11})P_2 = 0, \quad \Phi(A_{11}A_{12})P_2 = \Phi(A_{11})A_{12}, \quad \Phi(A_{12})A_{12} = 0.$$

Proof For any invertible operator $A_{11} \in \mathcal{A}_{11}$ with $A_{11}^{-1} \in \mathcal{A}_{11}$ and $B_{22}, C_{22} \in \mathcal{A}_{22}$ such that $B_{22}C_{22} = Z_{22}$, by (2.1) and

$$(A_{11} + t(A_{11}A_{12} + B_{22}))((A_{11}^{-1}Z - A_{12}C_{22}) + t^{-1}C_{22}) = Z, \quad \forall t \in \mathbb{Q},\ \forall A_{12} \in \mathcal{A}_{12}, \tag{2.2}$$

we have

$$\begin{aligned}\Phi(Z) &= \Phi((A_{11} + t(A_{11}A_{12} + B_{22})))((A_{11}^{-1}Z - A_{12}C_{22}) + t^{-1}C_{22}) \\ &= \Phi(A_{11})(A_{11}^{-1}Z - A_{12}C_{22}) + \Phi(A_{11}A_{12} + B_{22})C_{22} \\ &\quad + t\Phi(A_{11}A_{12} + B_{22})(A_{11}^{-1}Z - A_{12}C_{22}) + t^{-1}\Phi(A_{11})C_{22}.\end{aligned}$$

This implies

$$\Phi(A_{11})(A_{11}^{-1}Z - A_{12}C_{22}) + \Phi(A_{11}A_{12} + B_{22})C_{22} = \Phi(Z), \tag{2.3}$$

$$\Phi(A_{11}A_{12} + B_{22})(A_{11}^{-1}Z - A_{12}C_{22}) = 0, \tag{2.4}$$

$$\Phi(A_{11})C_{22} = 0. \tag{2.5}$$

By (2.3), (2.4) and replacing A_{12} with tA_{12} for arbitrary $t \in \mathbb{Q}$, we have

$$\Phi(A_{11})A_{12}C_{22} = \Phi(A_{11}A_{12})C_{22}, \quad \Phi(A_{11}A_{12})A_{12}C_{22} = 0, \tag{2.6}$$

$$\Phi(A_{11}A_{12})A_{11}^{-1}Z = \Phi(B_{22})A_{12}C_{22}. \tag{2.7}$$

Let $C_{22} = P_2$. Then (2.5) implies $\Phi(A_{11})P_2 = 0$ for any invertible operator $A_{11} \in \mathcal{A}_{11}$, thus

$$\Phi(A_{11})P_2 = 0, \quad \forall A_{11} \in \mathcal{A}_{11}. \tag{2.8}$$

Similarly, taking $C_{22} = P_2$ in the first equality of (2.6), we have

$$\Phi(A_{11})A_{12} = \Phi(A_{11}A_{12})P_2, \quad \forall A_{11} \in \mathcal{A}_{11},\ A_{12} \in \mathcal{A}_{12}. \tag{2.9}$$

Letting $A_{11} = P_1,\ C_{22} = P_2$ in the second equality of (2.6), one gets

$$\Phi(A_{12})A_{12} = 0, \quad \forall A_{12} \in \mathcal{A}_{12}. \tag{2.10}$$

The proof of this lemma is complete.

Lemma 2.3 *For $Z \in \mathcal{B}(H)$ such that $Z_{21} = 0$, if an additive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies*

$$\Phi(AB) = A\Phi(B) \quad \text{for any } A, B \in \mathcal{B}(H) \text{ with } AB = Z, \tag{2.11}$$

then for any $A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{22} \in \mathcal{A}_{22}$,

$$P_1\Phi(A_{22}) = 0, \quad P_1\Phi(A_{12}A_{22}) = A_{12}\Phi(A_{22}), \quad A_{12}\Phi(A_{12}) = 0.$$

Proof It follows from (2.2) and (2.11) that $(A_{11}+t(A_{11}A_{12}+B_{22}))\Phi((A_{11}^{-1}Z-A_{12}C_{22})+t^{-1}C_{22})=\Phi(Z),\ \forall t\in\mathbb{Q}$. This implies

$$A_{11}\Phi(A_{11}^{-1}Z-A_{12}C_{22})+(A_{11}A_{12}+B_{22})\Phi(C_{22})=\Phi(Z), \tag{2.12}$$

$$(A_{11}A_{12}+B_{22})\Phi(A_{11}^{-1}Z-A_{12}C_{22})=0, \tag{2.13}$$

$$A_{11}\Phi(C_{22})=0. \tag{2.14}$$

By (2.12) and replacing A_{12} with tA_{12} for arbitrary $t\in\mathbb{Q}$, we have

$$A_{11}\Phi(A_{12}C_{22})=A_{11}A_{12}\Phi(C_{22}). \tag{2.15}$$

$$A_{11}A_{12}\Phi(A_{12}C_{22})=0. \tag{2.16}$$

Let $B_{22}=Z_{22}D_{22}$ and $C_{22}=D_{22}^{-1}$ in (2.14), where $D_{22}\in\mathcal{A}_{22}$ is invertible in $\mathcal{A}_{22}$. Then $B_{22}C_{22}=Z_{22}$. Thus (2.14) entails $P_1\Phi(D_{22}^{-1})=0$ and

$$P_1\Phi(A_{22})=0,\quad \forall A_{22}\in\mathcal{A}_{22}. \tag{2.17}$$

Similarly, by (2.15) and (2.16), we have

$$P_1\Phi(A_{12}A_{22})=A_{12}\Phi(A_{22}),\quad A_{12}\Phi(A_{12})=0,\quad \forall A_{12}\in\mathcal{A}_{12},A_{22}\in\mathcal{A}_{22}. \tag{2.18}$$

The proof is complete.

By Lemmas 2.2 and 2.3, we can show an additive map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ is a centralizer if and only if it is centralized at an arbitrary but fixed operator, new equivalent characterization of centralizers on $\mathcal{B}(H)$ is obtained.

Theorem 2.4 *Assume H is a Hilbert space with $\dim H\geq 2$ and $Z\in\mathcal{B}(H)$ is an arbitrary but fixed operator. Then an additive map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ satisfies $\Phi(AB)=\Phi(A)B=A\Phi(B)$ for any $A,B\in\mathcal{B}(H)$ with $AB=Z$ if and only if $\Phi(AB)=\Phi(A)B=A\Phi(B)$, $\forall A,B\in\mathcal{B}(H)$.*

Proof The "if" part is obvious, we only check the "only if" part. If $Z=0$, the conclusion follows from main results in [1]. Next we assume $Z\neq 0$.

Case 1 $\overline{\mathrm{ran}(Z)}\neq H$, where $\mathrm{ran}(Z)$ denotes the range of Z.

Let P be the nontrivial orthogonal projection from $\mathcal{H}$ onto the closure of $\mathrm{ran}(Z)$. Then $(I-P)Z=0$. Set $P_1=P$ and $P_2=I-P$. Then the condition $(I-P)Z=P_2Z=0$ implies $Z_{21}=Z_{22}=0$, and thus by Lemmas 2.2–2.3 we see (2.7) holds for any $B_{22},C_{22}\in\mathcal{A}_{22}$ with $B_{22}C_{22}=0$. Letting $C_{22}=0$ and $A_1=P_1$ in (2.7), we get $\Phi(A_{12})P_1Z=0$. Since $\overline{\mathrm{ran}(Z)}=P_1$, thus

$$\Phi(A_{12})P_1=0,\quad \forall A_{12}\in\mathcal{A}_{12}. \tag{2.19}$$

It follows from $P_1Z=Z$ and $P_1(Z+A_{21})=Z$ that $\Phi(P_1)Z=P_1\Phi(Z)=\Phi(Z)$ and $\Phi(P_1)(Z+A_{21})=\Phi(Z)=P_1\Phi(Z+A_{21})$, and hence

$$\Phi(Z)=P_1\Phi(Z)=\Phi(P_1)Z,\quad \Phi(P_1)A_{21}=P_1\Phi(A_{21})=0,\quad \forall A_{21}\in\mathcal{A}_{21}. \tag{2.20}$$

For any $A_{12}\in\mathcal{A}_{12},A_{21}\in\mathcal{A}_{21}$, it follows from $(P_1+A_{12})(Z-A_{12}A_{21}-A_{12}+A_{21}+P_2)=Z$ that $(P_1+A_{12})\Phi(Z-A_{12}A_{21}-A_{12}+A_{21}+P_2)=\Phi(Z)$. By Lemmas 2.2–2.3 and (2.20), we have $-P_1\Phi(A_{12}A_{21})+A_{12}\Phi(Z)-A_{12}\Phi(A_{12}A_{21})+A_{12}\Phi(A_{21})=0$. Replacing A_{21} and A_{12} with sA_{21} and tA_{12} respectively for arbitrary $s,t\in\mathbb{Q}$, one gets

$$A_{12}\Phi(Z)=0,\quad P_1\Phi(A_{12}A_{21})=A_{12}\Phi(A_{21}),\quad \forall A_{12}\in\mathcal{A}_{12},A_{21}\in\mathcal{A}_{21}. \tag{2.21}$$

Since $\mathcal{B}(H)$ is a prime algebra, thus (2.20) and (2.21) imply $P_2\Phi(Z) = 0$, $P_2\Phi(P_1)Z = 0$ and $P_2\Phi(P_1)P_1 = 0$ since $\overline{\operatorname{ran}(Z)} = P_1$. By Lemma 2.2, we have $P_2\Phi(A_{12})P_2 = P_2\Phi(P_1)A_{12} = 0$. This and (2.19) imply $\Phi(A_{12}) \in \mathcal{A}_{12}$. Lemmas 2.2–2.3 entail $P_2\Phi(A_{11}A_{12})P_2 = P_2\Phi(A_{11})A_{12} = 0$ and $P_1\Phi(A_{12}A_{22})P_1 = A_{12}\Phi(A_{22})P_1 = 0$, thus $P_2\Phi(A_{11})P_1 = 0$ and $P_2\Phi(A_{22})P_1 = 0$. Therefore, $\Phi(A_{11}) \in \mathcal{A}_{11}$, $\Phi(A_{22}) \in \mathcal{A}_{22}$, and $\Phi(I) = \Phi(P_1)+\Phi(P_2) = P_1\Phi(P_1)P_1+P_2\Phi(P_2)P_2$. Now by (2.21), we get $A_{12}\Phi(A_{21})P_2 = P_1\Phi(A_{12}A_{21})P_2 = 0$, and hence $P_2\Phi(A_{21})P_2 = 0$. This and (2.20) imply $\Phi(A_{21}) \in \mathcal{A}_{21}$.

Next we show Φ is a centralizer. Lemmas 2.2–2.3 entail

$$P_1\Phi(A_{12})P_2 = A_{12}\Phi(P_2) = \Phi(P_1)A_{12}, \quad \forall A_{12} \in \mathcal{A}_{12}. \tag{2.22}$$

and $P_1\Phi(A_{11}A_{12})P_2 = \Phi(P_1)A_{11}A_{12} = A_{11}A_{12}\Phi(P_2)P_2 = A_{11}\Phi(P_1)A_{12}$. On the other hand, Lemma 2.2 and (2.11) imply $P_1\Phi(A_{11}A_{12})P_2 = P_1\Phi(A_{11})A_{12}$. Thus

$$\begin{aligned} &P_1\Phi(A_{11})P_1 = P_1\Phi(P_1)A_{11} = A_{11}\Phi(P_1)P_1, \\ &\Phi(A_{11}) = A_{11}\Phi(P_1) = \Phi(P_1)A_{11}, \ \ \forall A_{11} \in \mathcal{A}_{11}. \end{aligned} \tag{2.23}$$

Similarly,

$$\Phi(A_{22}) = A_{22}\Phi(P_2) = \Phi(P_2)A_{22}, \quad \forall A_{22} \in \mathcal{A}_{22}. \tag{2.24}$$

Now (2.21)–(2.22) imply

$$\begin{aligned} A_{12}\Phi(A_{21})P_1 &= P_1\Phi(A_{12}A_{21})P_1 = A_{12}A_{21}\Phi(P_1)P_1 \\ &= A_{12}A_{21}\Phi(P_1) = \Phi(P_1)P_1A_{12}A_{21} = A_{12}\Phi(P_2)A_{21}. \end{aligned}$$

Since $\mathcal{B}(H)$ is a prime algebra, thus

$$\Phi(A_{21}) = A_{21}\Phi(P_1) = \Phi(P_2)A_{21}, \ \ \forall A_{21} \in \mathcal{A}_{21}. \tag{2.25}$$

Therefore, for any $A = \sum_{i,j=1}^{2} A_{ij}, \ i,j = 1,2$, we have

$$\begin{aligned} \Phi(A) &= \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}) \\ &= A_{11}\Phi(P_1) + A_{12}\Phi(P_2) + A_{21}\Phi(P_1) + A_{22}\Phi(P_2) \\ &= (A_{11} + A_{12} + A_{21} + A_{22})(\Phi(P_1) + \Phi(P_2)) \\ &= \Phi(P_1)A_{11} + \Phi(P_1)A_{12} + \Phi(P_2)A_{21} + \Phi(P_2)A_{22} \\ &= (\Phi(P_1) + \Phi(P_2))(A_{11} + A_{12} + A_{21} + A_{22}), \end{aligned}$$

and $\Phi(A) = \Phi(I)A = A\Phi(I), \forall A \in \mathcal{B}(H)$. Now it is easy to verify $\Phi(AB) = \Phi(A)B = A\Phi(B), \forall A, B \in \mathcal{B}(H)$, that is, Φ is a centralizer.

Case 2 $\overline{\operatorname{ran}(Z)} = H$.

If Z is an injective operator, by Lemma 2.1 Φ is a centralizer.

If Z is not an injective operator, then $\overline{\operatorname{ran}(Z^*)} \neq H$. Define an additive map $\Phi^* : \mathcal{B}(H) \to \mathcal{B}(H)$ by $\Phi^*(A^*) = \Phi(A)^*$, $\forall A \in \mathcal{B}(H)$. Then for any $A, B \in \mathcal{B}(H)$ with $B^*A^* = Z^*$ and $AB = Z$, we have

$$\begin{aligned} \Phi^*(B^*A^*) &= \Phi^*((AB)^*) = \Phi(AB)^* = (\Phi(A)B)^* = (A\Phi(B))^* \\ &= B^*\Phi(A)^* = \Phi(B)^*A^* = B^*\Phi^*(A^*) = \Phi^*(B^*)A^*, \end{aligned}$$

that is, Φ^* is centralized at Z^*. Therefore, Φ^* is a centralizer by case 1, which implies Φ is a centralizer. The proof of this theorem is complete.

Next we show each operator is a full-centralized point of a Hilbert space nest algebra $\mathrm{Alg}\mathcal{N}$. This result has been got in [5], here we give a short proof of it.

Theorem 2.5 *Let* $\mathrm{Alg}\mathcal{N}$ *be a Hilbert space nest algebra. Assume* $Z \in \mathrm{Alg}\mathcal{N}$ *is an arbitrary but fixed operator. Then an additive map* $\Phi : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ *satisfies* $\Phi(AB) = \Phi(A)B = A\Phi(B)$ *for any* $A, B \in \mathrm{Alg}\mathcal{N}$ *with* $AB = Z$ *if and only if* $\Phi(AB) = \Phi(A)B = A\Phi(B)$, $\forall A, B \in \mathrm{Alg}\mathcal{N}$.

Proof Only the "only if" part needs to be checked. If $\mathcal{N} = \{0, I\}$, then $\mathrm{Alg}\mathcal{N} = \mathcal{B}(H)$, the conclusion follows from Theorem 2.4.

If there is a nontrivial projection $P \in \mathcal{N}$, let $P_1 = P$ and $P_2 = I - P$. We can decompose $\mathcal{A} = \mathrm{Alg}\mathcal{N}$ as $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{22}$, $\mathcal{A}_{ij} = P_i\mathrm{Alg}\mathcal{N}P_j$, $1 \le i \le j \le 2$. Since $P_1\mathcal{B}(H)P_2 \subseteq P_1\mathrm{Alg}\mathcal{N}P_2$, thus for $A_{11} \in \mathcal{A}_{11}$ and $A_{22} \in \mathcal{A}_{22}$, the condition $A_{11}A_{12} = 0$, $\forall A_{12} \in \mathcal{A}_{12}$ implies $A_{11} = 0$, and $A_{12}A_{22} = 0$, $\forall A_{12} \in \mathcal{A}_{12}$ implies $A_{22} = 0$. Therefore by the fact $P_2\mathrm{Alg}\mathcal{N}P_1 = \{0\}$ and similar arguments as that in the proof of Theorem 2.4, we can show Φ is a centralizer. The proof is complete.

3 Left or Right Centralizers on $\mathcal{B}(H)$

In this section, we characterize left or right centralizers on $\mathcal{B}(H)$ by square zero operators and involutions, and generalize main results in [2] to infinite-dimensional case. To prove our main result, the following two theorems are necessary.

Theorem 3.1 ([6]) *Let* H *be an infinite-dimensional Hilbert space. Then every operator* $A \in \mathcal{B}(H)$ *is a sum of five square-zero operators.*

Recall that a von Neumann algebra $\mathcal{A}$ is a C^*-subalgebra of some $\mathcal{B}(H)$, which satisfies the double commutant property: $\mathcal{A}'' = \mathcal{A}$, where $\mathcal{A}' = \{T \mid T \in \mathcal{B}(H),\ \ TA = AT,\ \forall A \in \mathcal{A}\}$ and $\mathcal{A}'' = \{\mathcal{A}'\}'$. A von Neumann algebra is called properly infinite if it contains no nonzero finite central projection (see [3]). In particular, $\mathcal{B}(H)$ is a properly infinite von Neumann algebra if H is an infinite dimensional Hilbert space.

Theorem 3.2 ([6]) *Let* $\mathcal{A}$ *be a properly infinite von Neumann algebra. Then every operator* $A \in \mathcal{A}$ *is a sum of five idempotents.*

It is well known that if H is infinite-dimensional, $\mathcal{B}(H)$ does not admit a nontrivial finite trace. Comparing with main results in [2], we have

Theorem 3.3 *Let* H *be an infinite-dimensional Hilbert space, and* $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *be an additive map. Then*

(1) $A\Phi(A) = 0$ *for any* $A \in \mathcal{B}(H)$ *with* $A^2 = 0$ *if and only if* $\Phi(A) = A\Phi(I)$, $\forall A \in \mathcal{B}(H)$.

(2) $\Phi(A)A = 0$ *for any* $A \in \mathcal{B}(H)$ *with* $A^2 = 0$ *if and only if* $\Phi(A) = \Phi(I)A$, $\forall A \in \mathcal{B}(H)$.

(3) $A\Phi(A) = \Phi(A)A = 0$ *for any* $A \in \mathcal{B}(H)$ *with* $A^2 = 0$ *if and only if* $\Phi(A) = A\Phi(I) = \Phi(I)A$, $\forall A \in \mathcal{B}(H)$.

Proof (1) and (2) imply (3). We only check (1), (2) can be obtained by similar argument. The "if" part of (1) is obvious, its "only if" part need to be checked.

Claim $\Phi(P) = P\Phi(I)$, $\forall P \in \mathcal{B}(H)$, $P^2 = P$.

Case 1 P is an idempotent with infinite-dimensional range and infinite-dimensional kernel.

Let $H = H_1 + H_2$, $H_1 = P(H)$, $H_2 = (I-P)(H)$. Denote the restriction of P to H_1 and that of $I-P$ to H_2 by $\widetilde{P}$ and $\widetilde{I-P}$ respectively. Then $\widetilde{P} \in \mathcal{B}(H_1), \widetilde{I-P} \in \mathcal{B}(H_2)$, Theorem 3.1 implies both $\widetilde{P}$ and $\widetilde{I-P}$ can be written as sum of five square-zero operators, $\widetilde{P} = \widetilde{A_1} + \widetilde{A_2} + \cdots + \widetilde{A_5}$, $\widetilde{I-P} = \widetilde{B_1} + \widetilde{B_2} + \cdots + \widetilde{B_5}$ with $\widetilde{A_i} \in \mathcal{B}(H_1)$, $\widetilde{B_i} \in \mathcal{B}(H_2)$, $\widetilde{A_i}^2 = 0$ and $\widetilde{B_i}^2 = 0$ for each $i = 1, 2, \ldots, 5$. For each $i = 1, 2, \ldots, 5$, let A_i and B_i be the extensions of $\widetilde{A_i}$ and $\widetilde{B_i}$ to H such that $PA_iP = \widetilde{A_i}, (I-P)B_i(I-P) = \widetilde{B_i}$. Then we have $P = A_1 + A_2 + \cdots + A_5$ and $I - P = B_1 + B_2 + \cdots + B_5$, with $A_i^2 = 0$, $B_i^2 = 0$, $\forall i = 1, 2, \ldots, 5$. Thus $A_i\Phi(A_i) = 0$, $B_j\Phi(B_j) = 0$, $(A_i + B_j)^2 = 0$ and $0 = (A_i + B_j)\Phi(A_i + B_j) = A_i\Phi(A_i) + B_j\Phi(B_j) + A_i\Phi(B_j) + B_j\Phi(A_i) = A_i\Phi(B_j) + B_j\Phi(A_i)$, $1 \le i, j \le 5$. Consequently,

$$0 = P\Phi(I-P) + (I-P)\Phi(P) = P\Phi(I) - 2P\Phi(P) + \Phi(P).$$

Multiplying this equality by P from the left side, we get $P\Phi(P) = P\Phi(I)$ and $\Phi(P) = P\Phi(I)$.

Case 2 P is an idempotent with finite-dimensional range or finite-dimensional kernel.

If P has finite-dimensional range, let K_1 and K_2 be orthogonal infinite dimensional subspaces of H with $(I-P)(H) = K_1 \oplus K_2$ and Q_i be the orthogonal projection onto K_i, $i = 1, 1$. Then $PQ_i = Q_iP = 0$, $i = 1, 2$, $Q_1Q_2 = Q_2Q_1 = 0$, $I = P + Q_1 + Q_2$ and Q_1, Q_2 have infinite dimensional ranges and and infinite-dimensional kernel. By Case 1, we have $\Phi(Q_i) = Q_i\Phi(I)$, $i = 1, 2$. Thus $\Phi(I-P) = \Phi(Q_1 + Q_2) = Q_1\Phi(I) + Q_2\Phi(I) = (I-P)\Phi(I)$ and $\Phi(P) = P\Phi(I)$.

If $P \in \mathcal{B}(H)$ is an idempotent with finite-dimensional kernel, we have $\Phi(P) = P\Phi(I)$ by considering $I - P$.

These facts imply $\Phi(P) = P\Phi(I)$ for any idempotent $P \in \mathcal{B}(H)$. Therefore, $\Phi(A) = A\Phi(I)$, $\forall A \in \mathcal{B}(H)$ by Theorem 3.2. The proof of this theorem is complete.

Next we characterize left or right centralizers on properly infinite von Neumann algebras by involutions.

Theorem 3.4 *Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $\Phi : \mathcal{A} \to \mathcal{A}$ be an additive map. Then*

(1) $\Phi(A^2) = A\Phi(A)$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = A\Phi(I)$, $\forall A \in \mathcal{A}$.

(2) $\Phi(A^2) = \Phi(A)A$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = \Phi(I)A$, $\forall A \in \mathcal{A}$.

(3) $\Phi(A^2) = A\Phi(A) = \Phi(A)A$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = A\Phi(I) = \Phi(I)A$, $\forall A \in \mathcal{A}$.

Proof We only check the "only of" part of (1). For any idempotent $P \in \mathcal{A}$, it follows from $(I-2P)^2 = I$ that $(I-2P)\Phi(I-2P) = \Phi((I-2P)^2) = \Phi(I)$. This implies $\Phi(P) + P\Phi(I) = 2P\Phi(P)$, and thus $\Phi(P) = P\Phi(I)$ and $\Phi(A) = A\Phi(I)$, $\forall A \in \mathcal{A}$ by Theorem 3.2. The proof is complete.

Using the fact that any von Neumann algebra is the closed linear span of its projection and by similar argument as that in the proof of Theorem 3.4, we have

Theorem 3.5 *Let $\mathcal{A}$ be a von Neumann algebra and $\Phi : \mathcal{A} \to \mathcal{A}$ be a bounded linear map. Then*

(1) $\Phi(A^2) = A\Phi(A)$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = A\Phi(I)$, $\forall A \in \mathcal{A}$.

(2) $\Phi(A^2) = \Phi(A)A$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = \Phi(I)A$, $\forall A \in \mathcal{A}$.

(3) $\Phi(A^2) = A\Phi(A) = \Phi(A)A$ *for any* $A \in \mathcal{A}$ *with* $A^2 = I$ *if and only if* $\Phi(A) = A\Phi(I) = \Phi(I)A$, $\forall A \in \mathcal{A}$.

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