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Fractional Wavelet Packet Transformations Involving Hankel–Clifford Integral Transformations

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Abstract In this paper, we introduce the fractional wavelet transformations (FrWT) involving Hankel–Clifford integral transformation (HClIT) on the positive half line and studied some of its basic properties. Also we obtain Parseval's relation and an inversion formula. Examples of fractional powers of Hankel–Clifford integral transformation (FrHClIT) and FrWT are given. Then, we introduce the concept of fractional wavelet packet transformations FrBWPT and FrWPIT, and investigate their properties.

Keywords Wavelet packet transformation, fractional Hankel–Clifford integral transformation, fractional Hankel–Clifford integral convolution, wavelet

MR(2010) Subject Classification 65T60, 42A85, 47B35

1 Introduction

Let us define the space $L^p_{\nu+\mu}(I)$, $1 \le p < \infty$ and $I = (0, \infty)$, consisting all those real valued measurable function f on I such that the integral $\int_0^\infty |f(t)|^p t^{\frac{\mu+\nu}{2}} dt$ exists and finite. Also the space $L^\infty(I)$ is the collection of almost everywhere bounded integrable functions. Hence, the norm is endowed with

$$\|f\|_{L^p_{\nu+\mu}} = \begin{cases} \left(\int_0^\infty |f(t)|^p t^{\frac{\mu+\nu}{2}} dt \right)^{1/p}, & 1 \le p < \infty,\ \mu, \nu \in \mathbb{R}, \\ \operatorname*{ess\,sup}_{t \in I} |f(t)|, & p = \infty. \end{cases} \tag{1.1}$$

As per fractionalization of conventional Hankel transforms [5, 7], the two variants of fractional powers α $(0 < \alpha < \pi)$ of Hankel–Clifford transformations of order $\nu \ge 0$ are generalization of a pair of Hankel–Clifford transformations [2, 6] defined by Prasad and Kumar [11]. The modified versions of [11] are defined as:

The first FrHClIT of function f is defined by

$$(h^\alpha_{1,\nu,\mu} f)(\omega) = \hat{f}(\omega) = \int_0^\infty K^\alpha_1(t, \omega) f(t) dt, \tag{1.2}$$

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where

$$K_1^\alpha(t,\omega)=\begin{cases}\gamma_{\nu,\mu}^\alpha C_{\nu,\mu}(t\omega\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+\omega)\cot\alpha}\omega^\mu, & \alpha\neq n\pi,\\ C_{\nu,\mu}(t\omega)\omega^\mu, & \alpha=\frac{\pi}{2},\\ \delta(t-\omega), & \alpha=n\pi,\end{cases}\tag{1.3}$$

where $n\in\mathbb{Z}$, $\gamma_{\nu,\mu}^\alpha=\frac{\mathrm{e}^{\mathrm{i}(\nu+1)(\alpha-\frac{\pi}{2})}}{(\sin\alpha)^{\mu+1}}$, $C_{\nu,\mu}(t)=t^{-\mu/2}J_\nu(2\sqrt{t})$ and J_ν is the Bessel function of first kind of order ν.

Analogously, the second FrHClIT of function g is defined by

$$(h_{2,\nu,\mu}^\alpha g)(\omega)=\tilde{g}(\omega)=\int_0^\infty K_2^\alpha(t,\omega)g(t)dt=\omega^{-\mu}(h_{1,\nu,\mu}^\alpha(t^\mu g))(\omega),\tag{1.4}$$

where

$$K_2^\alpha(t,\omega)=\begin{cases}\gamma_{\nu,\mu}^\alpha C_{\nu,\mu}(t\omega\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+\omega)\cot\alpha}t^\mu, & \alpha\neq n\pi,\\ C_{\nu,\mu}(t\omega)t^\mu, & \alpha=\frac{\pi}{2},\\ \delta(t-\omega), & \alpha=n\pi,\end{cases}\tag{1.5}$$

with n and $\gamma_{\nu,\mu}^\alpha$ being the same as above.

The inverse of (1.2) and (1.4) are respectively defined as follows:

$$f(t)=((h_{1,\nu,\mu}^\alpha)^{-1}\hat{f})(t)=(h_{1,\nu,\mu}^{-\alpha}\hat{f})(t)=\int_0^\infty K_1^{*\alpha}(\omega,t)\hat{f}(\omega)d\omega\tag{1.6}$$

and

$$g(t)=((h_{2,\nu,\mu}^\alpha)^{-1}\tilde{g})(t)=(h_{2,\nu,\mu}^{-\alpha}\tilde{g})(t)=\int_0^\infty K_2^{*\alpha}(\omega,t)\tilde{g}(\omega)d\omega,\tag{1.7}$$

where $K_1^{*\alpha}(\omega,t)$ and $K_2^{*\alpha}(\omega,t)$ are complex conjugate of $K_1^\alpha(\omega,t)$ and $K_2^\alpha(\omega,t)$, that is, $K_1^{*\alpha}(\omega,t)=K_1^{-\alpha}(\omega,t)$ and $K_2^{*\alpha}(\omega,t)=K_2^{-\alpha}(\omega,t)$ respectively. The first and second FrHClIT's are adjoint of each other. We note that $(h_{1,\nu,\mu}^{\pi/2})^{-1}=h_{1,\nu,\mu}^{\pi/2}$ and $(h_{2,\nu,\mu}^{\pi/2})^{-1}=h_{2,\nu,\mu}^{\pi/2}$.

The Dirac-delta function is a very peculiar kind of function; it vanishes for all $t\neq 0$, and it tends to infinity in an appropriate way when $t=0$. Here we express the Dirac-delta function $\delta(t-a)$ as

$$\delta(t-a)=a^\mu\int_0^\infty\omega^\mu C_{\mu,\nu}(t\omega)C_{\mu,\nu}(a\omega)d\omega.\tag{1.8}$$

Now, we have interesting properties of kernels stated in the following lemma:

Lemma 1.1 *If $K_2^\alpha(t,\omega)$ is the kernel of the transformation* (1.4), *then*

(i) $\int_0^\infty K_2^\alpha(t,\omega)K_2^\beta(\omega,z)d\omega=K_2^{\alpha+\beta}(t,z)$,

(ii) $\int_0^\infty K_2^\alpha(a,\omega)K_2^{*\alpha}(\omega,t)d\omega=\delta(t-a)$.

Proof (i) Using (1.5) and $C_{\nu,\mu}(x)=x^{-\mu/2}J_\nu(2\sqrt{x})$, we have

$$\begin{aligned}&\int_0^\infty K_2^\alpha(t,\omega)K_2^\beta(\omega,z)d\omega\\&=\frac{\mathrm{e}^{\mathrm{i}(\nu+1)(\alpha+\beta-\pi)}}{(\sin\alpha\sin\beta)^{\mu+1}}\mathrm{e}^{\mathrm{i}(t\cot\alpha+z\cot\beta)}(tz\csc^2\alpha\csc^2\beta)^{-\mu/2}t^\mu\\&\quad\times\int_0^\infty\mathrm{e}^{\mathrm{i}(\cot\alpha+\cot\beta)\omega}J_\nu(2\csc\alpha\sqrt{t\omega})J_\nu(2\csc\beta\sqrt{\omega z})d\omega.\end{aligned}$$

Using [1, p. 55] the last integral can be solved and then on simplification yields the required result.

(ii) An application of (1.8) gives the result. □

Remark 1.2 If $K_1^\alpha(t,\omega)$ is the kernel of the transformation (1.2), then
(i) $\int_0^\infty K_1^\alpha(t,\omega)K_1^\beta(\omega,z)d\omega = K_1^{\alpha+\beta}(t,z)$,
(ii) $\int_0^\infty K_1^\alpha(a,\omega)K_1^{*\alpha}(\omega,t)d\omega = \delta(t-a)$.

1.1 Fractional Hankel–Clifford Integral Translation and Convolution

Let $\Delta(t,\omega,z)$ denote the area of a triangle having sides t,ω,z [19]. Then for $\nu \geq 0$, arbitrary real parameter μ and $0<\alpha<\pi$, we define

$$D_{\nu,\mu,\alpha}(t,\omega,z) = \frac{\left(\gamma_{\nu,\mu}^\alpha\right)^* \Delta^{2\nu-1}\mathrm{e}^{-\mathrm{i}(t+\omega+z)\cot\alpha}}{2^{2\nu}(\csc\alpha)^{3\mu-\nu+2}(t\omega z)^{\frac{\nu+\mu}{2}}\Gamma(\nu+1/2)\sqrt{\pi}}, \tag{1.9}$$

where if such a triangle exists and zero otherwise. Clearly, $D_{\nu,\mu,\alpha}(t,\omega,z) \geq 0$ and is symmetric in t,ω,z.

Lemma 1.3 *For $0<t,\omega,z<\infty$, we have*

$$\begin{aligned}\gamma_{\nu,\mu}^\alpha &\int_0^\infty C_{\nu,\mu}(\omega s\csc^2\alpha)\mathrm{e}^{\mathrm{i}(\omega+s)\cot\alpha}\omega^\mu D_{\nu,\mu,\alpha}(t,\omega,z)d\omega \\ &= \mathrm{e}^{-\mathrm{i}(t+z)\cot\alpha}C_{\nu,\mu}(ts\csc^2\alpha)C_{\nu,\mu}(zs\csc^2\alpha)\mathrm{e}^{\mathrm{i}s\cot\alpha}s^{\frac{\mu-\nu}{2}},\end{aligned} \tag{1.10}$$

and

$$\gamma_{\nu,\mu}^\alpha\int_0^\infty D_{\nu,\mu,\alpha}(t,\omega,z)\mathrm{e}^{\mathrm{i}\omega\cot\alpha}\omega^{\frac{\mu+\nu}{2}}d\omega = \frac{\mathrm{e}^{-\mathrm{i}(t+z)\cot\alpha}(tz\csc^2\alpha)^{\frac{\nu-\mu}{2}}}{\Gamma(\nu+1)}. \tag{1.11}$$

Proof From Prasad et al. [14], we have

$$\int_0^\infty J_\nu(2\sqrt{ts})J_\nu(2\sqrt{\omega s})J_\nu(2\sqrt{zs})(\sqrt{s})^{-\nu}ds = \frac{\Delta^{2\nu-1}}{2^{2\nu}(t\omega z)^{(\nu/2)}\Gamma(\nu+1/2)\sqrt{\pi}}.$$

Then by using the relation $C_{\nu,\mu}(t) = t^{-\mu/2}J_\nu(2\sqrt{t})$, we may write

$$\begin{aligned}&\left(\gamma_{\nu,\mu}^\alpha\right)^*\mathrm{e}^{-\mathrm{i}(t+\omega+z)\cot\alpha}\int_0^\infty C_{\nu,\mu}(ts\csc^2\alpha)C_{\nu,\mu}(\omega s\csc^2\alpha)C_{\nu,\mu}(zs\csc^2\alpha)s^{\frac{3\mu-\nu}{2}}ds \\ &= \frac{\left(\gamma_{\nu,\mu}^\alpha\right)^*\Delta^{2\nu-1}\mathrm{e}^{-\mathrm{i}(t+\omega+z)\cot\alpha}}{2^{2\nu}(\csc\alpha)^{3\mu-\nu+2}(t\omega z)^{\frac{\mu+\nu}{2}}\Gamma(\nu+1/2)\sqrt{\pi}} \\ &= D_{\nu,\mu,\alpha}(t,\omega,z).\end{aligned}$$

Now, applying the second FrHClIT (1.4), the relation (1.10) is established and then setting $s=0$ we obtain (1.11).

This completes the proof of Lemma 1.3. □

Let $\phi,\psi \in L^1_{\nu+\mu}(I)$. Then, the fractional Hankel–Clifford integral convolution is defined by

$$(\psi\#_\alpha\phi)(t) = \gamma_{\nu,\mu}^\alpha\int_0^\infty ({}_\alpha\tau_t\psi)(\omega)\mathrm{e}^{\mathrm{i}\omega\cot\alpha}\omega^{\frac{\mu+\nu}{2}}\phi(\omega)d\omega, \tag{1.12}$$

where the fractional Hankel–Clifford integral translation ${}_\alpha\tau_t$ of ψ is defined by

$$({}_\alpha\tau_t\psi)(\omega) = \psi_\alpha(t,\omega)$$

$$= \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} D_{\nu,\mu,\alpha}(t,\omega,z) e^{iz\cot\alpha} z^{\frac{\mu+\nu}{2}} \psi(z) dz, \tag{1.13}$$

provided that the above integrals exists and $0 < t, \omega, z < \infty$.

Parseval's identity If $\tilde{f}$ and $\tilde{g}$ are the second FrHClIT of $f(t)$ and $g(t)$ respectively, then under certain assumptions,

$$\int_0^{\infty} f(t) g^*(t) t^{\mu} dt = \int_0^{\infty} \tilde{f}(\omega) \tilde{g}^*(\omega) \omega^{\mu} d\omega \tag{1.14}$$

holds.

2 The Fractional Bessel–Clifford Wavelets

Fractional wavelets (see [13, 16]) are a family of functions constructed from translation, dilation and chirp modulation of the mother wavelet $\psi \in L^2(\mathbb{R})$. In the present work based on the ideas of fractional wavelets and inspired by the work of [8, 9, 12, 17], mathematically we define the fractional wavelet $\psi_{b,a,\alpha}$ of function (Bessel–Clifford wavelet) $\psi \in L^2_{\nu+\mu}(I)$ with dilation and translation parameters $a > 0, b \geq 0$ respectively and $0 < \alpha < \pi$, as

$$\begin{aligned} \psi_{b,a,\alpha}(t) &= \mathcal{D}_a({}_{\alpha}\tau_b \psi)(t) = \mathcal{D}_a \psi_{\alpha}(b,t) \\ &= a^{-\mu-1} e^{i(\frac{1}{a}-1)(t+b)\cot\alpha} \psi_{\alpha}(b/a, t/a) \\ &= a^{-\mu-1} e^{i(\frac{1}{a}-1)(t+b)\cot\alpha} \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} D_{\nu,\mu,\alpha}(b/a, t/a, z) e^{iz\cot\alpha} z^{\frac{\mu+\nu}{2}} \psi(z) dz, \end{aligned} \tag{2.1}$$

where $\mathcal{D}_a$ denotes the fractional dilation operator.

A Bessel–Clifford wavelet is a function $\psi \in L^2_{\nu+\mu}(I)$, which satisfies the condition

$$\mathcal{C}^{\psi}_{\nu,\mu,\alpha} = \int_0^{\infty} \omega^{\mu-\nu-1} \Big| (h^{\alpha}_{2,\nu,\mu} e^{-iz\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z))(\omega) \Big|^2 d\omega < \infty, \tag{2.2}$$

then $\mathcal{C}^{\psi}_{\nu,\mu,\alpha}$ is known as the admissibility condition of the Bessel–Clifford wavelet.

Proposition 2.1 *If $\psi \in L^2_{\nu+\mu}(I)$, then the second* FrHClIT *of $\psi_{b,a,\alpha}$ is given by*

$$\begin{aligned} (h^{\alpha}_{2,\nu,\mu} \psi_{b,a,\alpha})(\omega) = (a\omega)^{\frac{\mu-\nu}{2}} e^{-i[(a-1)\omega+b]\cot\alpha} C_{\nu,\mu}(b\omega \csc^2\alpha) \\ \times h^{\alpha}_{2,\nu,\mu}[e^{-iz\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z)](a\omega). \end{aligned}$$

Proof Let $\psi \in L^2_{\nu+\mu}(I)$. Then using (2.1), (1.10) and (1.3), we have

$$\begin{aligned} (h^{\alpha}_{2,\nu,\mu} \psi_{b,a,\alpha})(\omega) &= \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} C_{\nu,\mu}(t\omega \csc^2\alpha) e^{i(t+\omega)\cot\alpha} t^{\mu} \psi_{b,a,\alpha}(t) dt \\ &= \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} C_{\nu,\mu}(t\omega \csc^2\alpha) e^{i(t+\omega)\cot\alpha} t^{\mu} a^{-\mu-1} e^{i(t+b)(\frac{1}{a}-1)\cot\alpha} \\ &\quad \times \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} \psi(z) D_{\nu,\mu,\alpha}(b/a, t/a, z) e^{iz\cot\alpha} z^{\frac{\mu+\nu}{2}} dz dt \\ &= \gamma^{\alpha}_{\nu,\mu} \int_0^{\infty} e^{i(\omega-b)\cot\alpha} C_{\nu,\mu}(b\omega \csc^2\alpha) C_{\nu,\mu}(za\omega \csc^2\alpha) (a\omega)^{\frac{\mu-\nu}{2}} \psi(z) z^{\frac{\mu+\nu}{2}} dz \\ &= (a\omega)^{\frac{\mu-\nu}{2}} e^{-i[(a-1)\omega+b]\cot\alpha} C_{\nu,\mu}(b\omega \csc^2\alpha) h^{\alpha}_{2,\nu,\mu}[e^{-iz\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z)](a\omega). \end{aligned}$$

This proves Proposition 2.1. □

3 Fractional Wavelet Transformation (FrWT)

In this section, we now introduce the fractional wavelet transformation (FrWT) associated with the second FrHClIT of a function $f \in L^2_{\nu+\mu}(I)$ with respect to Bessel–Clifford wavelet $\psi \in L^2_{\nu+\mu}(I)$ as

$$(\mathrm{HW}^{\alpha}_{\psi} f)(b,a) = \int_0^{\infty} \psi^*_{b,a,\alpha}(t) f(t) t^{\mu} dt.$$
$$= a^{-\mu-1} \int_0^{\infty} \mathrm{e}^{-\mathrm{i}(\frac{1}{a}-1)(t+b)\cot\alpha} \psi^*_{\alpha}(b/a, t/a) f(t) t^{\mu} dt. \tag{3.1}$$

Using Parseval's formula (1.14) and Proposition 2.1, the fractional wavelet transformation of a function $f \in L^2_{\nu+\mu}(I)$ can be written in the following form

$$(\mathrm{HW}^{\alpha}_{\psi} f)(b,a) = \frac{1}{\gamma^{\alpha}_{\nu,\mu}} h^{\alpha}_{2,\nu,\mu}[(a\omega)^{\frac{\mu-\nu}{2}} \mathrm{e}^{\mathrm{i}(a-2)\omega\cot\alpha} \tilde{f}(\omega)$$
$$\times (h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z)))^*(a\omega)](b). \tag{3.2}$$

Theorem 3.1 *If ψ and ϕ are Bessel–Clifford wavelets and $f, g \in L^2_{\nu+\mu}(I)$, then*

(i) $\left(\mathrm{HW}^{\alpha}_{\psi}(\beta_1 f + \beta_2 g)\right)(b,a) = \beta_1 (\mathrm{HW}^{\alpha}_{\psi} f)(b,a) + \beta_2 (\mathrm{HW}^{\alpha}_{\psi} g)(b,a)$;

(ii) $\left(\mathrm{HW}^{\alpha}_{\beta_1\phi+\beta_2\psi} f)\right)(b,a) = \beta_1^* \left(\mathrm{HW}^{\alpha}_{\phi} f\right)(b,a) + \beta_2^* \left(\mathrm{HW}^{\alpha}_{\psi} f\right)(b,a)$;

(iii) $\left(\mathrm{HW}^{\alpha}_{\psi}({}_{\alpha}\tau_c f)\right)(b,a) = a^{-\mu-1} \mathrm{e}^{-\mathrm{i}(c-b)(1-1/a)} \left(\mathrm{HW}^{\alpha}_{f}({}_{\alpha}\tau_{b/a}\psi)\right)^* (c/a, 1/a), \quad c > 0$;

(iv) $\left(\mathrm{HW}^{\alpha}_{\mathcal{D}_c\psi} f)\right) b, a) = \left(\mathrm{HW}^{\alpha}_{\psi} f\right)(b, ac)$, $c > 0$; *where β_1, β_2 are scalars.*

Proof The proof is obvious and is omitted. □

Theorem 3.2 *Let ϕ and ψ be two Bessel–Clifford wavelets and $(\mathrm{HW}^{\alpha}_{\phi} f)(b,a)$ and $(\mathrm{HW}^{\alpha}_{\psi} g)(b, a)$ denote the* FrWT *of f and $g \in L^2_{\nu+\mu}(I)$ respectively. Then*

$$\int_0^{\infty}\int_0^{\infty} \left(\mathrm{HW}^{\alpha}_{\phi} f\right)(b,a) \left(\mathrm{HW}^{\alpha}_{\psi} g\right)^*(b,a) b^{\mu} db \frac{da}{a} = (\sin^2\alpha)^{\mu+1} \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} \int_0^{\infty} f(t) g^*(t) t^{\mu} dt,$$

where

$$\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} = \int_0^{\infty} \omega^{\mu-\nu-1} [h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \phi(z))]^*(\omega)$$
$$\times [h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z))](\omega) d\omega < \infty. \tag{3.3}$$

Proof Using (3.2) and Parseval's identity (1.14), we have

$$\int_0^{\infty}\int_0^{\infty} \left(\mathrm{HW}^{\alpha}_{\phi} f\right)(b,a) \left(\mathrm{HW}^{\alpha}_{\psi} g\right)^*(b,a) b^{\mu} db \frac{da}{a}$$
$$= (\sin^2\alpha)^{\mu+1} \int_0^{\infty}\int_0^{\infty} (a\omega)^{\frac{\mu-\nu}{2}} \tilde{f}^{\alpha}(\omega) (\tilde{g}^{\alpha})^*(\omega) \omega^{\mu} [h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \phi(z))]^*(a\omega)$$
$$\times [h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi(z))](a\omega) d\omega da$$
$$= (\sin^2\alpha)^{\mu+1} \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} \int_0^{\infty} \tilde{f}^{\alpha}(\omega) (\tilde{g}^{\alpha})^*(\omega) \omega^{\mu} d\omega.$$

Then, by utilizing Parseval's identity (1.14), Theorem 3.2 is established. □

Remark 3.3 The following are its deductions:

(i) If $f = g$, then

$$\int_0^{\infty}\int_0^{\infty} \left(\mathrm{HW}^{\alpha}_{\phi} f\right)(b,a) \left(\mathrm{HW}^{\alpha}_{\psi} f\right)^*(b,a) b^{\mu} db \frac{da}{a} = (\sin^2\alpha)^{\mu+1} \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} \int_0^{\infty} |f(t)|^2 t^{\mu} dt.$$

(ii) If $\phi = \psi$, then

$$\int_0^\infty \int_0^\infty \left(\mathrm{HW}_\psi^\alpha f\right)(b,a)\left(\mathrm{HW}_\psi^\alpha g\right)^*(b,a) b^\mu db \frac{da}{a} = (\sin^2\alpha)^{\mu+1}\mathcal{C}_{\nu,\mu,\alpha}^{\psi}\int_0^\infty f(t)g^*(t)t^\mu dt.$$

(iii) If $\phi = \psi$ and $f = g$, then

$$\int_0^\infty \int_0^\infty |\left(\mathrm{HW}_\psi^\alpha f\right)(b,a)|^2 b^\mu db \frac{da}{a} = (\sin^2\alpha)^{\mu+1}\mathcal{C}_{\nu,\mu,\alpha}^{\psi}\int_0^\infty |f(t)|^2 t^\mu dt,$$

where $\mathcal{C}_{\nu,\mu,\alpha}^{\psi}$ is given by (2.2).

Theorem 3.4 (Inversion formula) *Let $f \in L^2_{\nu+\mu}(I)$. If ϕ and ψ are Bessel–Clifford wavelets, then f can be reconstructed by the formula*

$$f(t) = \frac{(\csc^2\alpha)^{\mu+1}}{\mathcal{C}_{\nu,\mu,\alpha}^{\phi,\psi}}\int_0^\infty \int_0^\infty \left(\mathrm{HW}_\phi^\alpha f\right)(b,a)\psi_{b,a,\alpha}(t) b^\mu db \frac{da}{a}.$$

Proof Let $f \in L^2_{\nu+\mu}(I)$. Then for any arbitrary $g \in L^2_{\nu+\mu}(I)$, we have from Theorem 3.2,

$$\begin{aligned}&(\sin^2\alpha)^{\mu+1}\mathcal{C}_{\nu,\mu,\alpha}^{\phi,\psi}\int_0^\infty f(t)g^*(t)t^\mu dt\\ &= \int_0^\infty \left(\int_0^\infty \int_0^\infty \left(\mathrm{HW}_\phi^\alpha f\right)(b,a)\psi_{b,a,\alpha}(t) b^\mu db \frac{da}{a}\right) g^*(t)t^\mu dt,\end{aligned}$$

which on simplification yield the theorem. □

4 Examples of Second FrHClIT and FrWT

4.1 Table of the Second FrHClIT of Some Functions:

$f(t)$	$(h^\alpha_{2,\nu,\mu} f)(\omega)$
$\delta(x-p)$	$\gamma^\alpha_{\nu,\mu}\mathrm{e}^{\mathrm{i}(\omega+p)\cot\alpha} p^\mu C_{\nu,\mu}(p\omega\csc^2\alpha)$
$t^{-\nu/2}\mathrm{e}^{-\mathrm{i}t\cot\alpha}J_\mu(2\sqrt{pt})$	$\frac{\gamma^\alpha_{\nu,\mu}}{\Gamma(\nu-\mu)}\mathrm{e}^{\mathrm{i}\omega\cot\alpha}p^{\mu/2}(\omega\csc^2\alpha)^{-(\mu+\nu)/2}$ $\times(\omega\csc^2\alpha - p)^{\nu-\mu-1};\ p<\omega\csc^2\alpha<\infty$
$t^{-\mu/2}\mathrm{e}^{-\mathrm{i}t(\cot\alpha-\mathrm{i}q)}J_\nu(2\sqrt{pt})$	$\frac{\gamma^\alpha_{\nu,\mu}}{q}\mathrm{e}^{\mathrm{i}\omega\cot\alpha}(\omega\csc^2\alpha)^{-\mu/2}$ $\times\mathrm{e}^{-\frac{(\omega\csc^2\alpha+p)}{q}}I_\nu\left(\frac{2\sqrt{p\omega\csc^2\alpha}}{q}\right)$
$t^{(\nu-\mu)/2}\mathrm{e}^{-\mathrm{i}t(\cot\alpha-\mathrm{i})}L^\nu_n(t)$	$\gamma^\alpha_{\nu,\mu}\mathrm{e}^{\mathrm{i}\omega\cot\alpha}\frac{\sqrt{2}}{n!}(\omega\csc^2\alpha)^{(2n+\nu-\mu+1/2)/2}\mathrm{e}^{-\omega\csc^2\alpha}$
$t^{(\nu-\mu)/2}\mathrm{e}^{-\mathrm{i}t\cot\alpha}(p^2+t)^{-1}$	$\gamma^\alpha_{\nu,\mu}\mathrm{e}^{\mathrm{i}\omega\cot\alpha}(\omega\csc^2\alpha)^{-\mu/2}p^\nu K_\nu(2p\sqrt{\omega\csc^2\alpha})$
$t^{(\nu-\mu)/2}\mathrm{e}^{-\mathrm{i}t\cot\alpha}(p^2-t)^{-\nu-1/2}$	$\gamma^\alpha_{\nu,\mu}\frac{2^{2\nu+1/2}\Gamma(1/2-\nu)}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}\omega\cot\alpha}$ $\times(\omega\csc^2\alpha)^{\frac{\nu-\mu-1/2}{2}}\sin(2p\sqrt{\omega\csc^2\alpha})$

where $p>0$, $q>0$, n is an integer, $\mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1$, I_ν, K_ν and L^ν_n are known as modified Bessel functions of the first, third kind and generalized Laguerre polynomial respectively.

Example 4.1 Assume that $f(t) = \mathrm{e}^{-\mathrm{i}t(\cot\alpha - \mathrm{i}p)}$, $p>0$.

Then the FrWT of $f(t)$ is given by

$$(\mathrm{HW}_\psi^\alpha f(t))(b,a) = \int_0^\infty f(t)\psi_{b,a,\alpha}^*(t)t^\mu dt$$
$$= (\gamma_{\nu,\mu}^\alpha)^* a^{-\mu-1} \int_0^\infty \mathrm{e}^{-\mathrm{i}(t+b)(\frac{1}{a}-1)\cot\alpha}$$
$$\times \left[\int_0^\infty \psi(z) D_{\nu,\mu,\alpha}(b/a, t/a, z)\mathrm{e}^{\mathrm{i}z\cot\alpha} z^{\frac{\mu+\nu}{2}} dz\right]^* f(t)t^\mu dt.$$

Substituting the value of $D_{\nu,\mu,\alpha}(b/a,t/a,z)$ from (1.12) in the last integral and evaluating the resulting integral by means of the formula [3, p. 29], then we have

$$(\mathrm{HW}_\psi^\alpha f(t))(b,a)$$
$$= a^{-\nu/2-2} b^{-\mu/2} p^{-\nu} (\csc\alpha)^{2-3\mu+\nu} \mathrm{e}^{\mathrm{i}b\cot\alpha} \int_0^\infty \psi^*(z) z^{\nu/2}$$
$$\times \int_0^\infty \exp\left(\frac{-s\csc^2\alpha}{ap}\right) J_\nu(2\sqrt{zs\csc^2\alpha}) J_\nu(2\sqrt{(b/a)s\csc^2\alpha}) ds dz.$$

Now, evaluating the last integral by means of the formula [3, p. 23], we have

$$(\mathrm{HW}_\psi^\alpha f(t))(b,a) = a^{-\nu/2} b^{-\mu/2} p^{-\nu+1} (\csc\alpha)^{\nu-3\mu} \mathrm{e}^{\mathrm{i}b\cot\alpha}$$
$$\times \int_0^\infty \psi^*(z) z^{\nu/2} \exp(-(b+az)p) I_\nu(2p\sqrt{abz}) dz,$$

where $I_\nu(x)$ is the modified Bessel function of first kind of order ν.

5 Fractional Powers of Bessel–Clifford Wavelet Packet Transformation (FrBWPT)

Posch [10] studied the wave packet transformation as applied to signal processing using the Weyl operator and established some fruitful properties. Huang and Suter [4] introduced the fractional wave packet transformation associated with fractional Fourier transform and obtained some properties including the version of the Resolution of the Identity for fractional wave packet transformation. In this section, we introduce the fractional powers of Bessel–Clifford wavelet packet transformation (FrBWPT) involving second FrHClIT as

$$(\mathrm{BWP}_\psi^\alpha f)(u,b,a) = \int_0^\infty K_2^\alpha(t,u)\psi_{b,a,\alpha}^*(t) f(t) dt, \tag{5.1}$$

where $K_2^\alpha(t,u)$ and $\psi_{b,a,\alpha}$ are defined as (1.5) and (2.1) respectively.

More precisely, the FrBWPT is given by

$$(\mathrm{BWP}_\psi^\alpha f)(u,b,a) = \gamma_{\nu,\mu}^\alpha \int_0^\infty C_{\nu,\mu}(ut\csc^2\alpha)\mathrm{e}^{\mathrm{i}(u+t)\cot\alpha}\psi_{b,a,\alpha}^*(t) f(t) t^\mu dt. \tag{5.2}$$

Assuming

$$f_{u,\alpha}(t) = \gamma_{\nu,\mu}^\alpha C_{\nu,\mu}(ut\csc^2\alpha)\mathrm{e}^{\mathrm{i}(u+t)\cot\alpha} f(t),$$

and using Parseval's identity (1.14), (5.2) becomes

$$(\mathrm{BWP}_\psi^\alpha f)(u,b,a) = \int_0^\infty \tilde{f}_{u,\alpha}(\omega)\tilde{\psi}_{b,a,\alpha}^*(\omega)\omega^\mu d\omega. \tag{5.3}$$

We see that

$$\tilde{f}_{u,\alpha}(\omega) = \gamma_{\nu,\mu}^\alpha \int_0^\infty C_{\nu,\mu}(t\omega\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+\omega)\cot\alpha} t^\mu f_{u,\alpha}(t) dt$$

$$= \left(\gamma^{\alpha}_{\nu,\mu}\right)^2 \int_0^\infty C_{\nu,\mu}(t\omega \csc^2\alpha) C_{\nu,\mu}(ut\csc^2\alpha) \mathrm{e}^{\mathrm{i}(u+\omega+2t)\cot\alpha} f(t)t^{\mu}dt$$
$$= \left(\gamma^{\alpha}_{\nu,\mu}\right)^2 \mathrm{e}^{\mathrm{i}2(\omega+u)\cot\alpha} \int_0^\infty (\mathrm{e}^{-\mathrm{i}(\omega+u)\cot\alpha} C_{\nu,\mu}(t\omega\csc^2\alpha) C_{\nu,\mu}(tu\csc^2\alpha)$$
$$\times \mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\mu-\nu}{2}}) \mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\mu+\nu}{2}} f(t)dt.$$

Using (1.10) and (1.13), we have

$$\tilde{f}_{u,\alpha}(\omega) = \left(\gamma^{\alpha}_{\nu,\mu}\right)^3 \mathrm{e}^{\mathrm{i}2(\omega+u)\cot\alpha} \int_0^\infty \int_0^\infty C_{\nu,\mu}(tz\csc^2\alpha) \mathrm{e}^{\mathrm{i}(t+z)\cot\alpha} D^{\alpha}_{\nu,\mu,\alpha}(u,\omega,z)$$
$$\times z^{\mu} \mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\mu+\nu}{2}} f(t)dzdt$$
$$= \left(\gamma^{\alpha}_{\nu,\mu}\right)^2 \mathrm{e}^{\mathrm{i}2(\omega+u)\cot\alpha} \int_0^\infty h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f(t))(z) D^{\alpha}_{\nu,\mu,\alpha}(u,\omega,z) z^{\mu} dz$$
$$= \gamma^{\alpha}_{\nu,\mu} \mathrm{e}^{\mathrm{i}2(\omega+u)\cot\alpha} {}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f(t))(z))(\omega). \tag{5.4}$$

Hence, from (5.4) and Proposition 2.1, (5.3) may be written as

$$(\mathrm{BWP}^{\alpha}_{\psi} f)(u,b,a)$$
$$= \gamma^{\alpha}_{\nu,\mu} \mathrm{e}^{\mathrm{i}2u\cot\alpha} \int_0^\infty C_{\nu,\mu}(b\omega\csc^2\alpha) \mathrm{e}^{\mathrm{i}(\omega+b)\cot\alpha} (a\omega)^{\frac{\mu-\nu}{2}} \mathrm{e}^{\mathrm{i}a\omega\cot\alpha}$$
$$\times \{h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi)(a\omega)\}^*$$
$$\times\ {}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f)(z))(\omega)\omega^{\mu} d\omega$$
$$= \mathrm{e}^{\mathrm{i}2u\cot\alpha} h^{\alpha}_{2,\nu,\mu}[(a\omega)^{\frac{\mu-\nu}{2}} \mathrm{e}^{\mathrm{i}a\omega\cot\alpha} \{h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi)(a\omega)\}^*$$
$$\times\ {}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f)(z))(\omega)](b). \tag{5.5}$$

Theorem 5.1 *If ϕ and ψ are two Bessel–Clifford wavelets and $(\mathrm{BWP}^{\alpha}_{\phi} f)(u,b,a)$ and $(BWP^{\alpha}_{\psi} g)(u,b,a)$ denote the* FrBWPT*'s of f and $g \in L^2_{\nu+\mu}(I)$ respectively, then*

$$\int_0^\infty \int_0^\infty (\mathrm{BWP}^{\alpha}_{\phi} f)(u,b,a)(\mathrm{BWP}^{\alpha}_{\psi} g)^*(u,b,a) b^{\mu} db \frac{da}{a}$$
$$= \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} \int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2 f(t) g^*(t) t^{\mu} dt, \tag{5.6}$$

where $\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}$ is as (3.3).

Proof Using (5.5), we have

$$\int_0^\infty \int_0^\infty (\mathrm{BWP}^{\alpha}_{\phi} f)(u,b,a)(\mathrm{BWP}^{\alpha}_{\psi} g)^*(u,b,a) b^{\mu} db \frac{da}{a}$$
$$= \int_0^\infty \int_0^\infty (a\omega)^{\mu-\nu} {}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f)(z))(\omega)$$
$$\times \{h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \phi)\}^*(a\omega) \{h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{-\mathrm{i}z\cot\alpha} z^{\frac{\nu-\mu}{2}} \psi)\}(a\omega)$$
$$\times \left({}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} g)(z))\right)^*(\omega)\omega^{\mu} d\omega \frac{da}{a}$$
$$= \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha} \int_0^\infty {}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} f)(z))(\omega)$$
$$\times \left({}_{\alpha}\tau_u (z^{\frac{\mu-\nu}{2}} \mathrm{e}^{-\mathrm{i}z\cot\alpha} h^{\alpha}_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha} t^{\frac{\nu-\mu}{2}} g)(z))\right)^*(\omega)\omega^{\mu} d\omega.$$

Hence by using Parseval's identity (1.14), then the above expression becomes

$$\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)(\mathrm{BWP}^\alpha_\psi g)^*(u,b,a)b^\mu db\frac{da}{a}$$
$$= \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}\int_0^\infty [h^\alpha_{2,\nu,\mu}{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z))(\omega)](t)$$
$$\times[h^\alpha_{2,\nu,\mu}{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}g)(z))(\omega)]^*(t)t^\mu dt. \tag{5.7}$$

Now, we see that

$$[h^\alpha_{2,\nu,\mu}{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z))(\omega)](t)$$
$$= \gamma^\alpha_{\nu,\mu}\int_0^\infty C_{\nu,\mu}(t\omega\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+\omega)\cot\alpha}\omega^\mu$$
$$\times{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z))(\omega)d\omega$$
$$= (\gamma^\alpha_{\nu,\mu})^2\int_0^\infty C_{\nu,\mu}(t\omega\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+\omega)\cot\alpha}\int_0^\infty D^\alpha_{\nu,\mu,\alpha}(u,\omega,z)$$
$$\times h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z)z^\mu dz)\omega^\mu d\omega. \tag{5.8}$$

Therefore, using (1.10) and (1.6) in (5.8), we obtain

$$[h^\alpha_{2,\nu,\mu}{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z))(\omega)](t)$$
$$= \gamma^\alpha_{\nu,\mu}\int_0^\infty \mathrm{e}^{-\mathrm{i}(u+z)\cot\alpha}C_{\nu,\mu}(ut\csc^2\alpha)C_{\nu,\mu}(zt\csc^2\alpha)\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\mu-\nu}{2}}$$
$$\times h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}f)(z)z^\mu dz$$
$$= \mathrm{e}^{2\mathrm{i}(1+\nu)(\alpha-\pi/2)}\mathrm{e}^{-\mathrm{i}(u-3t)\cot\alpha}C_{\nu,\mu}(ut\csc^2\alpha)f(t). \tag{5.9}$$

Similarly, we get

$$[h^\alpha_{2,\nu,\mu}{}_\alpha\tau_u(z^{\frac{\mu-\nu}{2}}\mathrm{e}^{-\mathrm{i}z\cot\alpha}h^\alpha_{2,\nu,\mu}(\mathrm{e}^{\mathrm{i}t\cot\alpha}t^{\frac{\nu-\mu}{2}}g)(z))(\omega)]^*(t)$$
$$= \mathrm{e}^{-2\mathrm{i}(1+\nu)(\alpha-\pi/2)}\mathrm{e}^{\mathrm{i}(u-3t)\cot\alpha}C_{\nu,\mu}(ut\csc^2\alpha)g^*(t). \tag{5.10}$$

Hence, using (5.9) and (5.10) in (5.7), we have

$$\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)(\mathrm{BWP}^\alpha_\psi g)^*(u,b,a)b^\mu db\frac{da}{a}$$
$$= \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}\int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2 f(t)g^*(t)t^\mu dt. \tag{5.11}$$

Thus, we obtain the required result. □

Remark 5.2 The following are its deductions:

(i) If $f = g$, then

$$\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)(\mathrm{BWP}^\alpha_\psi f)^*(u,b,a)b^\mu db\frac{da}{a}$$
$$= \mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}\int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2|f(t)|^2 t^\mu dt,$$

(ii) If $\phi = \psi$, then

$$\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\psi f)(u,b,a)(\mathrm{BWP}^\alpha_\psi g)^*(u,b,a)b^\mu db\frac{da}{a}$$

$$= \mathcal{C}^{\psi}_{\nu,\mu,\alpha}\int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2 f(t)g^*(t)t^\mu dt,$$

(iii) If $\phi=\psi$ and $f=g$, then

$$\int_0^\infty\int_0^\infty |(\mathrm{BWP}^\alpha_\psi f)(u,b,a)|^2 b^\mu db\frac{da}{a} = \mathcal{C}^{\psi}_{\nu,\mu,\alpha}\int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2|f(t)|^2t^\mu dt$$

where $\mathcal{C}^{\psi}_{\nu,\mu,\alpha}$ is as in (2.2).

Theorem 5.3 (Inversion formula) *Let* $f\in L^2_{\nu+\mu}(I)$ *and* $(\mathrm{BWP}^\alpha_\psi f)(u,b,a)$ *be the corresponding* FrBWPT. *Then* f *can be reconstructed by the formula*

$$f(t) = \frac{\mathrm{e}^{-\mathrm{i}(t+u)\cot\alpha}\left(\gamma^\alpha_{\nu,\mu}\right)^*}{\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}C_{\nu,\mu}(ut\csc^2\alpha)}\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)\psi_{b,a,\alpha}(t)b^\mu db\frac{da}{a}$$

or

$$f_{u,\alpha}(t) = \frac{(\csc^2\alpha)^{\mu+1}}{\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}}\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)\psi_{b,a,\alpha}(t)b^\mu db\frac{da}{a},$$

where $\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}$ *is as* (3.3).

Proof Let us consider an arbitrary function $g\in L^2_{\nu+\mu}(I)$. Then from Theorem 5.1, we have

$$\begin{aligned}
&\mathcal{C}^{\phi,\psi}_{\nu,\mu,\alpha}\int_0^\infty (C_{\nu,\mu}(ut\csc^2\alpha))^2 f(t)g^*(t)t^\mu dt\\
&= \int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)(\mathrm{BWP}^\alpha_\psi g)^*(u,b,a)b^\mu db\frac{da}{a}\\
&= \int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)\left(\int_0^\infty K^\alpha_2(t,u)\psi^*_{b,a,\alpha}(t)g(t)dt\right)^* b^\mu db\frac{da}{a}\\
&= \left(\gamma^\alpha_{\nu,\mu}\right)^*\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{BWP}^\alpha_\phi f)(u,b,a)C_{\nu,\mu}(ut\csc^2\alpha)\mathrm{e}^{-\mathrm{i}(t+u)\cot\alpha}\\
&\quad\times\psi_{b,a,\alpha}(t)b^\mu db\frac{da}{a}g^*(t)t^\mu dt,
\end{aligned}$$

which on comparing yields the required result. □

Note 5.4 Similar results of Sections 2–5 can be found by using first FrHClIT.

6 Example of FrBWPT

Example 6.1 Assume that $f(t)=\frac{\delta(t-c)}{t}$, $0<c<\infty$, then the FrBWPT is given by

$$\begin{aligned}
(\mathrm{BWP}^\alpha_\psi f)(u,b,a) &= \int_0^\infty K^\alpha_2(t,u)\psi^*_{b,a,\alpha}f(t)dt\\
&= \gamma^\alpha_{\nu,\mu}\int_0^\infty C_{\nu,\mu}(tu\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+u)\cot\alpha}a^{-\mu-1}\mathrm{e}^{-\mathrm{i}(t+b)\left(\frac{1}{a^2}-1\right)\cot\alpha}\\
&\quad\times\psi^*_\alpha(b/a,t/a)f(t)t^\mu dt\\
&= \csc^2\alpha\int_0^\infty C_{\nu,\mu}(tu\csc^2\alpha)\mathrm{e}^{\mathrm{i}(t+u)\cot\alpha}a^{-\mu-1}\mathrm{e}^{-\mathrm{i}(t+b)\left(\frac{1}{a^2}-1\right)\cot\alpha}\\
&\quad\times\left[\int_0^\infty D_{\nu,\mu,\alpha}(b/a,t/a,z)\mathrm{e}^{\mathrm{i}z\cot\alpha}z^{\frac{\mu+\nu}{2}}\psi(z)dz\right]^* f(t)t^\mu dt.
\end{aligned}$$

Using (1.9), we have

$$(\mathrm{BWP}^\alpha_\psi f)(u,b,a)$$

$$= \frac{\gamma^{\alpha}_{\nu,\mu} a^{-\mu-1} e^{i(u+b)\cot\alpha}}{2^{2\nu}\Gamma(\nu+1/2)\sqrt{\pi}(\csc^2\alpha)^{\frac{3\mu-\nu}{2}}} \int_0^\infty \int_0^\infty \frac{\Delta^{2\nu-1}}{(bt/a^2)^{\frac{\nu+\mu}{2}}} C_{\nu,\mu}(tu\csc^2\alpha)$$
$$\times e^{i2t\cot\alpha}\psi^*(z)t^{\mu}\frac{\delta(t-c)}{t}dtdz$$
$$= \gamma^{\alpha}_{\nu,\mu}(\csc^2\alpha)^{(\nu-3\mu)/2} b^{-(\mu+\nu)/2} c^{(\mu-\nu-2)/2} e^{i(b+u+2c)\cot\alpha} C_{\nu}(cu\csc^2\alpha)$$
$$\times \frac{a^{\nu-1}}{2^{2\nu}\Gamma(\nu+1/2)\sqrt{\pi}} \int_0^\infty \psi^*(z)\Delta^{2\nu-1}dz,$$

where Δ denotes the area of a triangle having sides b/a, c/a and z if such a triangle exists.

7 Fractional Bessel Integral Transformation

It is seen that for $\mu = 0$ the two variants of FrHClIT's defined in (1.2) and (1.4) coincide that is these are act as a self-adjoint transformation. We named this transformation as fractional Bessel integral transformation, which is denoted by $\mathfrak{H}^{\alpha}$ and defined on function $f \in L^1_{\nu}$ as

$$(\mathfrak{H}^{\alpha} f)(\omega) = \int_0^\infty \mathcal{K}_{\alpha}(t,\omega) f(t)dt, \tag{7.1}$$

where the kernel

$$\mathcal{K}_{\alpha}(t,\omega) = \begin{cases} \gamma^{\alpha}_{\nu} J_{\nu}(2\csc\alpha\sqrt{t\omega})e^{i(t+\omega)\cot\alpha}, & \alpha \neq n\pi, \\ \delta(t-\omega), & \alpha = n\pi, \end{cases} \tag{7.2}$$

and $\gamma^{\alpha}_{\nu} = e^{i(\nu+1)(\alpha-\frac{\pi}{2})}\csc\alpha$.

The corresponding inversion formula is given by

$$f(t) = \int_0^\infty \mathcal{K}^*_{\alpha}(t,\omega)(\mathfrak{H}^{\alpha} f)(\omega)d\omega, \tag{7.3}$$

where the kernel $\mathcal{K}^*_{\alpha}(t,\omega)$ is complex conjugate of $\mathcal{K}_{\alpha}(t,\omega)$.

The $\mathcal{K}_{\alpha}(t,\omega)$ has the following properties:

$$\mathcal{K}_{\alpha}(t,\omega) = \mathcal{K}_{\alpha}(\omega,t),$$
$$\int_0^\infty \mathcal{K}_{\alpha}(t,\omega)\mathcal{K}_{\beta}(\omega,z)d\omega = \mathcal{K}_{\alpha+\beta}(t,z),$$
$$\int_0^\infty \mathcal{K}_{\alpha}(t,a)\mathcal{K}^*_{\alpha}(t,\omega)dt = \delta(\omega-a).$$

We note that

$$\mathcal{K}^*_{\alpha}(t,\omega) = \mathcal{K}_{-\alpha}(t,\omega).$$

Let $\Phi, \Psi \in L^1(I)$. Then we define the fractional Bessel integral convolution $(\Phi \star_{\alpha} \Psi)(t)$ by

$$(\Phi \star_{\alpha} \Psi)(t) = \gamma^{\alpha}_{\nu} \int_0^\infty ({}_{\alpha}\varsigma_t \Phi)(\omega)\Psi(\omega)e^{i\omega\cot\alpha}d\omega, \quad 0 < \omega < \infty, \tag{7.4}$$

where the fractional Bessel integral translation $({}_{\alpha}\varsigma_t\Phi)(\omega)$ is given by

$$({}_{\alpha}\varsigma_t\Phi)(\omega) = \gamma^{\alpha}_{\nu} \int_0^\infty D_{\nu,\alpha}(t,\omega,z)e^{iz\cot\alpha}\Phi(z)dz, \quad 0 < t,\omega < \infty, \tag{7.5}$$

provided the integrals convergent, and

$$D_{\nu,\alpha}(t,\omega,z) = \frac{(\gamma^{\alpha}_{\nu})^* \Delta^{2\nu-1} e^{-i(t+\omega+z)\cot\alpha}}{2^{2\nu}(\csc\alpha)^{(-\nu+2)}(t\omega z)^{\nu/2}\Gamma(\nu+1/2)\sqrt{\pi}}, \tag{7.6}$$

where $\Delta(t,\omega,z)$ denotes the area of a triangle having sides t,ω,z [19] if such a triangle exists and zero otherwise. Clearly, $D_{\nu,\alpha}(t,\omega,z)\geq 0$ and is symmetric in t,ω,z.

We now prove the following inequality which shall be used in the sequel:

Lemma 7.1 *If $\Phi\in L^2(I)$ and $\|\Phi\|=1$, then*

$$\int_0^\infty |({}_\alpha\varsigma_t\Phi)(\omega)|^2 d\omega \leq \frac{(t\csc^2\alpha)^\nu}{(\Gamma(\nu+1))^2}.$$

Proof By employing (7.5), using the Cauchy–Schwartz inequality and by Lemma 1.3 for $\mu=0$, we have

$$|({}_\alpha\varsigma_t\Phi)(\omega)|^2 \leq |\gamma_\nu^\alpha|\frac{(t\omega\csc^2\alpha)^{\nu/2}}{\Gamma(\nu+1)}\int_0^\infty |D_{\nu,\alpha}(t,\omega,z)|z^{-\nu/2}|\Phi(z)|^2 dz. \tag{7.7}$$

Again using Lemma 1.3 for $\mu=0$, we get the required inequality. □

8 Fractional Wavelet Packet Integral Transformation (FrWPIT)

In this section, we study the FrWPIT. Let us define the fractional wavelets $\Psi_{b,a,\alpha}$ of any Bessel wavelet $\Psi\in L^2(I)$ by

$$\begin{aligned}\Psi_{b,a,\alpha}(t) &= \mathfrak{D}_a({}_\alpha\varsigma_b\Psi)(t)\\ &= \frac{\Gamma(\nu+1)}{(\csc\alpha)^\nu}a^{(\nu-1)/2}({}_\alpha\varsigma_{b/a}\Psi)(t/a)\mathrm{e}^{-\mathrm{i}(t-a)\cot\alpha}\end{aligned} \tag{8.1}$$

for all $0<a; b\in\mathbb{R}$ and $\mathfrak{D}_a$ denotes the fractional dilation operator.

Now, the fractional wavelet transformation of a function $f\in L^2_\nu(I)$ with respect to wavelet $\Psi\in L^2(I)$ is defined by

$$(\mathscr{W}_\Psi^\alpha f)(b,a) = \int_0^\infty \Psi^*_{b,a,\alpha}(t)f(t)dt.$$

Wave packet transform is a generalization of wavelet transformation which have applications in signal processing [10], pattern recognition [18] and many more. The fractional wave packet transform (FrWPT) associated with Fourier/fractional Fourier transform have been studied [4, 10, 15]. Motivated by their works, in this paper we introduce the fractional wavelet packet integral transformation (FrWPIT) involving fractional Bessel integral transformation, denoted by $(\mathfrak{P}_\alpha f)(\omega,b,a)$ for a given signal $f(t)$ as follows:

$$(\mathfrak{P}^\alpha f)(\omega,b,a) = \int_0^\infty \mathcal{K}_\alpha(t,\omega)\Psi^*_{b,a,\alpha}(t)f(t)dt. \tag{8.2}$$

It is important to see that FrWPIT is a function of time, frequency and scale with parameter α. Now, we will show a version of Resolution of Identity for the transformation (8.2) stated as follows:

Proposition 8.1 *If $\Psi\in L^2(I)$, $(\mathfrak{P}^\alpha f)(\omega,b,a)$ and $(\mathfrak{P}^\alpha g)(\omega,b,a)$ denote the* FrWPIT*'s of functions f and $g\in L^2_\nu(I)$ respectively, then*

$$\int_0^\infty\int_0^\infty (\mathfrak{P}^\alpha f)(\omega,b,a)(\mathfrak{P}^\alpha g)^*(\omega,b,a)d\omega db \leq \int_0^\infty f(t)g^*(t)t^\nu dt.$$

Proof Using (8.1) and by an application of Lemma 7.1, we have

$$\int_0^\infty\int_0^\infty (\mathfrak{P}^\alpha f)(\omega,b,a)(\mathfrak{P}^\alpha g)^*(\omega,b,a)d\omega db$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \int_0^\infty \mathcal{K}_\alpha(t,\omega)\Psi^*_{b,a,\alpha}(t)f(t)dt \int_0^\infty \mathcal{K}^*_\alpha(t',\omega)\Psi_{b,a,\alpha}(t')g^*(t')dt'd\omega db \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \Psi^*_{b,a,\alpha}(t)\Psi_{b,a,\alpha}(t')f(t)g^*(t') \int_0^\infty \mathcal{K}_\alpha(t,\omega)\mathcal{K}^*_\alpha(t',\omega)d\omega dt'dtdb \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \Psi^*_{b,a,\alpha}(t)\Psi_{b,a,\alpha}(t')f(t)g^*(t')\delta(t'-t)dt'dtdb \\
&= \int_0^\infty \int_0^\infty |\Psi_{b,a,\alpha}(t)|^2 f(t)g^*(t)dtdb \\
&= \left(\frac{\Gamma(\nu+1)}{(\csc\alpha)^\nu}\right)^2 a^{\nu-1} \int_0^\infty \int_0^\infty |({}_\alpha\varsigma_{b/a}\Psi)(t/a)|^2 f(t)g^*(t)dtdb \\
&= \left(\frac{\Gamma(\nu+1)}{(\csc\alpha)^\nu}\right)^2 a^{\nu} \int_0^\infty f(t)g^*(t)dt \int_0^\infty |({}_\alpha\varsigma_{t/a}\Psi)(b/a)|^2 db \\
&\le \int_0^\infty f(t)g^*(t)t^\nu dt.
\end{aligned}
$$

This proves the proposition. □

Remark 8.2 As a consequence, we have the energy-preserving property by putting $f = g$ in Proposition 8.1:

$$\int_0^\infty \int_0^\infty \left|(\mathfrak{P}^\alpha f)(\omega, b, a)\right|^2 d\omega db \le \int_0^\infty |f(t)|^2 t^\nu dt.$$

9 Conclusion

In the beginning, we established some properties of the kernels $K_1^\alpha(x,y)$ and $K_2^\alpha(x,y)$. Then, we introduced the concept of FrWT and studied their basic properties. Also, a table of second FrHClIT and examples of FrWT have been obtained. Thereafter, combining the idea of FrHClIT and FrWT, we defined the FrBWPT. The Parseval's relation and an inversion formula of FrBWPT have been established. Moreover, we have observed that for $\mu = 0$ the two variant of FrHClIT coincide, which give forth the fractional Bessel integral transformation. In the last we introduced the concept of FrWPIT and derived a version of the resolution of the identity.

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