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Some Class 1 Graphs on g_c-colorings

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Abstract An edge-coloring of a graph G is an assignment of colors to all the edges of G. A g_c-coloring of a graph G is an edge-coloring of G such that each color appears at each vertex at least $g(v)$ times. The maximum integer k such that G has a g_c-coloring with k colors is called the g_c-chromatic index of G and denoted by $\chi'_{g_c}(G)$. In this paper, we extend a result on edge-covering coloring of Zhang and Liu in 2011, and give a new sufficient condition for a simple graph G to satisfy $\chi'_{g_c}(G) = \delta_g(G)$, where $\delta_g(G) = \min_{v\in V(G)}\{\lfloor \frac{d(v)}{g(v)} \rfloor\}$.

Keywords Edge-coloring, g_c-coloring, g_c-chromatic index, edge covering, classification problem

MR(2010) Subject Classification 05C15

1 Introduction

All graphs considered in this paper are finite, simple and undirected. An *edge-coloring* of a graph G is an assignment of colors to all the edges of G. Let G be a graph and g be a function which assigns a positive integer $g(v)$ to each vertex $v \in V(G)$. Let $C = \{c_1, c_2, \dots, c_k\}$ be an available color set. A g_c-*coloring* of a graph G is an edge-coloring of G such that each color of C appears at each vertex $v \in V(G)$ at least $g(v)$ times. The maximum integer k such that G has a g_c-coloring with k colors is called the g_c-*chromatic index* of G and denoted by $\chi'_{g_c}(G)$. If $g(v) = 1$ for all $v \in V(G)$, the g_c-coloring is reduced to the *edge-covering coloring*, in which each color appears at each vertex at least once, and the g_c-chromatic index of G is reduced to the *edge-covering coloring chromatic index* of G and denoted by $\chi'_c(G)$. Denote the maximum and minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively.

We define

$$\delta_g(G) = \min_{v\in V(G)} \left\{ \left\lfloor \frac{d(v)}{g(v)} \right\rfloor \right\},$$

$$V_{\delta_g}(G) = \{v \in V(G) : d(v) = g(v)\delta_g(G)\}.$$

Clearly, G has no g_c-coloring if and only if $\delta_g(G)$=0. If $\delta_g(G)$=1, then $\chi'_{g_c}(G)$=1. In the rest of the article, we concentrate ourselves on the case that $\delta_g(G) \ge 2$. We call the subgraph induced by $V_{\delta_g}(G)$ the g_c-*core* of G and denote it by G_{δ_g}.

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Lemma 1.1 ([3]) *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$. Then*

$$\delta_g(G) - 1 \le \chi'_{g_c}(G) \le \delta_g(G).$$

G is said to be of g_c-*class* 1 if $\chi'_{g_c}(G)$=$\delta_g(G)$ and of g_c-*class* 2 otherwise. In this paper, this classification problem on g_c-coloring will be concerned. Zhang and Liu [6] studied the edge-covering colorings of graphs and got some sufficient conditions for $\chi'_c(G)$=$\delta(G)$. In this article, we extend one of their results to g_c-colorings of graphs and obtain a similar conclusion. Also, we find a new sufficient condition for a graph G to satisfy $\chi'_{g_c}(G)$=$\delta_g(G)$.

Let $k \ge 2$ be an integer and $C = \{c_1, c_2, \dots, c_k\}$. Given an edge-coloring of G with k colors in C, for each $v \in V(G)$, let $c_i(v)$ denote the number of edges incident with v and colored with c_i $(1 \le i \le k)$. Call an edge-coloring of G with k colors in C *equitable* if $|c_i(v) - c_j(v)| \le 1$ for any $v \in V(G)$ and i, j with $1 \le i < j \le k$.

Let t be a positive integer. The notation $t|d(v)$ denotes that t divides $d(v)$. Define that $V_t(G) = \{v \in V(G) : t|d(v)\}$. Call the subgraph of G induced by $V_t(G)$ the t-*core* of G. We call a graph G t-*peelable*, if all the vertices of G can be iteratively peeled off in an order $v_1, v_2, \dots, v_n$ using the following t-*peeling operation*: For each $1 \le i \le n$, peel off vertex v_i if v_i has at most one neighbor v' satisfying that $v' \in V_t(G)$ and $d_{G_{i-1}}(v') = d_G(v')$, where $G_{i-1} = G - \{v_1, v_2, \dots, v_{i-1}\}$ $(2 \le i \le n)$ and $G_0 = G$.

Theorem 1.2 ([5]) *Let G be a graph and $k \ge 2$ be an integer. If G is k-peelable, then G has an equitable edge-coloring with k colors.*

Corollary 1.3 *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$ and $\delta_g(G) \ge 2$. If G is $\delta_g(G)$-peelable, then G is of g_c-class* 1.

Proof By Theorem 1.2, G has an equitable edge-coloring with $\delta_g(G)$ colors. Assume that there exist a vertex $v \in V(G)$ and a color $\alpha \in C$ such that $\alpha(v) < g(v)$. Since $d(v) \ge \delta_g(G)g(v)$ for each $v \in V(G)$ by the definition of $\delta_g(G)$, there must exist another color $\beta \in C$ which has $\beta(v) > g(v)$. This means that $\beta(v) - \alpha(v) \ge 2$, a contradiction. So G is of g_c-class 1. □

By imitating [6], we give the following concept. We call a graph G *weakly-$\delta_g(G)$-peelable*, if all the vertices of G can be iteratively peeled off in an order $v_1, v_2, \dots, v_n$ using the following *weakly-$\delta_g(G)$-peeling operation*: For each $1 \le i \le n$, peel off vertex v_i if v_i has at most one neighbor v' satisfying that $v' \in V_{\delta_g}(G)$ and $d_{G_{i-1}}(v') = d_G(v')$ in G_{i-1}, where $G_{i-1} = G - \{v_1, v_2, \dots, v_{i-1}\}$ $(2 \le i \le n)$ and $G_0 = G$.

Claim 1 Let G be a graph associated a positive integer function $g : V(G) \to Z^+$ and $\delta_g(G) \ge 2$. If G is $\delta_g(G)$-peelable, then G is weakly-$\delta_g(G)$-peelable. Conversely, this is not always true.

Proof Let $k = \delta_g(G)$. For a graph G, it is easy to see $V_k(G) \supseteq V_{\delta_g}(G)$ according to their definitions. So if G is $\delta_g(G)$-peelable, then G is weakly-$\delta_g(G)$-peelable. Conversely, we can not ensure $V_k(G) = V_{\delta_g}(G)$ for every graph. For example, K_5 has vertices $v_1, v_2, \dots, v_5$, in which $g(v_1) = g(v_2) = 2$, $g(v_3) = g(v_4) = g(v_5) = 1$. It is easy to verify that this graph is weakly-2-peelable but not 2-peelable. □

By Claim 1, the following result extends Corollary 1.3.

Theorem 1.4 *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$ and $\delta_g(G) > 0$. If G is weakly-$\delta_g(G)$-peelable, then G is of g_c-class* 1.

Let G be a graph which is not weakly-$\delta_g(G)$-peelable. Call $R(G)$ *the remaining subgraph of* G, which is obtained from G by peeling as many vertices as possible using weakly-$\delta_g(G)$-peeling operation. Construct a new graph $R'(G)$ by sticking some pendent edges to each vertex $v \in R(G)$ so that $d_{R'(G)}(v) = d_G(v)$. We call $R'(G)$ *the degree restoration* of the remaining subgraph of G (the DRRS of G, for short). It is easy to see that both $R(G)$ and $R'(G)$ are determined uniquely by G and the associated function g. When each vertex of G has at least two neighbors in G_{δ_g}, $R(G) = R'(G) = G$. If g is a nonnegative integer function, then an edge-coloring of G satisfying that each color of C appears at each $v \in V(G)$ at least $g(v)$ times is called a *general g_c-coloring of* G. When g is a positive integer function, a general g_c-coloring of G is a g_c-coloring of G. For the DRRS of G, $R'(G)$, define that

$$g_{R'(G)}(v) = \begin{cases} g_G(v), & v \in V(R'(G)) \cap V(G); \\ 0, & v \in V(R'(G)) \backslash V(G). \end{cases}$$

Theorem 1.5 *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$ and $\delta_g(G) \geq 2$, G be not weakly-$\delta_g(G)$-peelable. If the* DRRS *of G has a general g_c-coloring with $\delta_g(G)$ colors, then G is of g_c-class* 1.

2 The Proofs of Main Theorems

The reader refers to [1] for usual definitions and notations. Some concepts and notations are needed before proving the results. For two colors α, β and $w \in V(G)$, let $G(w; \alpha, \beta)$ be the component of G, which is induced by the edges colored with α and β and contains w. An (α, β)-*alternating walk* K of G is a sequence $v_0e_1v_1e_2 \cdots v_{r-1}e_rv_r$ of vertices and edges of G in which

(i) for $1 \leq i \leq r$, the vertices v_{i-1} and v_i are distinct and are both incident with the edge e_i;

(ii) the edges are all distinct and are colored alternately by α and β;

(iii) e_1 is colored by α and $\alpha(v_0) > \beta(v_0)$. $\gamma(v_r) > \overline{\gamma}(v_r)$, where γ denotes the color of e_r and $\overline{\gamma}$ denotes the other color of α, β.

Proof of Theorem 1.4 The case $\delta_g(G) = 1$ is trivial. So we assume that $\delta_g(G) \geq 2$ and the vertices of G can be peeled off in this order $v_1, v_2, \ldots, v_n$, where $n = |V(G)|$, using the weakly-$\delta_g(G)$-peeling operation. Let $k = \delta_g(G)$.

If $V_k(G) = V_{\delta_g}(G)$, then G is $\delta_g(G)$-peelable. By Corollary 1.3, G is of g_c-class 1.

Otherwise, $V_k(G) \supset V_{\delta_g}(G)$. We denote $T = V_k(G) \backslash V_{\delta_g}(G)$. For each vertex $v \in T$, $d_G(v) = m(v) \times \delta_g(G)$, where $m(v) > g(v)$ is an integer. Construct a new graph G' from G as follows: add to G an arbitrary $(k+1)$-regular graph H, where $V(H) = \{w_1, w_2, \ldots, w_h\}$, and join $w_1 \in V(H)$ to each vertex in T by a new edge. Let $g(v) = 1$ for each $v \in V(H)$. Then $\delta_g(G') = \delta_g(G)$. Clearly, $V_k(G') \cap (V(G) \backslash V_k(G)) = \emptyset$. Since $v \notin V_k(G')$ for any $v \in T$, $V_k(G') \backslash \{w_1\} = V_{\delta_g}(G)$ whether vertex w_1 belongs to $V_k(G')$ or not. Thus the vertices of G' can be peeled off in the order $w_1, w_2, \ldots, w_h, v_1, v_2, \ldots, v_n$ using the k-peeling operation, that is, graph G' is k-peelable. By Theorem 1.2, G' has an equitable edge-coloring with $k = \delta_g(G)$ colors. Denote an equitable edge-coloring of G' with $\delta_g(G)$ colors by η. Note that each color appears at each vertex $v \in V(G')$ at least $g(v)$ times in η. In particular, each color appears at each vertex $v \in T$ at least $g(v)+1$ times. Therefore, we can obtain a g_c-coloring of G with $\delta_g(G)$ colors by restricting η to graph G. Then $\chi'_{g_c}(G) = \delta_g(G)$. □

By Theorem 1.4, we easily get next result.

Corollary 2.1 *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$. If G_{δ_g} is a forest, then G is of g_c-class 1.*

Proof Since G_{δ_g} is a forest, there is a vertex of degree one in G_{δ_g}, saying v_1. We can first peel off v_1. $G_{\delta_g} - \{v_1\}$ is also a forest including a vertex of degree one, saying v_2. We continue peeling off v_2. By this analogy, we can peel off all the vertices of G_{δ_g} and then peel off the other vertices of G. So G is weakly-$\delta_g(G)$-peelable. By Theorem 1.4, G is of g_c-class 1. □

Lemma 2.2 ([2]) *Let G be a connected graph. Then G has a 2-edge coloring ζ such that*

(a) *If G is Eulerian and $|E(G)|$ is odd, then for an arbitrary vertex $u \in V(G)$, we have $|1(u) - 2(u)| = 2$ and $1(v) - 2(v) = 0$ for all $v \in V(G)\backslash\{u\}$.*

(b) *If G is Eulerian and $|E(G)|$ is even, then $1(v) - 2(v) = 0$ for all $v \in V(G)$.*

(c) *If G is not Eulerian, then $|1(v) - 2(v)| \le 1$ for all $v \in V(G)$.*

Let η be an edge-coloring of G with the colors in C and $H = G(u; \alpha, \beta)$, where $\alpha, \beta \in C$. We call H an *obstruction* (to g_c-coloring) if H is an odd Eulerian graph, $\alpha(u) = g(u) + 1$, $\beta(u) = g(u) - 1$, and $\alpha(v) = \beta(v) = g(v)$ for all $v \in V(H)\backslash\{u\}$. So any obstruction does not contain a vertex v with $g(v) = 0$. For an edge-coloring, let $\sigma(v)$ denote the number of different colors appearing at v at least $g(v)$ times.

Lemma 2.3 *Let G be a graph associated a nonnegative integer function $g : V(G) \to Z_0^+$, $u \in V(G)$, $C = \{c_1, c_2, \dots, c_k\}$ and $\alpha, \beta \in C$. Assume that η is an edge-coloring of G satisfying that $\alpha(u) + \beta(u) \ge 2g(u)$, $\min\{\alpha(u), \beta(u)\} \ge g(u) - 1$, and $\sigma(v) = k$ for each $v \in V(G)\backslash\{u\}$. Then we can recolor subgraph $H = G(u; \alpha, \beta)$ such that $\min\{\alpha(v), \beta(v)\} \ge g(v)$ for each $v \in V(H)$ if H is not an obstruction.*

Proof If H is not an Eulerian graph, by Lemma 2.2, we can recolor H such that $|\alpha(v) - \beta(v)| \le 1$ for each $v \in V(H)$. So we get $\min\{\alpha(v), \beta(v)\} \ge g(v)$ for each $v \in V(H)$. If H is an Eulerian graph and $E(H)$ is even, by Lemma 2.2, we can recolor H such that $\alpha(v) - \beta(v) = 0$ for all $v \in V(H)$. We are done, too. If H is an Eulerian graph and $E(H)$ is odd, then $d_H(v)$ is even for each $v \in V(H)$. If $d_H(v) = 2g(v)$ for each $v \in V(H)$, then $\alpha(v) = \beta(v) = g(v)$ for each $v \in V(H)\backslash\{u\}$ according to $\sigma(v) = k$ for each $v \in V(G)\backslash\{u\}$. If $\alpha(u) = \beta(u) = g(u)$, then the number of edges colored with α (or β) in H is $\frac{\sum_{v \in V(H)} g(v)}{2} = \frac{E(H)}{2}$, a contradiction with $E(H)$ odd. Thus we must have $\max\{\alpha(u), \beta(u)\} = g(u) + 1$, $\min\{\alpha(u), \beta(u)\} = g(u) - 1$. This means that H is an obstruction, a contradiction. Then there must exist a vertex $x \in V(H)$ so that $d_H(x) \ge 2g(x) + 2$. By Lemma 2.2, we can recolor H so that $|\alpha(x) - \beta(x)| = 2$, $\alpha(v) - \beta(v) = 0$ for all $v \in V(H)\backslash\{x\}$. We also get $\min\{\alpha(v), \beta(v)\} \ge g(v)$ for each $v \in V(H)$ whether $x = u$ or not. □

Next, we give the proof of Theorem 1.5.

Proof of Theorem 1.5 Let $C = \{c_1, c_2, \dots, c_k\}$ be a color set, where $k = \delta_g(G)$. $\delta_g(G) \ge 2$ and $g > 0$ imply that $d(v) \ge 2g(v)$ for each $v \in V(G)$, that is, G has no pendent edge. Assume that $u_1, u_2, \dots, u_s$ $(s < n)$ can be iteratively peeled off using the weakly-$\delta_g(G)$-peeling operation, and $R(G) = G - \{u_1, u_2, \dots, u_s\}$. Let $R' = R'(G)$ and η be a general g_c-coloring of R' with the colors in C. If $R(G) = G$, we are done. So we next assume that $1 \le s < n$. Add u_s to R'. Join u_s to $v \in V(R')$ if v is adjacent to u_s in G and delete a pendent edge at v. Use the color of the pendent edge to color edge $u_s v$. Stick some new pendent edges to u_s such that the degree of u_s equals $d_G(u_s)$. Color the pendent edges at u_s with the colors in C. The resulting graph is denoted by G'_{s-1}. In particular, we always define $g(v) = 0$ for each vertex v with degree one in these new pendent edges. Next, we give a general g_c-coloring of G'_{s-1}.

According to the definition of weakly-$\delta_g(G)$-peeling operation, we just need to discuss two cases.

Case 1 u_s is not adjacent to a vertex, which belongs to $V_{\delta_g}(G)$ and has no pendent edge in G'_{s-1}.

If each color appears at least $g(u_s)$ times at u_s, we get a required edge-coloring of G'_{s-1}. Otherwise, $\min_{c\in C}\{c(u_s)\} < g(u_s)$ and $\max_{c\in C}\{c(u_s)\} > g(u_s)$. If $\beta(u_s) = \min_{c\in C}\{c(u_s)\} \le g(u_s) - 2$, then there exists a color $\alpha_0 \in C$ such that $\alpha_0(u_s) \ge g(u_s) + 1$. Choose an (α_0, β)-alternating walk K starting at u_s. Two cases will be considered.

(a) K stops at u_s.

(b) K does not stop at u_s.

If (a) happens, exchange the two colors on K. Then $\alpha_0(u_s)$ decreases by 2 and $\alpha_0(u_s) \ge g(u_s) - 1$, $\beta(u_s)$ increases by 2. If (b) happens, exchange the two colors on K. Then $\alpha_0(u_s)$ decreases by 1 and $\alpha_0(u_s) \ge g(u_s)$, $\beta(u_s)$ increases by 1. If we still have $\beta(u_s) \le g(u_s) - 2$, then there exists a color $\alpha_1 \in C$ such that $\alpha_1(u_s) \ge g(u_s) + 1$. ($\alpha_1$ may be the same as α_0 if there is still $\alpha_0(u_s) \ge g(u_s) + 1$.) Continue finding an (α_1, β)-alternating walk K starting at u_s and exchange the two colors on K. This way goes on till $\beta(u_s) \ge g(u_s) - 1$. In short, we can recolor G'_{s-1} such that $\min_{c\in C}\{c(u_s)\} \ge g(u_s) - 1$. Assume that $\min_{c\in C}\{c(u_s)\} = g(u_s) - 1$. Let $\sigma(v) = |\{c \in C : c(v) \ge g(v)\}|$. So $\sigma(u_s) < k$. Let $\alpha_0(u_s) = \max_{c\in C}\{c(u_s)\}$ and $\beta(u_s) = \min_{c\in C}\{c(u_s)\}$. Next, we prove that G'_{s-1} could be recolored such that $\sigma(u_s)$ increases by 1 and $\sigma(v)$ does not decrease for each $v \in V(G'_{s-1})\backslash\{u_s\}$.

If $\alpha_0(u_s) \ge g(u_s) + 2$, noting that $\beta(u_s) = \min_{c\in C}\{c(u_s)\} = g(u_s) - 1$, then $H_0 = G'_{s-1}(u_s; \alpha_0, \beta)$ is not an obstruction. By Lemma 2.3, we can get a required edge-coloring of G'_{s-1}.

Now we assume $\beta(u_s) = g(u_s) - 1$, $\alpha_0(u_s) = g(u_s) + 1$.

If $H_0 = G'_{s-1}(u_s; \alpha_0, \beta)$ is not an obstruction, we can get a required edge-coloring of G'_{s-1} by Lemma 2.3. So we just consider that H_0 is an obstruction with $\alpha_0(u_s) = g(u_s)+1$, $\beta(u_s) = g(u_s) - 1$ and $\alpha_0(v) = \beta(v) = g(v)$ for each $v \in V(H_0)\backslash\{u_s\}$. We give the following iterative process for stage $i\,(i \ge 1)$. At the start of the i-th stage, $H_{i-1} = G'_{s-1}(u_s; \alpha_{i-1}, \beta)$ is an obstruction. There are two cases to be considered.

Case 1.1 At u_s, there exists an α_{i-1}-edge $e_{i-1} = u_s y_{i-1}$ such that $y_{i-1} \notin V_{\delta_g}(G)$.

We can choose a color α_i such that $\alpha_i(y_{i-1}) \ge g(y_{i-1}) + 1$. Clearly, such α_i exists by $d(y_{i-1}) > g(y_{i-1})\delta_g(G)$. If $\alpha_i(y_{i-1}) > g(y_{i-1}) + 1$, recolor e_{i-1} with β. Then $\alpha_{i-1}(u_s) = \beta(u_s) = g(u_s)$, $\alpha_{i-1}(y_{i-1}) = g(y_{i-1}) - 1$. Therefore $\sigma(y_{i-1}) = k - 1$, $\sigma(u_s)$ increases by 1 and $T = G'_{s-1}(y_{i-1}; \alpha_i, \alpha_{i-1})$ is not an obstruction. Note that there are $\alpha_i(u_s) \ge g(u_s) - 1$ and $\min\{\alpha_i(v), \alpha_{i-1}(v)\} \ge g(v)$ for each $v \in V(T)\backslash\{y_{i-1}, u_s\}$ $(*)$. If $u_s \notin V(T)$, by Lemma 2.3, we can get a required edge-coloring of G'_{s-1}. If $u_s \in V(T)$, by Lemma 2.2, we discuss three cases.

(i) When T is an odd Eulerian graph implying that $\alpha_i(y_{i-1}) \geq g(y_{i-1}) + 3$ and $\alpha_i(u_s) \geq g(u_s)$, we can recolor T such that $\min\{\alpha_i(y_{i-1}), \alpha_{i-1}(y_{i-1})\} \geq g(y_{i-1})$ and $\alpha_i(v) = \alpha_{i-1}(v) \geq g(v)$ for each $v \in V(T)\backslash\{y_{i-1}\}$.

(ii) When T is an even Eulerian graph, we can recolor T such that $\alpha_i(v) = \alpha_{i-1}(v) \geq g(v)$ for each $v \in V(T)$.

(iii) When T is not an Eulerian graph, we can recolor T such that $|\alpha_i(v) - \alpha_{i-1}(v)| \leq 1$ for each $v \in V(T)$. (For $v \in V(T)\backslash\{u_s\}$, $(*)$ implies that $\min\{\alpha_i(v), \alpha_{i-1}(v)\} \geq g(v)$. For u_s, $(*)$ implies that $\sigma(u_s)$ does not decrease whether $\alpha_i(u_s) = g(u_s) - 1$ or $\alpha_i(u_s) \geq g(u_s)$ before T is recolored.)

In any case, we make $\sigma(y_{i-1}) = k$ and $\sigma(u_s)$ increase by 1 comparing to that one before recoloring e_{i-1}. That is to say, we get a required edge-coloring of G'_{s-1}. Thus, we assume $\alpha_i(y_{i-1}) = g(y_{i-1}) + 1$. We argue three cases, noting that $\min_{c\in C}\{c(u_s)\} = g(u_s) - 1$ and $\max_{c\in C}\{c(u_s)\} = g(u_s) + 1$.

(1) $\alpha_i(u_s) = g(u_s) - 1$.

$G'_{s-1}(u_s; \alpha_i, \alpha_{i-1})$ contains y_{i-1}, $\alpha_i(y_{i-1}) = \alpha_{i-1}(y_{i-1}) + 1 = g(y_{i-1}) + 1$ and $\alpha_{i-1}(u_s) = g(u_s) + 1$. Thus, $G'_{s-1}(u_s; \alpha_{i-1}, \alpha_i)$ is not an obstruction. By Lemma 2.3, we can get a required edge-coloring of G'_{s-1}.

(2) $\alpha_i(u_s) = g(u_s) + 1$.

In $G'_{s-1} - e_{i-1}$, choose an (α_i, α_{i-1})-alternating walk K starting at y_{i-1}, K must be one of the following three cases.

(a) K does not stop at u_s or y_{i-1}.

(b) K stops at u_s.

(c) K stops at y_{i-1} with an edge colored α_i.

When (a) or (b) happens, we first exchange the two colors on K, then recolor e_{i-1} by β. Clearly, the resulting coloring is a required edge-coloring of G'_{s-1}. When (c) happens, we first exchange the two colors on K, then recolor e_{i-1} by α_i. Thus, $\alpha_i(u_s) = g(u_s) + 2$, which means that $G'_{s-1}(u_s; \alpha_i, \beta)$ is not an obstruction. By Lemma 2.3, we can get a required edge-coloring of G'_{s-1}.

(3) $\alpha_i(u_s) = g(u_s)$.

In $G'_{s-1} - e_{i-1}$, choose an (α_i, α_{i-1})-alternating walk K starting at y_{i-1}. K must be one of the following cases.

(a) K does not stop at y_{i-1} or u_s.

(b) K stops at y_{i-1} with an edge colored α_i.

When (a) happens, first exchange the two colors on K and then recolor edge e_{i-1} by β. The resulting coloring is a required edge-coloring of G'_{s-1}. If (b) happens, exchange the two colors on K and then recolor e_{i-1} by α_i. If $H_i = G'_{s-1}(u_s; \alpha_i, \beta)$ is not an obstruction, we have a required edge-coloring of G'_{s-1} by Lemma 2.3. If H_i is an obstruction, consider a maximal (α_{i-1}, β)-alternating walk X starting at u_s in G'_{s-1}. We claim that each X does not end at u_s. If X returns to u_s with an α_{i-1}-edge, then we can extend it with a new β-edge by $\alpha_{i-1}(u_s) = \beta(u_s) + 1$. If there exists an X ending at a vertex $v \notin \{y_{i-1}, u_s\}$, then exchange the two colors on X. Thus, we have $\alpha_i(u_s) - 1 = \alpha_{i-1}(u_s) + 1 = \beta(u_s) = g(u_s)$ and $\alpha_i(y_{i-1}) = \alpha_{i-1}(y_{i-1}) - 1 = \beta(y_{i-1}) = g(y_{i-1})$. Subgraph $H = G(u_s; \alpha_{i-1}, \alpha_i)$ is not an obstruction, because H contains the odd degree vertex y_{i-1}. So we can get a required edge-coloring of G'_{s-1} by Lemma 2.3. If every X ends at y_{i-1}, we denote this case by (A). Since H is an obstruction when (A) happens, there exists an edge colored α_i incident with u_s different from e_{i-1}. Denote the edge by e_i and the other end of e_i by y_i. So we can go to the $(i+1)$-th stage.

Case 1.2 For each α_{i-1}-edge $u_s w$, $w \in V_{\delta_g}(G)$.

We get an α_{i-1}-edge $e_{i-1} = u_s y_{i-1}$. Here $y_{i-1} \in V_{\delta_g}(G)$ and y_{i-1} has at least one pendent edge in G'_{s-1}. Choose such a pendent edge and denote it by $y_{i-1}y'_{i-1}$. Let α_i be the color of $y_{i-1}y'_{i-1}$. Since $H_{i-1} = G'_{s-1}(u_s; \alpha_{i-1}, \beta)$ is an obstruction, $\alpha_i \neq \alpha_{i-1}$. We consider the following three cases.

(1) $\alpha_i(u_s) = g(u_s) - 1$.

Exchange the two colors on walk $u_s y_{i-1} y'_{i-1}$. Clearly, the resulting coloring is a required edge-coloring of G'_{s-1}.

(2) $\alpha_i(u_s) = g(u_s) + 1$.

We exchange the two colors on walk $u_s y_{i-1} y'_{i-1}$, then $\alpha_i(u_s) = g(u_s) + 2$, $\beta(u_s) = g(u_s) - 1$. We know $G'_{s-1}(u_s; \alpha_i, \beta)$ is not an obstruction. So we can get a required edge-coloring of G'_{s-1} by Lemma 2.3.

(3) $\alpha_i(u_s) = g(u_s)$.

Exchange the two colors on walk $u_s y_{i-1} y'_{i-1}$. If $H_i = G'_{s-1}(u_s; \alpha_i, \beta)$ is not an obstruction, we can get a required edge-coloring of G'_{s-1} by Lemma 2.3. If H_i is an obstruction, there exists an α_i-edge incident with u_s different from e_{i-1}. Denote the edge by e_i and the other vertex incident with e_i by y_i. Then go to the $(i+1)$-th stage. Denote this case by (B).

Hence, at the start of the i-th stage ($i \geq 2$), the following conditions have been satisfied.

(I) Distinct vertices $y_0, y_1, \ldots, y_{i-1}$.

(II) Distinct edges $e_0, e_1, \ldots, e_{i-1}$, where $e_j = u_s y_j$, $0 \leq j \leq i-1$.

(III) Distinct colors $\alpha_0, \alpha_1, \ldots, \alpha_{i-1}, \beta$. At the start of the i-th stage, e_{i-1} is colored with α_{i-1}. At the end of the j-th stage, e_{j-1} is colored with α_j, $1 \leq j \leq i-1$. When α_i is chosen distinct from $\alpha_0, \alpha_1, \ldots, \alpha_{i-1}$, there is no edge colored with $\alpha_0, \alpha_1, \ldots, \alpha_{i-2}$ or β is recolored in the i-th stage.

(IV) At the start of the i-th stage, $H_{i-1} = G'_{s-1}(u_s; \alpha_{i-1}, \beta)$ is an obstruction, in which $\alpha_{i-1}(u_s) - 1 = \beta(u_s) + 1 = g(u_s)$ and $\alpha_{i-1}(v) = \beta(v) = g(v)$ for each $v \in V(H_{i-1}) \backslash \{u_s\}$. Also, H_{i-1} contains e_{i-1}.

Since $|C| = k$, we could get a color $\alpha_i \in \{\alpha_0, \alpha_1, \ldots, \alpha_{i-1}\}$ in some stage i ($1 \leq i \leq k-1$).

If $\alpha_i \in \{\alpha_0, \alpha_1, \ldots, \alpha_{i-1}\}$ and (A) or (B) happens at the i-th stage ($i \geq 2$), then the iterative process would end. Assume that α_i occurs exactly once in the sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_{i-1}\}$. Let $\alpha_i = \alpha_p$ for $0 \leq p \leq i-2$. We know $\alpha_i \neq \alpha_{i-1}$, $y_{i-1} \neq y_p$, by the discussion above. In this case, we consider two subcases.

Case I (A) happens in the $(p+1)$-th stage and (A) or (B) happens in the i-th stage.

At the start of the $(p+1)$-th stage, $H_p = G'_{s-1}(u_s; \alpha_p, \beta)$ is an obstruction with $\alpha_p(u_s) = \beta(u_s) + 2 = g(u_s) + 1$. Since (A) happens in the $(p+1)$-th stage, e_p is recolored by α_{p+1}. Thus at the end of the $(p+1)$-th stage, we have $\alpha_p(u_s) = g(u_s)$, $\beta(u_s) = g(u_s) - 1$. We can find an (α_p, β)-alternating walk X starting at u_s and ending at y_p. Between the end of the $(p+1)$-th stage and the start of the i-th stage, the edges colored with α_p and β do not change, and X remains the same.

If (A) happens in the i-th stage, then exchange the two colors on X. We have $\alpha_p(u_s) = g(u_s) - 1$, $\beta(u_s) = g(u_s)$, $\alpha_{i-1}(u_s) = g(u_s) + 1$, and the value $\sigma(y_p)$ is unchanged. Since $G'_{s-1}(u_s; \alpha_{i-1}, \alpha_p)$ contains y_{i-1}, $\alpha_p(y_{i-1}) = g(y_{i-1})+1$, and $\alpha_{i-1}(y_{i-1}) = g(y_{i-1})$, $G'(u_s; \alpha_{i-1}, \alpha_p)$ is not an obstruction. By Lemma 2.3, we can get a required edge-coloring of G'_{s-1}.

If (B) happens in the i-th stage, then exchange the two colors on X. Thus $\alpha_p(u_s) = g(u_s)-1$, $\beta(u_s) = g(u_s)$, $\alpha_{i-1}(u_s) = g(u_s)+1$, and the value $\sigma(y_p)$ is unchanged. We recolor edge $u_s y_{i-1}$ by α_p, recolor edge $y_{i-1}y'_{i-1}$ by α_{i-1}. Thus we get a required edge-coloring of G'_{s-1}.

Case II (B) happens in the $(p+1)$-th stage and (A) or (B) happens in the i-th stage.

At the start of the $(p+1)$-th stage, $H_p = G'_{s-1}(u_s; \alpha_p, \beta)$ is an obstruction with $\alpha_p(u_s) = g(u_s) + 1$, $\alpha_{p+1}(u_s) = g(u_s)$, $\beta(u_s) = g(u_s) - 1$. Since (B) happens in the $(p+1)$-th stage, e_p is recolored by α_{p+1}. So we have $\alpha_{p+1}(u_s) = g(u_s) + 1$, $\alpha_p(u_s) = g(u_s)$, $\beta(u_s) = g(u_s) - 1$. We know that $H_p - e_p + y_p y'_p$ contains an (α_p, β)-alternating walk X starting at u_s and ending at y'_p at the end of $(p+1)$-th stage. X remains the same between the end of the $(p+1)$-th stage and the start of i-th stage.

If (A) happens in the i-th stage, exchange the two colors on X. Then we get $\alpha_{i-1}(u_s) = g(u_s)+1$, $\alpha_p(u_s) = g(u_s)-1$, $\beta(u_s) = g(u_s)$. Now $G'_{s-1}(u_s; \alpha_{i-1}, \alpha_p)$ contains y_{i-1}, $\alpha_p(y_{i-1}) = \alpha_{i-1}(y_{i-1})+1 = g(y_{i-1})+1$. So $G'_{s-1}(u_s; \alpha_{i-1}, \alpha_p)$ is not an obstruction. We can get a required edge-coloring of G'_{s-1} by Lemma 2.3.

If (B) happens in the i-th stage, exchange the two colors on X. Then we get $\alpha_{i-1}(u_s) = g(u_s)+1$, $\alpha_p(u_s) = g(u_s)-1$, $\beta(u_s) = g(u_s)$. Since $y_{i-1}y'_{i-1} \notin X$, it is still colored with α_p. We recolor the edge $u_s y_{i-1}$ by α_p, the edge $y_{i-1}y'_{i-1}$ by α_{i-1}. Thus we get a required edge-coloring of G'_{s-1}.

If $\sigma(u_s) < k$, we construct a new iterative process to make $\sigma(u_s)$ increase. Thus we can get a general g_c-coloring of G'_{s-1} in finite iterative processes.

Case 2 u_s is exactly adjacent to a vertex, which belongs to $V_{\delta_g}(G)$ and has no pendent edge in G'_{s-1}.

If u_s is adjacent to a vertex $w \in V_{\delta_g}(G)$ which has no pendent edge in G'_{s-1}, then we delete the edge $u_s w$ and add a pendent edge to u_s and w, respectively. Color the two pendent edges with the color of $u_s w$. The resulting graph is denoted by G''_{s-1}.

By Case 1, we can get a general g_c-coloring of G''_{s-1}. Join u_s and w, then delete a pendent edge at w and u_s, respectively. The resulting graph is G'_{s-1}. We denote the two pendent edges e_x (colored with λ) and e_y (colored with γ), respectively. Recolor the edge $u_s w$ by λ. If each color appears at least $g(u_s)$ times at u_s, we get a required edge-coloring. Otherwise, we have $\gamma(u_s) = g(u_s) - 1$ and $c(u_s) \geq g(u_s)$ for each $c \in C \backslash \{\gamma\}$. We can find a new color as α_0 such that $\alpha_0(u_s) = \max_{c \in C}\{c(u_s)\} \geq g(u_s) + 1$. If $G'_{s-1}(u_s; \alpha_0, \gamma)$ is not an obstruction, we can get a general g_c-coloring of G'_{s-1}. If $G'_{s-1}(u_s; \alpha_0, \gamma)$ is an obstruction, we can find an edge $u_s y_0$ colored with α_0 different from edge $u_s w$. Then we can construct a new iterative process same as Case 1, because u_s is not adjacent to a vertex, which belongs to $V_{\delta_g}(G)$ and has no pendent edge, except w. So this case is converted into Case 1.

In a word, we can get a general g_c-coloring of G'_{s-1}. Similarly, we can get a general g_c-

coloring of $G'_{s-2}, G'_{s-3}, \dots, G'_1, G'_0 = G$ with $\delta_g(G)$ colors. So G is of g_c-class 1. □

By the proof of Theorem 1.5, we can get a more general result.

Theorem 2.4 *Let G be a graph associated a positive integer function $g : V(G) \to Z^+$, and $\delta_g(G) \geq 2$. Let H denote a graph obtained from G by peeling off some vertices of G using weakly-$\delta_g(G)$-peeling operation. If the degree restoration of H has a general g_c-coloring of $\delta_g(G)$ colors, then G is of g_c-class* 1.

Song and Liu [4] obtain the following result.

Theorem 2.5 ([4]) *Let G be a graph. If $g(v)$ is positive and even for all $v \in V(G)$, then $\chi'_{g_c}(G) = \delta_g(G)$.*

By Theorems 1.5 and 2.5, we can get a result as below.

Corollary 2.6 *Let G be a graph, which is associated a positive integer function $g : V(G) \to Z^+$ and not weakly-$\delta_g(G)$-peelable. Denote the remaining subgraph by $R(G)$. For each vertex $v \in V(R(G))$, if $g(v) > 0$ and $g(v)$ is even, then G is of g_c-class* 1.

Proof Denote the DRRS of G by $R'(G)$. For each vertex $x \in V(R'(G))$ which has degree one, we know $g_{R'(G)}(x) = 0$. We construct a new graph $T(G)$ as follows based on graph $R'(G)$: stick to x a $4\delta_g(G)$-regular graph H_x in such a way that a vertex of H_x coincides with x, and define $g_{T(G)}(v) = 2$ for each $v \in V(H_x)$. Clearly, $\delta_g(T(G)) = \delta_g(G)$ and $g_{T(G)}(v)$ is positive and even for each $v \in V(T(G))$. By Theorem 2.5, $T(G)$ is of g_c-class 1. Let η be a g_c-coloring of $T(G)$ with $\delta_g(G)$ colors. Restrict η to graph $R'(G)$, we obtain a general g_c-coloring of $R'(G)$ with $\delta_g(G)$ colors. By Theorem 1.5, G is of g_c-class 1. □

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