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On Graph-Lagrangians and Clique Numbers of 3-Uniform Hypergraphs

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Abstract The paper explores the connection of Graph-Lagrangians and its maximum cliques for 3-uniform hypergraphs. Motzkin and Straus showed that the Graph-Lagrangian of a graph is the Graph-Lagrangian of its maximum cliques. This connection provided a new proof of Turán classical result on the Turán density of complete graphs. Since then, Graph-Lagrangian has become a useful tool in extremal problems for hypergraphs. Peng and Zhao attempted to explore the relationship between the Graph-Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain range. They showed that if G is a 3-uniform graph with m edges containing a clique of order $t-1$, then $\lambda(G) = \lambda([t-1]^{(3)})$ provided $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. They also conjectured: If G is an r-uniform graph with m edges not containing a clique of order $t-1$, then $\lambda(G) < \lambda([t-1]^{(r)})$ provided $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. It has been shown that to verify this conjecture for 3-uniform graphs, it is sufficient to verify the conjecture for left-compressed 3-uniform graphs with $m = \binom{t-1}{3} + \binom{t-2}{2}$. Regarding this conjecture, we show: If G is a left-compressed 3-uniform graph on the vertex set $[t]$ with m edges and $|[t-1]^{(3)} \setminus E(G)| = p$, then $\lambda(G) < \lambda([t-1]^{(3)})$ provided $m = \binom{t-1}{3} + \binom{t-2}{2}$ and $t \geq 17p/2 + 11$.

Keywords Lagrangians of hypergraphs, extremal problems in hypergraphs

MR(2010) Subject Classification 05C35, 05C65, 05D99

1 Introduction

In 1965, Motzkin and Straus [5] established a connection between the order of a maximum clique and the Graph-Lagrangian of a graph. This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique problem. This connection provided another proof of Turán's theorem [15] which pushed the development of extremal

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graph theory. More generally, the connection between Graph-Lagrangians and Turán densities can be used to give another proof of the fundamental result of Erdös–Stone–Simonovits on Turán densities of graphs (see Keevash's survey paper [4]). However, the obvious generalization of Motzkin and Straus' result to r-uniform hypergraphs is false, i.e., the Graph-Lagrangian of a hypergraph is not always the same as the Graph-Lagrangian of its maximum cliques. There are many examples of r-uniform hypergraphs other than complete r-uniform hypergraphs that do not achieve their Graph-Lagrangian on any proper subhypergraph. In spite of this, the Graph-Lagrangian has been a useful tool in extremal problems in combinatorics. In 1980's, Sidorenko [9] and Frankl and Füredi [1] developed the method of applying Graph-Lagrangians in determining hypergraph Turán densities. More recent applications of Graph-Lagrangians can be found in Keevash's survey paper [4] and [3]. In most applications in extremal combinatorics, we need an upper bound of Graph-Lagrangians of hypergraphs. In the course of estimating Turán densities of hypergraphs by applying the Graph-Lagrangians of related hypergraphs, Frankl and Füredi [1] asked the following question: Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Graph-Lagrangian of an r-graph with m edges be? They proposed the following conjecture: The r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Graph-Lagrangian of all r-graphs with m edges. The Moztkin–Straus result implies that this conjecture is true for $r = 2$. For $r \geq 3$, this conjecture seems to be very challenging. Talbot first confirmed this conjecture for some cases in [11]. Later Tang et al. confirmed this conjecture for some more cases in [10, 12–14].

Although the obvious generalization of Motzkin and Straus' result to r-uniform hypergraphs is false, in [7], Peng and Zhao attempted to explore the relationship between the Graph-Lagrangian of a hypergraph and the size of its maximum cliques for hypergraphs when the number of edges is in certain range. The authors proposed a pair of conjectures in [7] on the relationship between the Graph-Lagrangian of a hypergraph and the size of its maximum cliques for hypergraphs when the number of edges is in certain range. This pair of conjectures also refine Frankl and Füredi's conjecture for this range of edges numbers. The authors in [7] confirmed one of the conjectures for 3-uniform hypergraph, that is, the Graph-Lagrangian of a 3-uniform hypergraph is the Graph-Lagrangian of its maximum cliques when the number of edges is in certain range. In this paper, we show that for that range of the number of edges, if a 3-uniform hypergraph does not contain a clique of such a size, then the Graph-Lagrangian of such a 3-uniform hypergraph is strictly less than the Graph-Lagrangian of a complete 3-uniform hypergraph of such a size under some conditions.

2 Definitions and Main Results

For a set V and a positive integer r, let $V^{(r)}$ denote the family of all r-subsets of V. An r-uniform graph or r-graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1a_2 \ldots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The complement of an r-graph G is denoted by G^c. A complete r-graph on t vertices is also called a clique of order t. Let $\mathbb{N}$ be the set of all positive integers. For any integer $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $[n]^{(r)}$ represent the complete r-uniform graph on the vertex set $[n]$. When $r = 2$, an r-uniform graph is a simple graph. When $r \geq 3$, an r-graph is often called a hypergraph.

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $\max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example, since $\max(\{2,4,6\} \triangle \{1,5,6\}) \in \{1,5,6\}$, so we have $246 < 156$ in $\mathbb{N}^{(3)}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. Note that the first $\binom{t}{r}$ r-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$. Formally, we let $C_{r,m}$ denote the r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$.

For an r-graph $G = (V, E)$ and $i \in V$, let $E_i = \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. For a pair of vertices $i, j \in V$, let $E_{ij} = \{B \in V^{(r-2)} : B \cup \{i,j\} \in E\}$. Let $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \setminus E\}$ and $E_{ij}^c = \{B \in V^{(r-2)} : B \cup \{i,j\} \in V^{(r)} \setminus E\}$. Denote $E_{i \setminus j} = E_i \cap E_j^c$.

An r-tuple $i_1 i_2 \cdots i_r$ is called a *descendant* of an r-tuple $j_1 j_2 \cdots j_r$ if $i_s \leq j_s$ for each $1 \leq s \leq r$, and $i_1 + i_2 + \cdots + i_r < j_1 + j_2 + \cdots + j_r$. And $j_1 j_2 \cdots j_r$ is called an *ancestor* of $i_1 i_2 \cdots i_r$. The r-tuple $i_1 i_2 \cdots i_r$ is called a *direct descendant* of $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$ and $j_1 + j_2 + \cdots + j_r = i_1 + i_2 + \cdots + i_r + 1$. $j_1 j_2 \cdots j_r$ has lower hierarchy than $i_1 i_2 \cdots i_r$ if $j_1 j_2 \cdots j_r$ is an ancestor of $i_1 i_2 \cdots i_r$. This is a partial order on the set of all r-tuples.

Definition 2.1 *For an r-uniform graph G with the vertex set $[n]$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. The Graph-Lagrangian of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

The value x_i is called the *weight* of the vertex i. A vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a legal weighting for G if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an *optimal weighting* for G if $\lambda(G, \vec{y}) = \lambda(G)$. The following fact is easily implied by the definition of the Graph-Lagrangian.

Fact 2.2 Let G_1, G_2 be r-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.

In [5], Motzkin and Straus provided the following expression for the Graph-Lagrangian of a 2-graph.

Theorem 2.3 ([5]) *If G is a 2-graph in which a largest clique has order t, then $\lambda(G) = \lambda([t]^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.*

The obvious generalization of Motzkin and Straus' result to hypergraphs is false because there are many examples of hypergraphs (other than a complete hypergraph) that do not achieve their Graph-Lagrangians on any proper subhypergraph. In [7], the authors attempted to explore the relationship between the Graph-Lagrangian of a hypergraph and the order of its maximum cliques of hypergraphs when the number of edges is in a certain range. The following two conjectures are proposed in [7].

Conjecture 2.4 ([7]) *Let m and t be positive integers satisfying $\binom{t-1}{r} \le m \le \binom{t-1}{r} + \binom{t-2}{r-1}$. Let G be an r-graph with m edges and contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(r)})$.*

The upper bound $\binom{t-1}{r} + \binom{t-2}{r-1}$ in this conjecture is the best possible. For example, if $m = \binom{t-1}{r} + \binom{t-2}{r-1} + 1$, then $\lambda(C_{r,m}) > \lambda([t-1]^{(r)})$, where $C_{r,m}$ is the r-graph on the vertex set $[t]$ and with the edge set $[t-1]^{(r)} \cup \{i_1 \cdots i_{r-1}t, i_1 \cdots i_{r-1} \in [t-2]^{(r-1)}\} \cup \{1 \cdots (r-2)(t-1)t\}$. Take $\vec{x} = (x_1, \dots, x_t) \in S$, where $x_1 = x_2 = \cdots = x_{t-2} = \frac{1}{t-1}$ and $x_{t-1} = x_t = \frac{1}{2(t-1)}$. Then $\lambda(C_{r,m}) \ge \lambda(C_{r,m}, \vec{x}) > \lambda([t-1]^{(r)})$.

Conjecture 2.5 ([7]) *Let m and t be positive integers satisfying $\binom{t-1}{r} \le m \le \binom{t-1}{r} + \binom{t-2}{r-1}$. Let G be an r-graph with m edges and contain no clique of order $t-1$. Then $\lambda(G) < \lambda([t-1]^{(r)})$.*

Peng and Zhao proved that Conjecture 2.4 holds for $r = 3$ in [7].

Theorem 2.6 ([7]) *Let m and t be positive integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a 3-graph with m edges and G contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(3)})$.*

There are also some partial results in [6, 8, 10, 12, 13] to support Conjecture 2.5. This pair of conjectures are also related to a long-standing conjecture of Frankl and Füredi.

Conjecture 2.7 ([1]) *$C_{r,m}$ has the largest Graph-Lagrangian of all r-graphs with m edges. In particular, the r-graph with $\binom{t}{r}$ edges and the largest Graph-Lagrangian is $[t]^{(r)}$.*

The following result in [11] states that the value of $\lambda(C_{r,m})$ can be easily figured out when m is in certain ranges.

Lemma 2.8 ([11]) *For any integers m, t, and r satisfying $\binom{t-1}{r} \le m \le \binom{t-1}{r} + \binom{t-2}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([t-1]^{(r)})$.*

Clearly, Conjectures 2.4 and 2.5 refined Conjecture 2.7 when m is in this range.

Definition 2.9 *An r-graph $G = (V, E)$ on the vertex set $[n]$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \le j_p$ for every $p, 1 \le p \le r$.*

Remark 2.10 An r-graph $G = (V, E)$ is *left-compressed* if and only if $E_{j \setminus i} = \emptyset$ for any $1 \le i < j \le n$.

The following result showed in [8] says that we only need to consider left-compressed 3-graphs G on t vertices with $\binom{t-1}{3} + \binom{t-2}{2}$ edges to verify Conjecture 2.5 for $r = 3$.

Lemma 2.11 ([8]) *To verify Conjecture 2.5 for $r = 3$, it is sufficient to show that for a left–compressed 3-graph G on the vertex set $[t]$ with $\binom{t-1}{3} + \binom{t-2}{2}$ edges and not containing a clique of order $t-1$, the inequality $\lambda(G) < \lambda([t-1]^{(3)})$ holds.*

The following provide some partial results to Conjecture 2.5 for $r = 3$.

Theorem 2.12 ([8]) *Let G be a left-compressed 3-graph with $\binom{t-1}{3} + \binom{t-2}{2}$ edges and containing no clique of order $t-1$. If $6 \le t \le 13$, then $\lambda(G) < \lambda([t-1]^{(3)})$.*

Theorem 2.13 ([14]) *Let G be a left-compressed 3-graph on the vertex set $[t]$ with $\binom{t-1}{3} + \binom{t-2}{2}$ edges and $|[t-1]^{(3)} \setminus E(G)| = 1$. Then $\lambda(G) < \lambda([t-1]^{(3)})$ for $t \ge 6$.*

Regarding Conjecture 2.5, we show the following result in this paper.

Theorem 2.14 *Let m, p and t be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$ and $t \ge 17p/2 + 11$. Let G be a left-compressed 3-graph on the vertex set $[t]$ with m edges and $|[t-1]^{(3)} \setminus E(G)| = p$. Then $\lambda(G) < \lambda([t-1]^{(3)})$.*

The paper is organized as follows. In Sections 3, we give some useful results. In Section 4, we give some preliminary results to the proof of Theorem 2.14. In Sections 5, we give the proof of our main result.

3 Useful Results

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \ldots, x_n)$ for an r-graph G:

$$|\{i : x_i > 0\}| \text{ is minimal, i.e., if } \vec{y} \text{ is a legal weighting for } G \text{ satisfying}$$
$$|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \tag{3.1}$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G)$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following lemma gives some necessary conditions of an optimal vector of $\lambda(G)$.

Lemma 3.1 ([2]) *Let $G = (V, E)$ be an r-graph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal legal weighting for G with k $(\leq n)$ non–zero weights $x_1, x_2, \ldots, x_k$. Then for every $\{i, j\} \in [k]^{(2)}$,* (a) *$\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$.* (b) *If $\vec{x}$ satisfies the condition* (3.1)*, then there is an edge in E containing both i and j.*

Remark 3.2 Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal legal weighting for G with k $(\leq n)$ non-zero weights $x_1, x_2, \ldots, x_k$. Let $1 \leq i < j \leq k$.

(a) Lemma 3.1 part (a) implies that $x_j\lambda(E_{ij}, \vec{x}) + \lambda(E_{i\setminus j}, \vec{x}) = x_i\lambda(E_{ij}, \vec{x}) + \lambda(E_{j\setminus i}, \vec{x})$. In particular, if G is left-compressed, then $(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i\setminus j}, \vec{x})$.

(b) If G is left-compressed, then

$$x_i - x_j = \frac{\lambda(E_{i\setminus j}, \vec{x})}{\lambda(E_{ij}, \vec{x})} \tag{3.2}$$

holds. If G is left-compressed and $E_{i\setminus j} = \emptyset$, then $x_i = x_j$.

(c) By (3.2), if G is left-compressed, then

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0. \tag{3.3}$$

4 Some Preliminary Results Used in Proof of Theorem 2.14

In this section, we will show the following preliminary results and their implications.

Theorem 4.1 *Let m, t and p be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$ and $t \geq 3p+8$. Let G be a left compressed 3-graph on the vertex set $[t]$ with m edges and let the triple with the minimum colex ordering in the complement of G be $(t-2-p)(t-2)(t-1)$, then $\lambda(G) < \lambda([t-1]^{(3)})$.*

Theorem 4.2 *Let m, t and p be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$ and $t \geq 2p+16$. Let G be a left compressed 3-graph on the vertex set $[t]$ with m edges and let the triple with the minimum colex ordering in the complement of G be $(t-3-p)(t-3)(t-1)$, then $\lambda(G) < \lambda([t-1]^{(3)})$.*

The following results are implied by the above two theorems.

Corollary 4.3 *Let m, t be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a left-compressed 3-graph on the vertex set $[t]$ with m edges and $|[t-1]^{(3)} \setminus E(G)| = 2$. Then $\lambda(G) < \lambda([t-1]^{(3)})$.*

Corollary 4.4 *Let m, t and p be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a left-compressed 3-graph on the vertex set $[t]$ with m edges and $|[t-1]^{(3)} \setminus E(G)| = 3$, then $\lambda(G) < \lambda([t-1]^{(3)})$ for $t \geq 18$.*

Corollary 4.5 *Let m and t be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a left-compressed 3-graph on the vertex set $[t]$ with m edges and $|[t-1]^{(3)} \setminus E(G)| = 4$. Then $\lambda(G) < \lambda([t-1]^{(3)})$ for $t \geq 19$.*

4.1 Proofs of Theorem 4.1 and Corollary 4.3

Proof of Theorem 4.1 Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting of a left-compressed 3-graph $G = ([t], E)$ satisfying the condition in Theorem 4.1. Then by Remark 3.2, $x_1 \geq x_2 \geq \cdots \geq x_t$. If $x_t = 0$, then $\lambda(G) < \lambda([t-1]^{(3)})$. So we assume that $x_t > 0$.

We apply induction on p. When $p = 1$, this is contained in Theorem 2.13. Then suppose that $p \geq 2$ and the assertion holds for smaller values of p.

Let i be the largest integer such that $i(t-1)t \in E$. Note that $1 \leq i \leq t-3-p$ and G is left-compressed. Let $G' = G \cup \{(t-2-p)(t-2)(t-1)\} \backslash \{i(t-1)t\}$. By the induction assumption, we have $\lambda(G', \vec{x}) \leq \lambda(G') < \lambda([t-1]^{(3)})$. If we can prove that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, then consequently, $\lambda(G) < \lambda([t-1]^{(3)})$. Now we show that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$. Note that

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) = x_{t-2-p}x_{t-2}x_{t-1} - x_i x_{t-1} x_t = (x_{t-2-p}x_{t-2} - x_i x_t)x_{t-1}. \tag{4.1}$$

By Remark 3.2(b),

$$x_i = x_{t-2-p} + \frac{\lambda(E_{i\backslash(t-2-p)}, \vec{x})}{\lambda(E_{i(t-2-p)}, \vec{x})}, \tag{4.2}$$

and

$$x_{t-2} = x_t + \frac{\lambda(E_{(t-2)\backslash t}, \vec{x})}{\lambda(E_{(t-2)t}, \vec{x})}. \tag{4.3}$$

Combining equations (4.1)–(4.3), we get

$$\begin{aligned} &\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \\ &= x_{t-2-p}\left(x_t + \frac{\lambda(E_{(t-2)\backslash t}, \vec{x})}{\lambda(E_{(t-2)t}, \vec{x})}\right)x_{t-1} - \left(x_{t-2-p} + \frac{\lambda(E_{i\backslash(t-2-p)}, \vec{x})}{\lambda(E_{i(t-2-p)}, \vec{x})}\right)x_{t-1}x_t \\ &= \left(\frac{\lambda(E_{(t-2)\backslash t}, \vec{x})}{\lambda(E_{(t-2)t}, \vec{x})}x_{t-2-p} - \frac{\lambda(E_{i\backslash(t-2-p)}, \vec{x})}{\lambda(E_{i(t-2-p)}, \vec{x})}x_t\right)x_{t-1}. \end{aligned} \tag{4.4}$$

Since G is left-compressed, then $[t-1]^{(3)} \backslash E = \{j(t-2)(t-1) : t-3-p \leq j \leq t-3\}$, and $it, i(t-1), (t-2)(t-1)$ and $(t-2)t \notin E_{i\backslash(t-2)}$, hence $\lambda(E_{i\backslash(t-2)}, \vec{x}) \leq (1 - x_i - x_{t-2})x_{t-1} + (1 - x_i - x_{t-2})x_t$ and $\lambda(E_{i(t-2)}, \vec{x}) = 1 - x_i - x_{t-2}$, then

$$\begin{aligned} x_i = x_{t-2} + \frac{\lambda(E_{i\backslash(t-2)}, \vec{x})}{\lambda(E_{i(t-2)}, \vec{x})} &\leq x_{t-2} + \frac{(1 - x_i - x_{t-2})x_{t-1} + (1 - x_i - x_{t-2})x_t}{1 - x_i - x_{t-2}} \\ &= x_{t-2} + x_{t-1} + x_t. \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned}\lambda(E_{i(t-2-p)},\vec{x})-\lambda(E_{(t-2)t},\vec{x})&\geq(1-x_i-x_{t-2-p})-(1-x_t-\cdots-x_{t-2-p})\\&\geq x_{t-2}+x_{t-1}+x_t-x_i\geq 0\end{aligned}$$

and $\lambda(E_{i(t-2-p)},\vec{x})\geq\lambda(E_{(t-2)t},\vec{x})$. In view of (4.4), to show $\lambda(G,\vec{x})\leq\lambda(G',\vec{x})$, it is sufficient to show that

$$\lambda(E_{(t-2)\backslash t},\vec{x})\geq\lambda(E_{i\backslash(t-2-p)},\vec{x}). \tag{4.6}$$

Let $D=\{jk: jk\in[t-3]^{(2)}$ and $jkt\notin E\}$. Assume that $|D|=l$.

If $l\leq 2$, we claim that $(t-5-p+l)(t-1)t\notin E$. Otherwise $(t-5-p+l)(t-1)t\in E$, then $|E^c_{(t-2)t}|\leq p+2-l$. Since G is left–compressed, and $[t-5-p+l]\subseteq E_{(t-1)t}$, then

$$\begin{aligned}m=|E|&=|E\cap[t-1]^{(3)}|+|[t-2]^{(2)}\cap E_t|+|E_{(t-1)t}|\\&\geq\left(\binom{t-1}{3}-p\right)+\left(\binom{t-2}{2}-(p+2-l)-l\right)+(t-5-p+l)\\&\geq\binom{t-1}{3}+\binom{t-2}{2}+t-3p-7+l\\&\geq\binom{t-1}{3}+\binom{t-2}{2}+t-3p-7\\&\geq\binom{t-1}{3}+\binom{t-2}{2}+1,\end{aligned}$$

a contradiction. As $(t-5-p+l)(t-1)t\notin E$, then

$$\begin{aligned}\lambda(E_{(t-2)\backslash t},\vec{x})\ &\geq x_{t-2-p}\lambda(E_{(t-2-p)(t-2)}\cap E^c_{(t-2-p)t},\vec{x})+(\lambda(D,\vec{x})\\&\quad+x_{t-5-p+l}x_{t-1}+\cdots+x_{t-3-p}x_{t-1})\\&\geq(x_{t-2-p}\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2-p}x_{t-2}-x_{t-2-p}x_{t-1})+3x_{t-3-p}x_{t-1}\\&=x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+3x_{t-3-p}x_{t-1}\\&\geq x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+3x_{t-4}x_{t-1}.\end{aligned}$$

If $l\geq 3$, then $\lambda(D,\vec{x})\geq 3x_{t-4}x_{t-3}\geq 3x_{t-4}x_{t-1}$, and

$$\begin{aligned}\lambda(E_{(t-2)\backslash t},\vec{x})&\geq x_{t-2-p}\lambda(E_{(t-2-p)(t-2)}\cap E^c_{(t-2-p)t},\vec{x})+\lambda(D,\vec{x})\\&\geq(x_{t-2-p}\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2-p}x_{t-2}-x_{t-2-p}x_{t-1})+3x_{t-4}x_{t-3}\\&=x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+3x_{t-4}x_{t-3}\\&\geq x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+3x_{t-4}x_{t-1}.\end{aligned}$$

In both cases, $\lambda(E_{(t-2)\backslash t},\vec{x})\geq x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+3x_{t-4}x_{t-1}$.

On the other hand,

$$\begin{aligned}\lambda(E_{i\backslash(t-2-p)},\vec{x})&=x_t\lambda(E^c_{(t-2-p)t},\vec{x})+x_{t-2}x_{t-1}\\&=x_t(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})+x_{t-2}x_{t-1}+x_{t-2}x_t+x_{t-1}x_t,\end{aligned}$$

and

$$\lambda(E^c_{(t-2-p)t})\geq x_{t-2}+x_{t-1}.$$

Combining the above inequalities, we get that

$$\begin{aligned}\lambda(E_{(t-2)\setminus t},\vec{x})-\lambda(E_{i\setminus(t-2-p)},\vec{x})\geq{}& x_{t-2-p}(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})\\ &+3x_{t-4}x_{t-1}-x_t(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}\\ &-x_{t-1})-x_{t-2}x_{t-1}-x_{t-2}x_t-x_{t-1}x_t\\ \geq{}&(x_{t-2-p}-x_t)(\lambda(E^c_{(t-2-p)t},\vec{x})-x_{t-2}-x_{t-1})\geq 0.\end{aligned}$$

Therefore, (4.6) holds. This completes the proof of Theorem 4.1. □

Proof of Corollary 4.3 Since $|[t-1]^{(3)}\setminus E(G)|=2$ and G is left-compressed, then $[t-1]^{(3)}\setminus E(G)=\{(t-3)(t-2)(t-1),(t-4)(t-2)(t-1)\}$. If $t\geq 14$, then it is guaranteed by the case $p=2$ of Theorem 4.1. If $t\leq 13$, then Theorem 2.12 implies it. This completes the proof of Corollary 4.3. □

4.2 Proofs of Theorem 4.2 and Corollary 4.4

Proof of Theorem 4.2 Let $\vec{x}=(x_1,x_2,\ldots,x_t)$ be an optimal weighting of a left-compressed 3-graph $G=([t],E)$ satisfying the condition in Theorem 4.2. Then $x_1\geq x_2\geq\cdots\geq x_t$. If $x_t=0$, then $\lambda(G)<\lambda([t-1]^{(3)})$. So we assume that $x_t>0$.

We apply induction on p. When $p=1$, then $[t-1]^{(3)}\setminus E\supseteq\{(t-3)(t-2)(t-1),(t-4)(t-2)(t-1),(t-4)(t-3)(t-1)\}$. Let i be the largest integer such that $i(t-1)t\in E$. Note that $1\leq i\leq t-5$ and G is left-compressed. Let $G'=G\bigcup\{(t-4)(t-3)(t-1)\}\setminus\{i(t-1)t\}$. Then G' is left-compressed. By Theorem 4.1, $\lambda(G',\vec{x})\leq\lambda(G')<\lambda([t-1]^{(3)})$. If we can prove that $\lambda(G,\vec{x})\leq\lambda(G',\vec{x})$, then consequently, $\lambda(G)<\lambda([t-1]^{(3)})$. Now we show that $\lambda(G,\vec{x})\leq\lambda(G',\vec{x})$. Note that

$$\lambda(G',\vec{x})-\lambda(G,\vec{x})=x_{t-4}x_{t-3}x_{t-1}-x_ix_{t-1}x_t=(x_{t-4}x_{t-3}-x_ix_t)x_{t-1}. \tag{4.7}$$

By Remark 3.2(b),

$$x_i=x_{t-4}+\frac{\lambda(E_{i\setminus(t-4)},\vec{x})}{\lambda(E_{i(t-4)},\vec{x})}, \tag{4.8}$$

and

$$x_{t-3}=x_t+\frac{\lambda(E_{(t-3)\setminus t},\vec{x})}{\lambda(E_{(t-3)t},\vec{x})}. \tag{4.9}$$

Combining equations (4.7)–(4.9), we get

$$\begin{aligned}&\lambda(G',\vec{x})-\lambda(G,\vec{x})\\ &\quad=x_{t-4}(x_t+\frac{\lambda(E_{(t-3)\setminus t},\vec{x})}{\lambda(E_{(t-3)t},\vec{x})})x_{t-1}-(x_{t-4}+\frac{\lambda(E_{i\setminus(t-4)},\vec{x})}{\lambda(E_{i(t-4)},\vec{x})})x_{t-1}x_t\\ &\quad=\frac{\lambda(E_{(t-3)\setminus t},\vec{x})}{\lambda(E_{(t-3)t},\vec{x})}x_{t-4}x_{t-1}-\frac{\lambda(E_{i\setminus(t-4)},\vec{x})}{\lambda(E_{i(t-4)},\vec{x})}x_{t-1}x_t.\end{aligned} \tag{4.10}$$

Similar to (4.5),

$$\begin{aligned}x_i=x_{t-2}+\frac{\lambda(E_{i\setminus(t-2)},\vec{x})}{\lambda(E_{i(t-2)},\vec{x})}&\leq x_{t-2}+\frac{(1-x_i-x_{t-2})x_{t-1}+(1-x_i-x_{t-2})x_t}{1-x_i-x_{t-2}}\\ &\leq x_{t-2}+x_{t-1}+x_t.\end{aligned} \tag{4.11}$$

Hence

$$\begin{aligned}\lambda(E_{i(t-4)},\vec{x})-\lambda(E_{(t-3)t},\vec{x})&\geq(1-x_i-x_{t-4})-(1-x_{t-4}-x_{t-3}-x_{t-2}-x_{t-1}-x_t)\\&=x_{t-3}+x_{t-2}+x_{t-1}+x_t-x_i>0,\end{aligned}$$

so $\lambda(E_{i(t-4)},\vec{x})>\lambda(E_{(t-3)t},\vec{x})$. Clearly, $x_{t-4}\geq x_t$, and it is sufficient to show that

$$\lambda(E_{(t-3)\setminus t},\vec{x})\geq\lambda(E_{i\setminus(t-4)},\vec{x}). \tag{4.12}$$

Let $D=\{jk : jk\in[t-2]^{(2)},\ j\neq t-3,\ k\neq t-3 \text{ and } jkt\notin E\}$, let $F=\{j(t-3)t : j\in[t-2] \text{ and } j(t-3)t\notin E\}$. Assume that $|D|=l$, where $l\geq 1$. Clearly, $|F|\leq l+1$.

If $l\leq 3$, then $(t-8+l)(t-1)t\notin E$. Otherwise, $(t-8+l)(t-1)t\in E$. Since G is left-compressed, and $[t-8+l]\subseteq E_{(t-1)t}$, then

$$\begin{aligned}m=|E|&=|E\cap[t-1]^{(3)}|+|[t-2]^{(2)}\cap E_t|+|E_{(t-1)t}|\\&=|E\cap[t-1]^{(3)}|+\left(\binom{t-2}{2}-|D|-|F|\right)+|E_{(t-1)t}|\\&\geq\left(\binom{t-1}{3}-2-l\right)+\left(\binom{t-2}{2}-2l-1\right)+(t-8+l)\\&=\binom{t-1}{3}+\binom{t-2}{2}+t-2l-11\\&\geq\binom{t-1}{3}+\binom{t-2}{2}+1,\end{aligned}$$

a contradiction. As $(t-8+l)(t-1)t\notin E$, then

$$\begin{aligned}&\lambda(E_{(t-3)\setminus t},\vec{x})\\&\quad\geq x_{t-4}\lambda(E_{(t-4)(t-3)}\cap E^c_{(t-4)t},\vec{x})+\lambda(D,\vec{x})+x_{t-8+l}x_{t-1}+\cdots+x_{t-5}x_{t-1}\\&\quad\geq x_{t-4}\lambda(E_{(t-4)(t-3)}\cap E^c_{(t-4)t},\vec{x})+lx_{t-4}x_{t-2}+(4-l)x_{t-5}x_{t-1}\\&\quad\geq x_{t-4}\lambda(E^c_{(t-4)t},\vec{x})-x_{t-4}x_{t-3}-x_{t-4}x_{t-1}+lx_{t-4}x_{t-2}+(4-l)x_{t-5}x_{t-1}\\&\quad\geq x_{t-4}(\lambda(E^c_{(t-4)t},\vec{x})-x_{t-3})+3x_{t-4}x_{t-1}.\end{aligned}$$

If $l\geq 4$, then

$$\begin{aligned}\lambda(E_{(t-3)\setminus t},\vec{x})&\geq x_{t-4}\lambda(E_{(t-4)(t-3)}\cap E^c_{(t-4)t},\vec{x})+\lambda(D,\vec{x})\\&\geq(x_{t-4}\lambda(E^c_{(t-4)t},\vec{x})-x_{t-4}x_{t-3}-x_{t-4}x_{t-1})+lx_{t-4}x_{t-2}\\&\geq x_{t-4}(\lambda(E^c_{(t-4)t},\vec{x})-x_{t-3})+3x_{t-4}x_{t-1}.\end{aligned}$$

Hence $\lambda(E_{(t-3)\setminus t},\vec{x})\geq x_{t-4}(\lambda(E^c_{(t-4)t},\vec{x})-x_{t-3})+3x_{t-4}x_{t-1}$ holds in both cases.

On the other hand,

$$\begin{aligned}\lambda(E_{i\setminus(t-4)},\vec{x})&=x_t\lambda(E^c_{(t-4)t},\vec{x})+x_{t-3}x_{t-1}+x_{t-2}x_{t-1}\\&=x_t(\lambda(E^c_{(t-4)t},\vec{x})-x_{t-3})+x_{t-3}x_{t-1}+x_{t-2}x_{t-1}+x_{t-3}x_t,\end{aligned}$$

and

$$\lambda(E^c_{(t-4)t},\vec{x})\geq x_{t-3}+x_{t-2}+x_{t-1}.$$

Combining the above inequalities, we get that $\lambda(E_{(t-2)\setminus t},\vec{x})-\lambda(E_{i\setminus(t-4)},\vec{x})\geq 0$. Therefore (4.12) holds when $p=1$.

Suppose then that $p \geq 2$ and the assertion holds for smaller values of p. Let i be the largest integer such that $i(t-1)t \in E$. Note that $1 \leq i \leq t-4-p$ and G is left-compressed. Let $G' = G \cup \{(t-3-p)(t-3)(t-1)\} \backslash \{i(t-1)t\}$. Then G' is left-compressed. By the induction assumption, we have $\lambda(G', \vec{x}) \leq \lambda(G') < \lambda([t-1]^{(3)})$. If we can prove that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, then consequently, $\lambda(G) < \lambda([t-1]^{(3)})$. Now we show that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$. Note that

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) = x_{t-3-p}x_{t-3}x_{t-1} - x_i x_{t-1} x_t = (x_{t-3-p}x_{t-3} - x_i x_t)x_{t-1}. \tag{4.13}$$

By Remark 3.2(b),

$$x_i = x_{t-3-p} + \frac{\lambda(E_{i\backslash(t-3-p)}, \vec{x})}{\lambda(E_{i(t-3-p)}, \vec{x})}, \tag{4.14}$$

and

$$x_{t-3} = x_t + \frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}. \tag{4.15}$$

Combining equations (4.13)–(4.15), we get

$$\begin{aligned} &\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \\ &= x_{t-3-p}\left(x_t + \frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}\right)x_{t-1} - \left(x_{t-3-p} + \frac{\lambda(E_{i\backslash(t-3-p)}, \vec{x})}{\lambda(E_{i(t-3-p)}, \vec{x})}\right)x_{t-1}x_t \\ &= \left(\frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}x_{t-3-p} - \frac{\lambda(E_{i\backslash(t-3-p)}, \vec{x})}{\lambda(E_{i(t-3-p)}, \vec{x})}x_t\right)x_{t-1}. \end{aligned} \tag{4.16}$$

Since G is left-compressed, and $it, i(t-1), (t-2)(t-1)$ and $(t-2)t \notin E_{i\backslash(t-2)}$, hence $\lambda(E_{i\backslash(t-2)}, \vec{x}) \leq (1-x_i-x_{t-2})x_{t-1} + (1-x_i-x_{t-2})x_{t-2}$ and $\lambda(E_{i(t-2)}, \vec{x}) = 1 - x_i - x_{t-2}$, then

$$\begin{aligned} x_i = x_{t-2} + \frac{\lambda(E_{i\backslash(t-2)}, \vec{x})}{\lambda(E_{i(t-2)}, \vec{x})} &\leq x_{t-2} + \frac{(1-x_i-x_{t-2})x_{t-1} + (1-x_i-x_{t-2})x_t}{1-x_i-x_{t-3}} \\ &\leq x_{t-2} + x_{t-1} + x_t. \end{aligned} \tag{4.17}$$

Hence

$$\begin{aligned} &\lambda(E_{i(t-3-p)}, \vec{x}) - \lambda(E_{(t-3)t}, \vec{x}) \\ &\quad\geq (1 - x_i - x_{t-3-p}) - (1 - x_{t-3-p} - x_{t-4} - x_{t-3} - x_{t-2} - x_{t-1} - x_t) \\ &\quad\geq x_{t-4} + x_{t-3} + x_{t-2} + x_{t-1} + x_t - x_i > 0 \end{aligned}$$

and $\lambda(E_{i(t-3-p)}, \vec{x}) > \lambda(E_{(t-3)t}, \vec{x})$. Clearly $x_{t-3-p} \geq x_t$. In view of (4.16), to show $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, it is sufficient to show that

$$\lambda(E_{(t-3)\backslash t}, \vec{x}) \geq \lambda(E_{i\backslash(t-3-p)}, \vec{x}). \tag{4.18}$$

Let $I = \{jk : jk \in [t-2]^{(2)},\ j \neq t-3,\ k \neq t-3 \text{ and } jkt \notin E\}$, let $J = \{j(t-3) : j \in [t-2]$ and $j(t-3)t \notin E\}$. Assume that $|I| = l$, then $l \geq p$. Clearly, $|J| \leq l + 1$.

If $l \leq 3$, then $p \leq 3$. We claim that $(t-7-p+l)(t-1)t \notin E$. Otherwise, $(t-7-p+l)(t-1)t \in E$. Since G is left-compressed, and $[t-7-p+l] \subseteq E_{(t-1)t}$, then

$$\begin{aligned} m = |E| &= |E \cap [t-1]^{(3)}| + |[t-2]^{(2)} \cap E_t| + |E_{(t-1)t}| \\ &\geq \left(\binom{t-1}{3} - 1 - p - l\right) + \left(\binom{t-2}{2} - 2l - 1\right) + (t-7-p+l) \end{aligned}$$

$$= \binom{t-1}{3} + \binom{t-2}{2} + t - 2p - 2l - 9$$
$$\geq \binom{t-1}{3} + \binom{t-2}{2} + 1,$$

contradiction. As $(t-7-p+l)(t-1)t \notin E$, then

$$\begin{aligned}\lambda(E_{(t-3)\setminus t}, \vec{x}) &\geq x_{t-4}\lambda(E_{(t-4)(t-3)} \cap E^c_{(t-4)t}, \vec{x}) \\ &\quad + (\lambda(I, \vec{x}) + x_{t-7-p+l}x_{t-1} + \cdots + x_{t-4-p}x_{t-1}) \\ &\geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-4}x_{t-3} - x_{t-4}x_{t-1}) \\ &\quad + (lx_{t-4}x_{t-2} + (4-l)x_{t-4-p}x_{t-1}) \\ &\geq x_{t-4}\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-4}x_{t-3} - x_{t-4}x_{t-1} + 4x_{t-4}x_{t-1} \\ &\geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3}) + 3x_{t-4}x_{t-1}.\end{aligned}$$

If $l \geq 4$, then

$$\begin{aligned}\lambda(E_{(t-3)\setminus t}, \vec{x}) &\geq x_{t-4}\lambda(E_{(t-4)(t-3)} \cap E^c_{(t-4)t}, \vec{x}) + \lambda(I, \vec{x}) \\ &\geq x_{t-4}\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-4}x_{t-3} - x_{t-4}x_{t-1} + 4x_{t-4}x_{t-2} \\ &\geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3}) + 3x_{t-4}x_{t-1}.\end{aligned}$$

Hence $\lambda(E_{(t-3)\setminus t}, \vec{x}) \geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3}) + 3x_{t-4}x_{t-1}$ holds in both cases.

On the other hand,

$$\begin{aligned}\lambda(E_{i\setminus(t-4)}, \vec{x}) &= x_t\lambda(E^c_{(t-4)t}, \vec{x}) + x_{t-3}x_{t-1} + x_{t-2}x_{t-1} \\ &= x_t(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3}) + x_{t-3}x_{t-1} + x_{t-2}x_{t-1} + x_{t-3}x_t,\end{aligned}$$

and

$$\lambda(E^c_{(t-4)t} \geq x_{t-3} + x_{t-2} + x_{t-1}.$$

Combining the above inequalities, we get that $\lambda(E_{(t-3)\setminus t}, \vec{x}) - \lambda(E_{i\setminus(t-4)}, \vec{x}) \geq 0$. Therefore, (4.18) holds in this case. This completes the proof. □

Proof of Corollary 4.4 Since $|[t-1]^{(3)} \setminus E(G)| = 3$ and G is left-compressed, then $[t-1]^{(3)} \setminus E(G) = \{(t-3)(t-2)(t-1), (t-4)(t-2)(t-1), (t-5)(t-2)(t-1)\}$ or $[t-1]^{(3)} \setminus E(G) = \{(t-3)(t-2)(t-1), (t-4)(t-2)(t-1), (t-4)(t-3)(t-1)\}$. The first case follows from Theorem 4.1. The second case follows from Theorem 4.2 by taking $p = 1$. This completes the proof of Corollary 4.4. □

4.3 Proof of Corollary 4.5

Proof There are 3 possible cases for $[t-1]^{(3)}\setminus E$.

Case 1 $[t-1]^{(3)}\setminus E = \{(t-3)(t-2)(t-1), (t-4)(t-2)(t-1), (t-5)(t-2)(t-1), (t-6)(t-2)(t-1)\}$;

Case 2 $[t-1]^{(3)}\setminus E = \{(t-3)(t-2)(t-1), (t-4)(t-2)(t-1), (t-5)(t-2)(t-1), (t-4)(t-3)(t-1)\}$;

Case 3 $[t-1]^{(3)}\setminus E = \{(t-3)(t-2)(t-1), (t-4)(t-2)(t-1), (t-4)(t-3)(t-1), (t-4)(t-3)(t-2)\}$.

Cases 1 and 2 follow from Theorems 4.1 and 4.2 respectively. So what remains is to consider Case 3.

Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting of a left-compressed 3-graph $G = ([t], E)$ satisfying the conditions in Corollary 4.5. Then $x_1 \geq x_2 \geq \cdots \geq x_t$. If $x_t = 0$, then $\lambda(G) < \lambda([t-1]^{(3)})$. So we assume that $x_t > 0$. Let i be the largest integer such that $i(t-1)t \in E$. Let $G' = G \cup \{(t-4)(t-3)(t-2)\} \backslash \{i(t-1)t\}$. By Theorem 4.2, $\lambda(G', \vec{x}) \leq \lambda(G') < \lambda([t-1]^{(3)})$. If we can prove that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, then consequently, $\lambda(G) < \lambda([t-1]^{(3)})$. Now we show that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$. Note that

$$\begin{aligned} \lambda(G', \vec{x}) - \lambda(G, \vec{x}) &= x_{t-4}x_{t-3}x_{t-2} - x_i x_{t-1} x_t \geq \ x_{t-4}x_{t-3}x_{t-2} - x_i x_{t-2} x_t \\ &= (x_{t-4}x_{t-3} - x_i x_t) x_{t-2}. \end{aligned} \tag{4.19}$$

By Remark 3.2(b),

$$x_i = x_{t-4} + \frac{\lambda(E_{i\backslash(t-4)}, \vec{x})}{\lambda(E_{i(t-4)}, \vec{x})}, \tag{4.20}$$

and

$$x_{t-3} = x_t + \frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}. \tag{4.21}$$

Combining equations (4.19)–(4.21), we get

$$\begin{aligned} &\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \\ &\quad = x_{t-4}\left(x_t + \frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}\right)x_{t-2} - \left(x_{t-4} + \frac{\lambda(E_{i\backslash(t-4)}, \vec{x})}{\lambda(E_{i(t-4)}, \vec{x})}\right)x_{t-2}x_t \\ &\quad = \left(\frac{\lambda(E_{(t-3)\backslash t}, \vec{x})}{\lambda(E_{(t-3)t}, \vec{x})}x_{t-4} - \frac{\lambda(E_{i\backslash(t-4)}, \vec{x})}{\lambda(E_{i(t-4)}, \vec{x})}x_t\right)x_{t-2}. \end{aligned} \tag{4.22}$$

Since G is left-compressed, and $it, i(t-1), (t-2)(t-1)$ and $(t-2)t \notin E_{i\backslash(t-2)}$, hence $\lambda(E_{i\backslash(t-2)}, \vec{x}) \leq (1 - x_i - x_{t-2})x_{t-1} + (1 - x_i - x_{t-2})x_t + x_{t-4}x_{t-3}$ and $\lambda(E_{i(t-2)}, \vec{x}) = 1 - x_i - x_{t-2}$, then

$$\begin{aligned} x_i &= x_{t-2} + \frac{\lambda(E_{i\backslash(t-2)}, \vec{x})}{\lambda(E_{i(t-2)}, \vec{x})} \\ &\leq x_{t-2} + \frac{(1 - x_i - x_{t-2})x_{t-1} + (1 - x_i - x_{t-2})x_t + x_{t-4}x_{t-3}}{1 - x_i - x_{t-3}} \\ &\leq x_{t-3} + x_{t-2} + x_{t-1} + x_t. \end{aligned} \tag{4.23}$$

Hence

$$\begin{aligned} \lambda(E_{i(t-4)}, \vec{x}) - \lambda(E_{(t-3)t}, \vec{x}) &\geq (1 - x_i - x_{t-4}) - (1 - x_{t-4} - x_{t-3} - x_{t-2} - x_{t-1} - x_t) \\ &= x_{t-3} + x_{t-2} + x_{t-1} + x_t - x_i \\ &\geq 0 \end{aligned}$$

and $\lambda(E_{i(t-4)}, \vec{x}) \geq \lambda(E_{(t-3)t}, \vec{x})$. Clearly $x_{t-4} \geq x_t$. In view of (4.22), it is sufficient to show that

$$\lambda(E_{(t-3)\backslash t}, \vec{x}) \geq \lambda(E_{i\backslash(t-4)}, \vec{x}). \tag{4.24}$$

Let $D = \{jk : jk \in [t-2]^{(2)},\ j \neq t-3,\ k \neq t-3 \text{ and } jkt \notin E\}$, let $F = \{j(t-3) : j \in [t-2] \text{ and } j(t-3)t \notin E\}$. Assume that $|D| = l$, where $l \geq 1$. Clearly, $|F| \leq l + 1$.

If $l \leq 3$, then $(t-10+l)(t-1)t \notin E$. Otherwise $(t-10+l)(t-1)t \in E$. Since G is left-compressed, and $[t-10+l] \subseteq E_{(t-1)t}$, then

$$\begin{aligned} m = |E| &= |E \cap [t-1]^{(3)}| + |[t-2]^{(2)} \cap E_t| + |E_{(t-1)t}| \\ &\geq \left(\binom{t-1}{3} - 4\right) + \left(\binom{t-2}{2} - 2l - 1\right) + (t-10+l) \\ &= \binom{t-1}{3} + \binom{t-2}{2} + t - l - 15 \\ &\geq \binom{t-1}{3} + \binom{t-2}{2} + 1, \end{aligned}$$

contradiction. As $(t-10+l)(t-1)t \notin E$, then

$$\begin{aligned} &\lambda(E_{(t-3)\backslash t}, \vec{x}) \\ &\quad \geq x_{t-4}\lambda(E_{(t-4)(t-3)} \cap E^c_{(t-4)t}, \vec{x}) + (\lambda(D, \vec{x}) + x_{t-10+l}x_{t-1} + \cdots + x_{t-5}x_{t-1}) \\ &\quad \geq (x_{t-4}\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-4}x_{t-3} - x_{t-4}x_{t-2} - x_{t-4}x_{t-1}) \\ &\qquad + lx_{t-4}x_{t-2} + (6-l)x_{t-5}x_{t-1} \\ &\quad \geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3} - x_{t-2}) + x_{t-4}x_{t-2} + 4x_{t-4}x_{t-1}. \end{aligned}$$

If $l \geq 4$, as G is left-compressed, then $(t-6)(t-2)t \in E(G^c)$, $(t-6)(t-1)t \in E(G^c)$ and $(t-5)(t-1)t \in E(G^c)$, and hence,

$$\begin{aligned} \lambda(E_{(t-3)\backslash t}, \vec{x}) &\geq x_{t-4}\lambda(E_{(t-4)(t-3)} \cap E^c_{(t-4)t}, \vec{x}) + lx_{t-4}x_{t-2} + 2x_{t-5}x_{t-1} \\ &\geq x_{t-4}\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-4}x_{t-3} - x_{t-4}x_{t-2} - x_{t-4}x_{t-1} \\ &\quad + 4x_{t-4}x_{t-2} + 2x_{t-5}x_{t-1} \\ &\geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3} - x_{t-2}) + x_{t-4}x_{t-2} + 4x_{t-4}x_{t-1}. \end{aligned}$$

Hence $\lambda(E_{(t-3)\backslash t}, \vec{x}) \geq x_{t-4}(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3} - x_{t-2}) + x_{t-4}x_{t-2} + 4x_{t-4}x_{t-1}$ holds in both cases.

On the other hand,

$$\begin{aligned} \lambda(E_{i\backslash(t-4)}, \vec{x}) &= x_t\lambda(E^c_{(t-4)t}, \vec{x}) + x_{t-3}x_{t-2} + x_{t-3}x_{t-1} + x_{t-2}x_{t-1} \\ &= x_t(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3} - x_{t-2}) + x_{t-3}x_{t-2} + x_{t-3}x_{t-1} \\ &\quad + 2x_{t-2}x_{t-1} + x_{t-3}x_t + x_{t-2}x_t \\ &\leq x_t(\lambda(E^c_{(t-4)t}, \vec{x}) - x_{t-3} - x_{t-2}) + x_{t-4}x_{t-2} + 4x_{t-4}x_{t-1}, \end{aligned}$$

and

$$\lambda(E^c_{(t-4)t}, \vec{x}) \geq x_{t-3} + x_{t-2} + x_{t-1}.$$

Combining the above inequalities, we get that $\lambda(E_{(t-3)\backslash t}, \vec{x}) - \lambda(E_{i\backslash(t-4)}, \vec{x}) \geq 0$. Therefore, (4.24) holds in this case. This completes the proof of Theorem 4.5. □

5 Proof of Theorem 2.14

Proof of Theorem 2.14 We adopt mathematical induction on p. When $p = 1$, $\lambda(G) < \lambda([t-1]^{(3)})$ by Theorem 2.13. Assume that the result holds when $|[t-1]^{(3)} \backslash E(G)| \leq p-1$, we will show that it holds when $|[t-1]^{(3)} \backslash E| = p$.

Let m and t be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2}$ and $t \geq 17p/2 + 11$. Let $G = (V, E)$ be a left-compressed 3-graph with m edges on the vertex set $[t]$ and $|[t-1]^{(3)} \backslash E(G)| = p$.

Let $\vec{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting of left-compressed 3-graph $G = ([t], E)$. Then $x_1 \geq x_2 \geq \cdots \geq x_t$. If $x_t = 0$, then $\lambda(G) < \lambda([t-1]^{(3)})$ and the conclusion is true. So we assume that $x_t > 0$.

Let $(t-j)(t-k)(t-l)$ be the triple in G^c with the minimum colex ordering, where $j > k > l$. Let i be the largest integer such that $i(t-1)t \in G$. Let $G' = G \cup \{(t-j)(t-k)(t-l)\} \backslash \{i(t-1)t\}$. If we can prove that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, then $|[t-1]^{(3)} \backslash E(G')| \leq p - 1$ and by the induction assumption, we have that $\lambda(G', \vec{x}) < \lambda([t-1]^{(3)})$. Consequently, $\lambda(G) < \lambda([t-1]^{(3)})$. Now we show that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$. By Remark 3.2(b), $x_1 = \cdots = x_i$. Note that

$$\begin{aligned} \lambda(G', \vec{x}) - \lambda(G, \vec{x}) &= x_{t-j}x_{t-k}x_{t-l} - x_i x_{t-1} x_t = x_{t-j}x_{t-k}x_{t-l} - x_1 x_{t-1} x_t \\ &\geq x_{t-j}x_{t-k}x_{t-1} - x_1 x_{t-1} x_t. \end{aligned} \tag{5.1}$$

If $k + l = 3$, then $17p/2 + 11 \geq 3p + 8$. By Theorem 4.1, we have $\lambda(G) < \lambda([t-1]^{(3)})$.

So we assume that $k + l \geq 4$, noting that $k \geq 3$.

Claim 1 $p \geq 2j + k + l - 9$.

Proof If the first triple in colex ordering in G^c is $\{(t-j)(t-3)(t-1)\}$, or the first two triples in colex ordering in G^c are $\{(t-5)(t-3)(t-1), (t-5)(t-4)(t-1)\}$, or the first two triples in colex ordering in G^c are $\{(t-4)(t-3)(t-1), (t-4)(t-3)(t-2)\}$, then p is equal to the number of edges in $E^c_{(t-2)(t-1)}$ adding the number of $j + k + l - 7$ edges of $E^c - E^c_{(t-2)(t-1)}$, and $p = j - 3 + 1 + j + k + l - 7 = 2j + k + l - 9$. Otherwise, p is not less than the number of edges in $E^c_{(t-2)(t-1)}$ adding the number of at least $j + k + l - 6$ edges of $E^c - E^c_{(t-2)(t-1)}$, and $p \geq j - 3 + 1 + j + k + l - 6 = 2j + k + l - 8$. So the conclusion holds. □

Let us continue the proof of Theorem 2.14.

By Remark 3.2(b), we have

$$x_1 = x_{t-j} + \frac{\lambda(E_{1\backslash(t-j)}, \vec{x})}{\lambda(E_{1(t-j)}, \vec{x})}, \tag{5.2}$$

and

$$x_{t-k} = x_t + \frac{\lambda(E_{(t-k)\backslash t}, \vec{x})}{\lambda(E_{(t-k)t}, \vec{x})}. \tag{5.3}$$

Combining equations (5.1)–(5.3), we get

$$\begin{aligned} &\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \\ &\quad \geq x_{t-j}\left(x_t + \frac{\lambda(E_{(t-k)\backslash t}, \vec{x})}{\lambda(E_{(t-k)t}, \vec{x})}\right)x_{t-1} - \left(x_{t-j} + \frac{\lambda(E_{1\backslash(t-j)}, \vec{x})}{\lambda(E_{1(t-j)}, \vec{x})}\right)x_{t-1}x_t \\ &\quad = \frac{\lambda(E_{(t-k)\backslash t}, \vec{x})}{\lambda(E_{(t-k)t}, \vec{x})}x_{t-j}x_{t-1} - \frac{\lambda(E_{1\backslash(t-j)}, \vec{x})}{\lambda(E_{1(t-j)}, \vec{x})}x_{t-1}x_t. \end{aligned} \tag{5.4}$$

Since $p \geq j - 2$, $t \geq 17p/2 + 11$, $j \geq 4$, then $1 + p \leq t - 15p/2 - 10 \leq t - 15j/2 + 5 < t - j$, that is $1 + p \leq t - j - 1 < t - k - 1$, hence $px_{t-j} \leq x_2 + \cdots + x_{1+p} < x_2 + \cdots + x_{t-k-1} + x_{t-k+1} + \cdots + x_t$.

By Remark 3.2(b),

$$x_1 = x_{t-k} + \frac{\lambda(E_{1\backslash(t-k)}, \vec{x})}{\lambda(E_{1(t-k)}, \vec{x})}$$

$$\leq x_{t-k} + \frac{(x_2 + \cdots + x_{t-k-1} + x_{t-k+1} + \cdots + x_{t-2})x_t + px_{t-j}x_{t-l}}{x_2 + \cdots + x_{t-k-1} + x_{t-k+1} + \cdots + x_t}$$
$$< x_{t-k} + x_{t-l} + x_t < 3x_{t-k} \leq 3x_{t-k-1}. \tag{5.5}$$

Hence

$$\begin{aligned}\lambda(E_{1(t-j)}, \vec{x}) - \lambda(E_{(t-k)t}, \vec{x}) &\geq x_t + x_{t-1} + \cdots + x_{t-j+1} + x_{t-j-1} + \cdots + x_2 \\ &\quad -(x_1 + \cdots + x_{t-j-1}) \\ &= x_t + x_{t-1} + \cdots + x_{t-j+1} - x_1 > 0,\end{aligned} \tag{5.6}$$

so $\lambda(E_{1(t-j)}, \vec{x}) > \lambda(E_{(t-k)t}, \vec{x})$.

By (5.1) and (5.5), we have

$$\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \geq x_{t-j}x_{t-k}x_{t-1} - 3x_{t-k}x_{t-1}x_t. \tag{5.7}$$

If $x_{t-j} \geq 3x_t$, then $\lambda(G', \vec{x}) - \lambda(G, \vec{x}) \geq 0$, so the conclusion holds. Now let us assume that $x_{t-j} < 3x_t$. Then

$$x_{t-k} \leq x_{t-j} < 3x_t. \tag{5.8}$$

In view of (5.4) and (5.6), to show that $\lambda(G, \vec{x}) \leq \lambda(G', \vec{x})$, it is sufficient to show that

$$\lambda(E_{(t-k)\backslash t}, \vec{x}) \geq \lambda(E_{1\backslash(t-j)}, \vec{x}). \tag{5.9}$$

Denote $V_0 = [t]\backslash\{1, t, t-k, t-j\}$. In order to show (5.9), we estimate the terms $x_q x_s$ (with positive or negative sign) contributing to $\lambda(E_{(t-k)\backslash t}, \vec{x}) - \lambda(E_{1\backslash(t-j)}, \vec{x})$ by dividing into three cases: $qs \in V_0^{(2)}$, $qs \in \{1, t, t-k, t-j\} \times V_0$, or $qs \in \{1, t, t-k, t-j\}^{(2)}$.

Claim 2 If $q_1 \leq s_1, q_2 \leq s_2$, $q_1 s_1 \in E_{(t-k)\backslash t} \cap V_0^{(2)}$, $q_2 s_2 \in E_{1\backslash(t-j)} \cap V_0^{(2)}$, then $3x_{q_1}x_{s_1} \geq x_{q_2}x_{s_2}$, $3x_{q_1}x_{s_1} \geq x_{t-k}x_{t-l}$, and $x_{q_1}x_{s_1} \geq x_{t-k}x_t$.

Proof Since $q_1 s_1(t-k) \in E$, $(t-j)(t-k)(t-l) \notin E$, and $q_1 \leq s_1$, then $q_1 \leq t-k-1$. So $x_{q_1}x_{s_1} \geq x_{t-k}x_t$ and $3x_{q_1}x_{s_1} \geq 3x_{t-l}x_t \geq x_{t-l}x_{t-k}$ by (5.8).

Next we show that either $q_1 < q_2$ or $s_1 < s_2$. Otherwise, assume that $q_1 \geq q_2$, and $s_1 \geq s_2$. Since $q_1 s_1 \in E_{(t-k)\backslash t} \cap V_0^{(2)}$, and G is left-compressed, then $q_1 s_1(t-k) \in E$. Since $t-j < t-k$, then $q_2 s_2(t-j) \in E$, which is a contradiction. Hence either $q_1 < q_2$ or $s_1 < s_2$. If both $q_1 < q_2$ and $s_1 < s_2$ holds, then $x_{q_1}x_{s_1} \geq x_{q_2}x_{s_2}$. Therefore, what remains is to consider the case $q_1 < q_2$ and $s_1 \geq s_2$, and the case $q_1 \geq q_2$ and $s_1 < s_2$.

Case 1 $q_1 < q_2$, and $s_1 \geq s_2$.

Since $q_2 s_2 \in E_{1\backslash(t-j)}$, $(t-j)(t-k)(t-l) \notin E$, and $q_2 \leq s_2$, so by (5.8) $s_2 \geq t-k$, and $x_t \geq x_{t-k}/3$. Therefore, we have that $x_{q_1}x_{s_1} \geq x_{q_1}x_t \geq x_{q_2}x_t \geq x_{q_2}x_{t-k}/3 \geq x_{q_2}x_{s_2}/3$. That is $3x_{q_1}x_{s_1} \geq x_{q_2}x_{s_2}$.

Case 2 $q_1 \geq q_2$, and $s_1 < s_2$.

Recall that $q_1 \leq t-k-1$, and $x_1 \leq 3x_{t-k-1}$ by (5.5). Therefore, we have that $x_{q_1}x_{s_1} \geq x_{q_1}x_{s_2} \geq x_{t-k-1}x_{s_2} \geq x_1 x_{s_2}/3 \geq x_{q_2}x_{s_2}/3$. That is, $3x_{q_1}x_{s_1} \geq x_{q_2}x_{s_2}$.

We have shown that $3x_{q_1}x_{s_1} \geq x_{q_2}x_{s_2}$, $3x_{q_1}x_{s_1} \geq x_{t-k}x_{t-l}$, and $x_{q_1}x_{s_1} \geq x_{t-k}x_t$, which complete the proof. □

To continue the proof of Theorem 2.14, we estimate the terms x_qx_s (with positive or negative sign) contributing to $\lambda(E_{(t-k)\backslash t},\vec{x})-\lambda(E_{1\backslash(t-j)},\vec{x})$ by dividing into three cases: $qs\in V_0^{(2)}$, $qs\in\{1,t,t-k,t-j\}\times V_0$, or $qs\in\{1,t,t-k,t-j\}^{(2)}$.

Step 1 Let $qs\in V_0^{(2)}$. The terms x_qx_s in this type contribute to

$$\begin{aligned}&\lambda(E_{(t-k)\backslash t},\vec{x})-\lambda(E_{1\backslash(t-j)},\vec{x})\\ &\quad\text{is}\sum_{q_1s_1\in E_{(t-k)\backslash t}\cap V_0^{(2)}}x_{q_1}x_{s_1}-\sum_{q_2s_2\in E_{1\backslash(t-j)}\cap V_0^{(2)}}x_{q_2}x_{s_2}.\end{aligned}\tag{5.10}$$

So we need to estimate the number of terms in $E_{1\backslash(t-j)}\cap V_0^{(2)}$ and $E_{(t-k)\backslash t}\cap V_0^{(2)}$.

Claim 3 $(V_0^{(2)}\cap E_{(t-k)\backslash t})\cap(V_0^{(2)}\cap E_{1\backslash(t-j)})=\emptyset$.

Proof If $qs\in V_0^{(2)}\cap E_{(t-k)\backslash t}$, then $qs(t-k)\in E$, and $qs(t-j)\in E$, so $qs\notin V_0^{(2)}\cap E_{1\backslash(t-j)}$. Similarly, if $qs\in V_0^{(2)}\cap E_{1\backslash(t-j)}$, then $qs(t-j)\notin E$, and $qs(t-k)\notin E$, so $qs\notin V_0^{(2)}\cap E_{(t-k)\backslash t}$. Hence $(V_0^{(2)}\cap E_{(t-k)\backslash t})\cap(V_0^{(2)}\cap E_{1\backslash(t-j)})=\emptyset$. □

Claim 4 $|E_{1\backslash(t-j)}\cap V_0^{(2)}|\le p$.

Proof Since $|[t-1]^{(3)}\setminus E|=p$, then the number of pairs $qs\in V_0^{(2)}$ such that $qs(t-j)\notin E$ are less than or equal to p. □

Claim 5 $|E_{(t-k)\backslash t}\cap V_0^{(2)}|\ge(2t-5p-4)/3$.

Proof Note that $(t-k)s,ts\notin E_{(t-k)\backslash t}$. Since $1st\in E$, then $1s\notin E_{(t-k)\backslash t}$. Hence

$$|E_{(t-k)\backslash t}\cap V_0^{(2)}|=|E_{(t-k)\backslash t}|-|\{(t-j)s:(t-j)s\in E_{(t-k)\backslash t}\}|.\tag{5.11}$$

If $(t-j)s\in E_{(t-k)\backslash t}$, note that $s\ne t-k,t,t-l$, so $(t-j)st\in E^c$, $(t-k)st\in E^c$, and $(t-l)st\in E^c$. Hence $3|\{(t-j)s:(t-j)s\in E_{(t-k)\backslash t}\}|\le|E_t^c|=t-2-p$. Therefore,

$$\begin{aligned}|E_{(t-k)\backslash t}\cap V_0^{(2)}|&\ge|E_{(t-k)\backslash t}|-\frac{t-2-p}{3}\\ &=|E_t^c\backslash E_{t-k}^c|-\frac{t-2-p}{3}\\ &\ge t-2-2p-\frac{t-2-p}{3}=\frac{2t-4-5p}{3}.\end{aligned}\tag{5.12}$$

Step 2 Let $q\in\{1,t,t-k,t-j\}$, and $s\in V_0$. We estimate the terms x_qx_s in this type contributing to $\lambda(E_{(t-k)\backslash t},\vec{x})-\lambda(E_{1\backslash(t-j)},\vec{x})$.

Since $1st\in E$ and $1s(t-j)\in E$, then $1s\notin E_{(t-k)\backslash t}$, and $1s\notin E_{1\backslash(t-j)}$. So we only consider the cases that $q\in\{t,t-k,t-j\}$, and $s\in[t]\backslash\{t,t-k,t-j\}$. Then the terms x_qx_s in this type contribute to

$$\begin{aligned}&\lambda(E_{(t-k)\backslash t},\vec{x})-\lambda(E_{1\backslash(t-j)},\vec{x})\\ &\quad\text{is } x_{t-j}\lambda(E_{(t-j)(t-k)}\cap E^c_{(t-j)t},\vec{x})-x_t\lambda(E^c_{(t-j)t},\vec{x})-x_{t-k}\lambda(E^c_{(t-j)(t-k)},\vec{x})\\ &\quad\ge x_{t-j}(\lambda(E^c_{(t-j)t},\vec{x})-x_{t-1}-\cdots-x_{t-l}-x_{t-k})\\ &\qquad -x_t\lambda(E^c_{(t-j)t},\vec{x})-x_{t-k}(x_{t-1}+\cdots+x_{t-l})\\ &\quad\ge x_{t-j}(\lambda(E^c_{(t-j)t},\vec{x})-x_{t-1}-\cdots-x_{t-l}-x_{t-k})\\ &\qquad -x_t(\lambda(E^c_{(t-j)t},\vec{x})-x_{t-1}-\cdots-x_{t-l}-x_{t-k})\\ &\qquad -x_t(x_{t-1}+\cdots+x_{t-l}+x_{t-k})-x_{t-k}(x_{t-1}+\cdots+x_{t-l})\end{aligned}$$

$$\geq -x_t(x_{t-1} + \cdots + x_{t-l} + x_{t-k}) - x_{t-k}(x_{t-1} + \cdots + x_{t-l})$$
$$\geq -(l+1)x_{t-k}x_t - lx_{t-k}x_{t-l}. \tag{5.13}$$

Step 3 Let $qs \in \{1, t, t-k, t-j\}^{(2)}$. Estimate the terms $x_q x_s$ in this type contributing to $\lambda(E_{(t-k)\setminus t}, \vec{x}) - \lambda(E_{1\setminus(t-j)}, \vec{x})$.

Since the situation is similar to Step 2 when $q = 1$ or $s = 1$, then we only need to consider the case that $qs \in \{t, t-k, t-j\}^{(2)}$. So the terms qs in this type contribute to

$$\lambda(E_{(t-k)\setminus t}, \vec{x}) - \lambda(E_{1\setminus(t-j)}, \vec{x}) \text{ is } -x_{t-k}x_t. \tag{5.14}$$

By (5.10), (5.13) and (5.14), we get

$$\begin{aligned}&\lambda(E_{(t-k)\setminus t}, \vec{x}) - \lambda(E_{1\setminus(t-j)}, \vec{x}) \\ &\geq \sum_{q_1s_1 \in E_{(t-k)\setminus t} \cap V_0^{(2)}} x_{q_1}x_{s_1} - \sum_{q_2s_2 \in E_{1\setminus(t-j)} \cap V_0^{(2)}} x_{q_2}x_{s_2} \\ &\quad -(l+1)x_{t-k}x_t - lx_{t-k}x_{t-l} - x_{t-k}x_t.\end{aligned} \tag{5.15}$$

By Claims 2–5, to show that $\lambda(E_{(t-k)\setminus t}, \vec{x}) \geq \lambda(E_{1\setminus(t-j)}, \vec{x})$, it is sufficient to have $(2t-4-5p)/3 \geq 3(p+l)+(l+1)+1$, that is $t \geq 7p+6l+5$. Since $p \geq 2j+k+l-9 \geq 2(l+2)+l+1+l-9 = 4l-4$, then $l \leq p/4+1$ and $7p+6l+5 \leq 17p/2+11$. So $t \geq 17p/2+11$, which is guaranteed by the condition of this theorem. This completes the proof. □

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