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Minimum Genus Embeddings of the Complete Graph

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Abstract In this paper, the problem of construction of exponentially many minimum genus embeddings of complete graphs in surfaces are studied. There are three approaches to solve this problem. The first approach is to construct exponentially many graphs by the theory of graceful labeling of paths; the second approach is to find a current assignment of the current graph by the theory of current graph; the third approach is to find exponentially many embedding (or rotation) schemes of complete graph by finding exponentially many distinct maximum genus embeddings of the current graph. According to this three approaches, we can construct exponentially many minimum genus embeddings of complete graph K_{12s+8} in orientable surfaces, which show that there are at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct minimum genus embeddings for K_{12s+8} in orientable surfaces. We have also proved that K_{12s+8} has at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct minimum genus embeddings in non-orientable surfaces.

Keywords Maximum genus embedding, minimum genus embedding, complete graph, current graph

MR(2010) Subject Classification 05C10, 05C30

1 Introduction

For the former, there are many results on the maximum genus orientable embeddings, for instance, papers such as Xuong [16], Liu [10–12] and Skoviera [15] studied this problem. Graph embedding theory has been widely used in very large scale integrated circuit, see [12]. In the history, finding a genus embedding for a complete graph was a long and difficult way, as surveyed in Ringel's monograph [14], which completed the proof of the well known Heawood Conjecture and gave the birth of modern topological graph. Since then Lawrencenko [8] showed that K_7 has exactly one genus embedding in orientable surfaces; Lawrencenko et al. [9] obtained that K_{19} has at least three genus embeddings in orientable surfaces; Bonnington et al. [2] proved that, for $s \geq 2$, K_{12s+7} has at least two genus embeddings in orientable surfaces; Goddyn et al. [4] showed that, for $s \geq 2$, K_{12s+7} has at least 11^s genus embeddings in orientable surfaces; Bonnington et al. [3] proved that, for $n \equiv 7, 19 \pmod{36}$ and $n \equiv 19, 55 \pmod{108}$, K_n has, respectively, $2^{\frac{n^2}{54}-O(n)}$ and $2^{\frac{2n^2}{81}-O(n)}$ genus embeddings in orientable surfaces; Korzhik and Voss [6] obtained

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that both K_{12s+4} and K_{12s+7} has at least 4^s genus embeddings in orientable surfaces; Korzhik and Voss [7] showed that, for any $i \in \{1,2,3,\dots,11\} - \{3,4,7\}$, K_{12+i} (for $s \geq d(i) \in 1,2,3,4$), has at least $h(i)4^s$ distinct genus embeddings in orientable surfaces, $h(i) \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$; Ren and Bai [13] proved that the complete graph K_n with order $n \equiv 4,7,10 \pmod{12}$ has at least $C2^{\frac{n}{4}}$ distinct genus embeddings in orientable surfaces, where $C = 1, 2^{\frac{-3}{4}}$ or $2^{-\frac{3}{2}}$ with respect to $n \equiv 4,7,10 \pmod{12}$, respectively.

In this paper, we will show that the complete graph K_{12s+8} has at least $2^{s+4} \times (\frac{5}{3})^{2s+1} \times 2^{2s-3}$ distinct minimum genus embeddings in orientable surfaces, we will also show that the complete graph K_{12s+8} has at least $2^{s+5} \times (\frac{5}{3})^{2s+1} \times 2^{2s-4}$ distinct minimum genus embeddings in non-orientable surfaces. But Korzhik and Voss [7] have only showed that K_{12s+8} has at least 2^{2s-3} orientable genus embeddings. Korzhik and Voss [5] have only showed that K_{12s+8} has at least 2^{2s-4} nonorientable genus embeddings.

2 Construction of Current Graph

Let $G = (V, E)$ be a connected graph with $|V(G)|$ vertices and $|E(G)|$ edges. If there exists $f : V(G) \to \{1, 2, \dots, |E(G)|+1\}$ being an injective mapping, then we define an induced function $f' : E(G) \to \{1, 2, \dots, |E(G)|\}$ by setting $f'(e) = |f(u) - f(v)|$ for all $e = (u, v) \in E(G)$. If f' maps $E(G)$ onto $\{1, 2, \dots, |E(G)|\}$, then f is said to be a graceful labeling of G, where the graph G is said to be a graceful graph.

See Figure 1, let P be a path on $2s + 1$ vertices and let f be a graceful labeling of P, so that f is a bijection from the vertex set of P onto the set of labels $\{1, 2, \dots, 2s, 2s + 1\}$, v is one vertex of P with valence one. In the next steps, we add two new vertices b and t together with the edge bt and the $4s + 2$ edges of the form bu and ut where u ranges over all vertices of P. According to those steps, a new graph F is obtained, where F triangulates the plane; and F is embedded so that b and t are the "bottom" and "top" vertices while P shows up as a "horizontal" path. Now, we transform f to a function f' on the vertex set of F defined by $f'(b) = 1$, $f'(t) = 6s+4$, and $f'(u) = 3f(u)$ for each vertex u of P. Observe that f' is a graceful labeling of the graph F, since the induced edge-labels on the $2s$ edges of P and on the edge bt, on the $2s + 1$ edges bu, and on the $2s + 1$ edges ut, coincide with the sets of integers between 1 and $6s + 3$ which are congruent to $0, 2$, and 1, respectively. We direct each edge-labels of F from the vertex carrying the smaller value of f' to the vertex with the larger value.

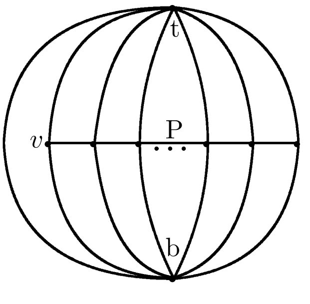

Figure 1 The graph F

In Figure 1, we identify the three vertices t, v and b of the graph F, a new graph G can be obtained, see Figure 2. For any edge e, $e \in E(F)$ and its corresponding edge $e \in E(G)$,

$f'(e(F)) = f'(e(G))$, i.e., the edge-label $f'(e)$ on the edge e of F and the edge-label $f'(e)$ on the edge e of G is the same, the direction of the edge-label on the edge e of F and the edge-label on the edge e of G is also the same.

Let G^* be the dual graph to the (plane) graph G, shown in Figure 2 in dashed lines. We now transform the edge-labels of the labeling f' of G into a current assignment f^* on G^* as follows. Firstly, we fix a clockwise orientation of every face of plane embedding of G; then, we direct every edge e^* of G^* towards the face whose orientation on the face boundary agrees with the orientation of e. Finally, for any directed edge e of G, the current $f^*(e^*)$ on the directed edge e^* of G^* dual to e is defined to be equal to $f'(e)$.

From the construction of the labels of the graph F, we know that, if $f'(v)$ is odd number, the current of the left top horizontal edge incident with the valence one of G^* is odd number, the current of the left bottom horizontal edge incident with the valence one of G^* is even number; if $f'(v)$ is even number, the current of the left top horizontal edge incident with the valence one of G^* is even number, the current of the left bottom horizontal edge incident with the valence one of G^* is odd number, and the current of right edge incident with vertex of valence one of G^* is $6s + 3$.

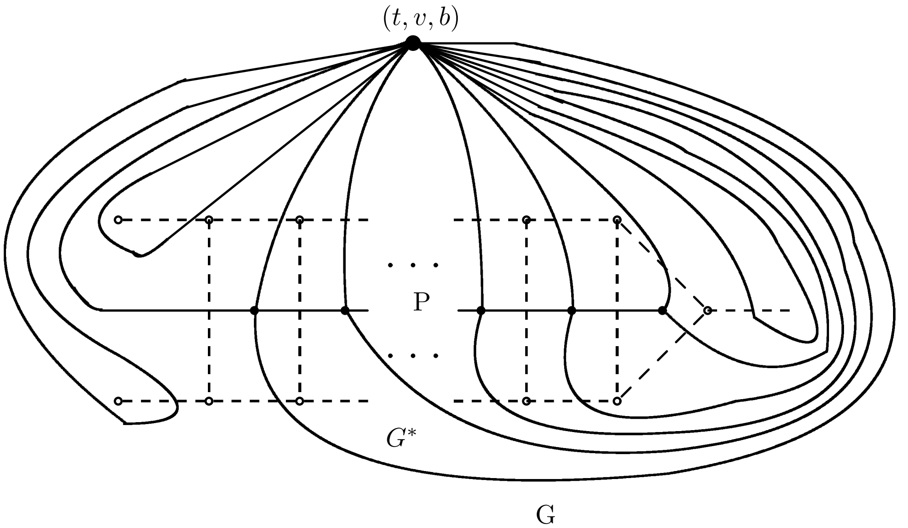

Figure 2 The graph G and its dual G^*

We consider any triangle $v_1v_2v_3$ in the plane embedding of F (see Figure 1). The way the labeling f' was introduced implies that we may always choose the notation for three vertices such that the boundary of the triangle consists of the directed path $v_1v_2v_3$ and the directed edges v_1v_3. Moreover, with this notation we have $f'(v_1v_2) + f'(v_2v_3) = f'(v_1v_3)$ for any triangular face of F. Therefore, at each vertex of valence three of G^*, we have either two incoming dual directed edges and one outgoing dual directed edge, or one incoming dual directed edge and two outgoing dual directed edges. In the first case, the sum of the currents on the incoming dual edges is equal to the current on the outgoing dual edge; the situation in the second case is analogous. Therefore, the current assignment f^* satisfies the Kirchhoff's Current Law at each vertex of valence three of G^*.

3 Rotation System

A spanning tree T in a graph G is called *optimal* if the number of odd components, denoted by $\omega(T)$, of $G-T$ is smallest among all spanning trees of G. It is well known that $\omega(T)$ is related to graph embeddings, especially in the maximum orientable embeddings of graphs.

Theorem 3.1 ([11, 16]) *The maximum genus of a graph G in orientable surfaces is*

$$\gamma_{M}(G)=\frac{\beta(G)-\omega(T)}{2},$$

where $\beta(G)$ is the Betti-number of G and $\omega(T)$ is the number of odd components in an optimal tree T in G.

Theorem 3.2 ([12, 16]) *Let T be an optimal tree in a graph G with $\omega(T)$ odd components in $G-T$. Then edges of $E(G)-E(T)$ may be partitioned as follows*

$$E(G)-E(T)=\bigcup_{i=1}^{s}\{e_{2i-1},e_{2i}\}\cup\{f_1,f_2,\ldots,f_m\},$$

where for each $i: 1\le i\le s$, $e_{2i-1}\cap e_{2i}\ne\phi$ and $\{f_1,f_2,\ldots,f_m\}$ is a matching of G and $s=\gamma_M(G)$, $m=\omega(T)$.

Fact Let H be a connected graph embedded in some orientable surfaces S_h with a single region whose boundary is W. Then for every vertex $x\in V(H)$, $d_H(x)=k\Leftrightarrow W$ visits x exactly k times (i.e., there are exactly $d_H(x)$ copies of x on W).

Theorem 3.3 *If G is a loopless 3-regular graph with n vertices. Then G has at least $2^{n-\gamma_M(G)}$ distinct orientable maximum genus embeddings, where $\gamma_M(G)$ is the orientable maximum genus of G.*

Proof Let T be an optimal tree of G. By Theorem 3.2, $E(G)-E(T)$ may be partitioned as follows

$$E(G)-E(T)=\bigcup_{i=1}^{s}\{e_{2i-1},e_{2i}\}\cup\{f_1,f_2,\ldots,f_m\},$$

where for each $i: 1\le i\le s$, $e_{2i-1}\cap e_{2i}\ne\phi$ and $\{f_1,f_2,\ldots,f_m\}$ is a matching of G and $s=\gamma_M(G)$, $m=\omega(T)$. Let $e_i=(x_i,y_i)(1\le i\le 2s)$ and $x_{2j-1}=x_{2j}\in e_{2j-1}\cap e_{2j}(1\le j\le s)$.

Let $v_1,v_2,\ldots,v_n$ be vertices of T, $v_1,v_2,\ldots,v_\alpha$ be its inner vertices. Any rotation scheme determines a planar embedding of T with a single region (face) whose boundary is W_0. For every inner vertex v_i, if $d_T(v_i)=2$, then there are two ways for an edge in $E(G)-E(T)$ to be incident to v_i ($d_T(v_i)=2$, the vertex v_i of T has two corner, G is loopless 3-regular graph, $d_G(v_i)=3$, an edge in $E(G)-E(T)$ to be incident to v_i in one corner); If $d_T(v_i)=3$, v_i will also contributes two ways in the rotation scheme of T (G is loopless 3-regular graph, $d_T(v_i)=d_G(v_i)=3$). Each case v_i will contribute two ways in constructing of maximum genus embeddings of G. Hence all the inner vertices in T give at least 2^α ways to construct an maximum genus embeddings of G.

Consider the vertex $x_1\in e_1\cap e_2$ and fix it at a copy of x_1 on W_0. Then choose a copy of y_1 and y_2, respectively, on W_0 (note that there are, respectively, $d_{G_0}(x_1), d_{G_0}(y_1)$ and $d_{G_0}(y_2)$ ways of doing so), where $G_0=T$. Then we find a new facial walk W_1 (containing W_0 and the edges e_1,e_2). We choose the labeling of y_1 and y_2 so that, in W_0, the selected copies of vertices

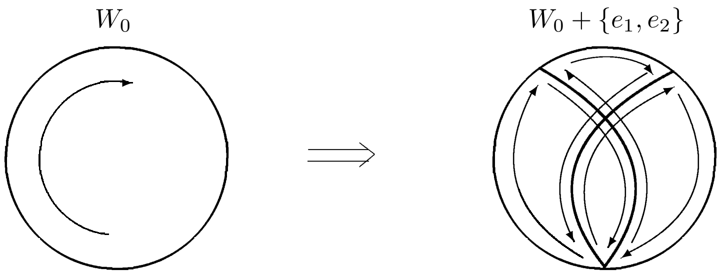

Figure 3 Adding a pair of "V" type of edges into a single face will result in another one-face graph in higher surface

x_1, y_1 and y_2 occur in the order x_1, y_1, y_2. Between the two consecutive edges at the current rotation at x_1, W_0 goes e, x_1, e'. We extend the rotation W_0 at x_1 from $\ldots, e, e', \ldots$ to $\ldots, e, x_1y_1, x_1y_2, e', \ldots$. Likewise we insert the edge x_1y_1 between the consecutive (at that copy of y_1) edges and the same for x_1y_2 at y_2 (as shown in Figure 3, $e_1 = x_1y_1$, $e_2 = x_1y_2$). Now we obtain $G_1 = G_0 + \{e_1, e_2\}$ to have its maximum genus embeddings on S_1, the torus, with a single region (face) bounded by edges of W_1. It is clear that there are at least $d_{G_0}(x_1)d_{G_0}(y_1)d_{G_0}(y_2)$ ways to construct the facial walk W_1 by the Fact.

Repeat the above procedure for $i = 1, 2, \ldots, s$, we can add all $\{e_{2i-1}, e_{2i}\}$. we can also add each $f_i \in \{f_1, f_2, \ldots, f_m\}$ into the inner region of W_s, a facial walk of G_s into S_s, each $f_i = (x_{2s+i}, y_{2s+i})$ has exactly $d_{G_s}(x_{2s+i})d_{G_s}(y_{2s+i})$ ways to be putted into W_s ($E(G) - E(T) = \cup_{i=1}^{s}\{e_{2i-1}, e_{2i}\} \cup \{f_1, f_2, \ldots, f_m\}$, $i : 1 \leq i \leq s$, $e_{2i-1} \cap e_{2i} \neq \phi$, $\{f_1, f_2, \ldots, f_m\}$ is a matching of G and $s = \gamma_{_M}(G)$, $m = \omega(T)$. We can add each f_i, but it does not yield a single face embedding of G_s.

Here G is a loopless 3-regular graph of order n, apart from the vertices $x_{2i-1} = x_{2i}$ ($i = 1, 2, \ldots, \gamma_{_M}(G)$), the remaining $n - \gamma_{_M}(G)$ vertices v each contributes a term $(d_G(v) - 1)! = 2$ to the orientable maximum genus of G. Therefore, G has at least $2^{n-\gamma_M(G)}$ distinct orientable maximum genus embeddings, $\gamma_M(G)$ is the orientable maximum genus of G.

This completes the proof of Theorem 3.3. □

4 Two Extremal Embeddings

From [14], we know, for each $\sigma \in \Pi$ (Π is a set of rotation systems), the pair (f^*, σ) determines a cyclic embedding of K_{12s+8}. Here, we first find the set Π of rotation systems of K_{12s+8}.

Rule Δ^* [14] If in line i has: $\cdots jk \cdots$, then in line k must have: $\cdots ij \cdots$.

Additive Rule [14] $\langle i \rangle = \langle 0 \rangle + i$; $* + i = *, * = x, y, z, \ldots$, $1 \leq i \leq n$, the number row $\langle i \rangle$ is obtained; the letter row $(\langle x \rangle, \langle y \rangle, \langle z \rangle, \ldots)$ can be obtained from number rows by Rule Δ^*.

Theorem 4.1 ([1]) *Let P_n be a path of order n, the number of graceful labeling of path P_n is larger than $(5/3)^n$ for sufficiently large n.*

Theorem 4.2 ([12]) *Let $\beta(G)$ be the Betti-number of G. If $\beta(G) = 0 \pmod 2$, then G can be embedded in an orientable surface with one face.*

Theorem 4.3 *The current graph G^* (See Figure 2, also see Figure 4) has at least 2^{3s+1} distinct orientable maximum genus embeddings.*

Proof Here G^* has $4s + 4$ vertices and $6s + 3$ edges, $\beta(G^*) = 0 \pmod{2}$, by Theorem 4.2, G^*

can be embedded in an orientable surface with one face.

See Figure 5, G^* has an optimal tree $T(G^*)$. Similar to the proof of Theorem 3.3, G^* has at least 2^{3s+1} distinct orientable maximum genus embeddings.

This completes the proof of Theorem 4.3. $\square$

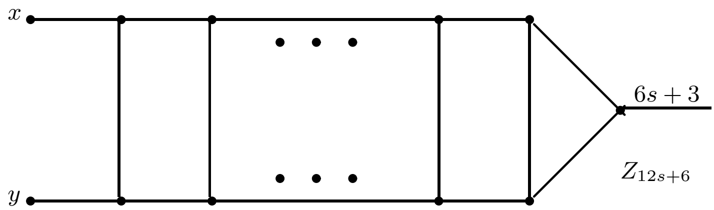

Figure 4 The current graph G^*

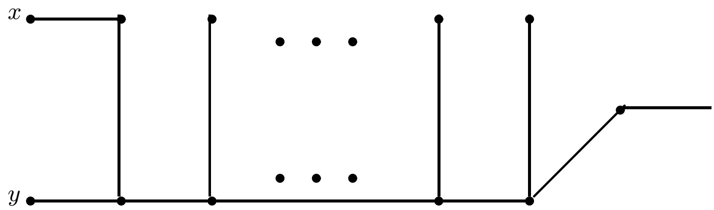

Figure 5 $T(G^*)$

Theorem 4.4 *To each graceful labeling f of a path on $2s+1$ vertices, $s \geq 2$, there corresponds a collection $\mathcal{C}_f$ of 2^{3s+1} distinct oriented genus embeddings of a complete graph on $12s+8$ vertices. Moreover, if f and g are distinct graceful labeling of a path on $2s+1$ vertices, then no genus embedding in $\mathcal{C}_f$ is isomorphic to a genus embedding in $\mathcal{C}_g$.*

Proof In Section 2, to each graceful labeling f of a path on $2s+1$ vertices, $s \geq 2$, there corresponds a graph G^* (see Figure 2), $\beta(G^*) = 0 \pmod 2$. By Theorem 4.2, G^* can be embedded in an oriented surface with one face. G^* has an optimal tree $T(G^*)$ (see Figure 5). $G^* - T(G^*)$ has $2s$ edges, and is connected. We now find an embedding of G^* in oriented surfaces with single face as follows. As we have done in the proof of Theorem 3.3, first, one find a planar embedding of the optimal tree $T(G^*)$ with a single face whose boundary is W_0, then, repeat adding s pairs "V"-type-edges into a single face. According to this, G^* can be embedded in an oriented surface with a single face, and the oriented surface with genus s.

We assume the current f^* on G^* is the group Z_{12s+6}, the additive group of integers modulo $12s+6$. G^* may be embedded in an oriented surface with a single face. As we know from [14], for each single face embedding of G^* has a circular rotation, the circular rotation induces exactly one circuit, record the currents sequentially, as they occur in the circuit. We take this currents sequentially and x, y as row 0 of a rotation scheme of K_{12s+8}. By Additive Rule, all row i ($i = 1, 2, \ldots, 12s+5, x, y$) can be obtained.

By Theorem 4.3, there exists a set Π of 2^{3s+1} different rotation systems σ for G^*. As we know from Subsection 7.6 of [14], for each $\sigma \in \Pi$, the pair (f^*, σ) determines a cyclic oriented genus embedding of K_{12s+8}.

To finish the proof of the theorem, it remains to show that for any two grace labeling f and

g of the path P and for any two distinct rotation systems σ, $\tau \in \Pi$, the pairs (f^*, σ) and (g^*, τ) determine isomorphic oriented genus embeddings if and only if $f^* = g^*$ and $\sigma = \tau$ (which also implies that each collection $\mathcal{C}_f$ of 2^{3s+1} pairwise non–isomorphic oriented embeddings).

According to Theorem 2 of [5], the case of orientable embeddings can be stated as follows:

The pairs (f^*, σ) and (g^*, τ) determine isomorphic oriented genus embeddings of K_{12s+8}, if and only if the graph G^* admits an automorphism φ (regard as a permutation of the arc set $A(G^*)$) and the group Z_{12s+8} admits an automorphism α such that $\varphi\sigma(e) = \tau\varphi(e)$ and $g^*\varphi(e) = \alpha f^*(e)$ for every arc $e \in A(G^*)$.

Because the current on the left two horizontal edges incident with the valence one of G^* (see Figure 4), one is odd number, another is even. Therefore, the current graph G^* has only one automorphism which, in Figure 4, is the identity. Therefore, to each graceful labeling f of a path on $2s + 1$ vertices, $s \geq 2$, there corresponds a collection $\mathcal{C}_f$ of 2^{3s+1} cyclic genus embeddings of a complete graph on $12s + 8$ vertices.

This complete the proof of Theorem 4.4. □

Theorem 4.5 *If s is sufficiently large, the complete graph K_{12s+8} has at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct genus embeddings in orientable surfaces.*

Proof By Theorems 4.1 and 4.4, Theorem 4.5 is obtained. □

We have proved that K_{12s+8} has at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct genus embeddings in orientable surfaces. We will prove a stronger result as follows.

Theorem 4.6 *If s is sufficiently large, the complete graph K_{12s+8} has at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct minimum genus embeddings in orientable surfaces.*

Proof Let $\gamma(G)$ be the minimal genus of G in orientable surfaces, in (4.13) of [14], $\gamma(K_n) = \{\frac{(n-3)(n-4)}{12}\}$ for $n \geq 3$.

In Figure 4, if the current on the edge incident with the vertex x of G^* is odd number, the current on the edge incident with the vertex y of G^* is even number. From Subsection 7.6 of [14], we know that the current graph (Figure 4) generates the row 0 and a scheme with rows $0, 1, 2, \ldots, 12s + 5, x, y_0$ and y_1. Since the current on the arc incident with y is even, which generates the even subgroup of Z_{12s+6}, the letter y splits into y_0 and y_1. The graph G_{12s+9} represented by this scheme is the following: The vertices $0, 1, 2, \ldots, 12s + 5$, x are all mutually adjacent, y_0 is adjacent to $0, 2, 4, \ldots$ and y_1 is adjacent to $1, 3, 5, \ldots$. There are no other arcs in the graph G_{12s+9}. The scheme represents a triangular embedding of G_{12s+9} into an orientable surface S. The genus p of S can be determined by using the formula $\alpha_1 = 3\alpha_0 - 3E(S) = 3\alpha_0 - 6 + 6p$ of Theorem 4.4 of [14], α_0 is the number of vertices, α_1 is the number of edges, $E(S)$ is Euler characteristic. We obtain $p = 12s^2 + 9s + 1$.

If we put $n = 12s + 8$ into the right-hand side of the formula $\gamma(K_n) = \{\frac{(n-3)(n-4)}{12}\}$, we get $12s^2 + 9s + 2$. This means we are forced to use one handle in order to manufacture the missing adjacency between x and y_0 and between y_0 and y_1. Then $\gamma(K_n) = \{\frac{(n-3)(n-4)}{12}\}$ will be proven for $n = 12s + 8$.

If the current on the edge incident with the vertex x of G^* is even number, the current on the edge incident with the vertex y of G^* is odd number. This case is analogous.

This complete the proof of Theorem 4.6. □

Theorem 4.7 *If s is sufficiently large, the complete graph K_{12s+8} has at least $\frac{10}{3} \times (\frac{200}{9})^s$ distinct minimum genus embeddings in non-orientable surfaces.*

Proof Let $\gamma^*(G)$ be the minimal genus of G in non-orientable surfaces, in (4.19) of [14], $\gamma^*(K_n) = \{\frac{(n-3)(n-4)}{6}\}$ for $n \neq 7$ and $n \geq 3$.

In Figure 4, if the current on the edge incident with the vertex x of G^* is odd number, the current on the edge incident with the vertex y of G^* is even number. Consider the dual map on S. By the proof of Theorem 4.6, we can construct a certain map M with countries $0, 1, 2, \dots, 12s+5, x, y_0, y_1$ on a certain orientable surface S, the genus of S equal $12s^2+9s+1$, $E(S) = -24s^2 - 18s$.

In the map M the country x must be adjacent to an even numbered country α and an odd numbered country β with α and β adjacent to each other. Moreover, the country y_0 must be adjacent to α and y_1 to β. Excise the interior of a closed 2-cell inside country α and identity opposite points on the boundary. Do the same for β. This construction adds two cross caps to S giving us the new surface S'. Therefore $E(S') = E(S) - 2$.

Erase the boundary between y_0 and y_1 and call this newly created country y. The map on S' has $n = 12s+8$ countries each adjacent to all the others. S' is non-orientable, $E(S') = 2-q$, the genus $q = 24s^2 + 18s + 4$. Then $\gamma^*(K_n) = \{\frac{(n-3)(n-4)}{6}\}$ will be proven for $n = 12s + 8$.

If the current on the edge incident with the vertex x of G^* is even number, the current on the edge incident with the vertex y of G^* is odd number. This case is analogous.

This complete the proof of Theorem 4.7. □

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References

[1] Aldred, R. E. L., Širáň, J., Širáň, M.: A note on the number of graceful labellings of paths. *Discrete Mathematics*, **261**, 27–30(2003)
[2] Bonnington, C. P., Grannell, M. J., Griggs, T. S., et al.: Face 2-colourable triangular embeddings of complete graphs. *Journal of Combinatorial Theory, Series B*, **74**, 8–19 (1998)
[3] Bonnington, C. P., Grannell, M. J., Griggs, T. S., et al.: Exponential families of non-isomorphic triangulations of complete graphs. *Journal of Combinatorial Theory, Series B*, **78**, 169–184 (2000)
[4] Goddyn, L. , Richter, R. B. , Širáň, J.: Triangular embeddings of complete graphs from graceful labellings of paths. *Journal of Combinatorial Theory, Series B*, **97**, 964–970 (2007)
[5] Korzhik, V. P. , Voss, H.-J.: Exponential families of nonisomorphic nonorientable genus embeddings of complete graphs. *Journal of Combinatorial Theory, Series B*, **91**, 253–287 (2004)
[6] Korzhik, V. P. , Voss, H.-J.: On the number nonisomorphic orientable regular embeddings of complete graphs. *Journal of Combinatorial Theory, Series B*, **81**, 58–76(2001)
[7] Korzhik, V. P. , Voss, H.-J.: Exponential families of nonisomorphic nontriangular orientable genus embeddings of complete graphs. *Journal of Combinatorial Theory, Series B*, **86**, 186–211 (2002)
[8] Lawrencenko, S.: The irreducible triangulations of the torus. *Ukrain. Geom.*, **30**, 52–62 (1987)
[9] Lawrencenkno, S., Negami, S., White, A. T.: Three nonisomorphic triangulations of an orientable surface with the same complete graph. *Discrete Mathematics*, **135**, 367–369(1994)
[10] Liu, Y. P.: The maximum orientable genus of some kinds of graph. *Acta Mathematica Sinica*, **24**, 817–132 (1981)
[11] Liu, Y. P.: The maximum orientable genus of a graph. *Scientia Sinica, Special Issue on Math*, 192–201 (1979)
[12] Liu, Y. P.: Embeddability in Graphs, Kluwer, Dordrecht/Boston/London, 1995

[13] Ren, H., Bai, Y.: Exponentially many maximum genus embeddings and genus embedings for complete graphs. *Sci. China Ser. A*, **51**(11), 2013–2019 (2008)

[14] Ringel, G.: Map Color Theorem, Springer-Verlag, Berlin, 1974

[15] Skoviera, M.: The maximum genus of graphs of diameter two. *Discrete Mathematics*, **87**, 175–180 (1991)

[16] Xuong, N. H.: How to determine the maximum genus of a graph. *Journal of Combinatorial Theory, Series B*, **23**, 217–225 (1979)