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On Submanifolds Whose Tubular Hypersurfaces Have Constant Higher Order Mean Curvatures

Tian Shou JIN
Information and Control Engineering Institute, ZheJiang Guangsha College of Applied Construction Technology, Dongyang 322100*, P. R. China*
E-mail: *mathmaticajts@163.com*

Jian Quan GE[1)]
School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875*, P. R. China*
E-mail: *jqge@bnu.edu.cn*

Abstract Motivated by the theory of isoparametric hypersurfaces, we study submanifolds whose tubular hypersurfaces have some constant higher order mean curvatures. Here a k-th order mean curvature Q_k^ν ($k \geq 1$) of a submanifold M^n is defined as the k-th power sum of the principal curvatures, or equivalently, of the shape operator with respect to the unit normal vector ν. We show that if all nearby tubular hypersurfaces of M have some constant higher order mean curvatures, then the submanifold M itself has some constant higher order mean curvatures Q_k^ν independent of the choice of ν. Many identities involving higher order mean curvatures and Jacobi operators on such submanifolds are also obtained. In particular, we generalize several classical results in isoparametric theory given by E. Cartan, K. Nomizu, H. F. Münzner, Q. M. Wang, et al. As an application, we finally get a geometrical filtration for the focal submanifolds of isoparametric functions on a complete Riemannian manifold.

Keywords Isoparametric hypersurface, constant mean curvature, austere submanifold

MR(2010) Subject Classification 53C42, 53C24

1 Introduction

A hypersurface M^n of a Riemannian manifold N^{n+1} is called *isoparametric*, if M^n is locally a regular level set of a function f, so-called *isoparametric function*, with the property that both $\|\nabla f\|^2$ and Δf are constant on the level sets of f. One can show that M^n is an isoparametric hypersurface of N^{n+1} if and only if its nearby parallel hypersurfaces have constant mean curvature (for recent progress and applications see for example in [9, 11, 24, 29–33]).

The theory of isoparametric hypersurfaces originated from studies on hypersurfaces of constant principal curvatures in real space forms. On this topic, Cartan started a series of researches by proving the following characterization (cf. [5–8]):

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1) Corresponding author

Theorem ([5]) *A hypersurface in a real space form has constant principal curvatures if and only if its nearby parallel hypersurfaces have constant mean curvature.*

Therefore, a hypersurface in a real space form has constant principal curvatures if and only if it is isoparametric. This characterization does not hold in more general ambient spaces; see [18, 34] and [13] where counterexamples are given in complex projective spaces and complex hyperbolic spaces. However, under an additional assumption that it is a curvature-adapted hypersurface, this characterization also holds in a locally rank one symmetric space as showed in [18, Theorem 1.4]. Note that in a real space form every hypersurface is curvature-adapted and thus this assumption is superfluous. In this paper, by applying the Riccati equation and some algebraic geometry, we will give a generalization of this characterization for these two cases by using "higher order mean curvature" instead of (1st order) mean curvature; see Theorems 1.1 and 1.2 later. Here higher order mean curvatures will be defined by power sum polynomials of the principal curvatures other than elementary symmetric polynomials as usual.

Noticing that parallel hypersurfaces of a hypersurface M can be looked as half-tubular hypersurfaces of M, we turn to consider submanifolds whose tubular hypersurfaces have some constant higher order mean curvatures. Recall that in the classical theory of isoparametric hypersurfaces in unit spheres, Nomizu [28] showed that each compact isoparametric hypersurface is a tubular hypersurface of some (exactly two) submanifolds, namely focal submanifolds, and by using the constancy of the mean curvature of these tubular hypersurfaces, he proved that the focal submanifolds are minimal. Later, as a fundamental step in his remarkable work, Münzner [25] proved that these focal submanifolds have constant principal curvatures which implies the austerity[2)] and also the minimality of the focal submanifolds. Here we say that a submanifold of higher codimension has *constant principal curvatures*, if the set of the eigenvalues of the shape operator S_ν at any point is independent of the choices of the unit normal vector ν and the point of the submanifold. This is different from that in [2] where the principal curvatures are constant with respect to a (local) parallel normal vector field and thus may depend on the choices of unit normal vectors.

When the ambient space is a general complete Riemannian manifold N^{n+1} and f is a global isoparametric function on N, Wang [35] showed that (1) there are at most two singular level sets, namely the focal varieties of f, and they are submanifolds (both may be disconnected and of different dimensions[3)]) of N; (2) each regular level set (isoparametric hypersurface) of f is a tubular hypersurface around either of the focal varieties; (3) (claimed without proof) the focal varieties are minimal. Based on the structural results (1)–(2) for the focal varieties, Wang's claim (3) just asserts the minimality of submanifolds whose tubular hypersurfaces have constant (1st order) mean curvature, which generalizes Nomizu's result to arbitrary Riemannian manifolds (see a more general result of this form for compact submanifolds in [23]). However, Münzner's result mentioned above does not hold in this general case, but it indeed holds for submanifolds whose tubular hypersurfaces have constant principal curvatures (and thus each

2) A submanifold of a Riemannian manifold is called an *austere submanifold* in the sense of [20] if its principal curvatures in any normal direction occur as pairs of opposite signs.

3) Henceforth, a connected component of the focal varieties of an isoparametric function f on a complete Riemannian manifold N will be called a *focal submanifold* of f.

higher order mean curvature is constant); see a proof of this assertion and Wang's claim (3) in [17]. In this paper, we will study submanifolds whose tubular hypersurfaces have some constant higher order mean curvatures in a general Riemannian manifold. By some technical treatment for the Taylor expansion formulae of higher order mean curvatures of the tubular hypersurfaces, we will show that such submanifolds must have some higher order mean curvatures and some curvature invariants involving the Jacobi operator of the ambient space being constant, which in particular will generalize the results mentioned above given by [25, 28, 35] and [17]; see Theorem 1.4 later. As an application, we finally get a geometrical filtration for the focal submanifolds of isoparametric functions on a complete Riemannian manifold according to the filtration of isoparametric functions introduced by [18]; see Theorem 1.6 later.

To state the theorems explicitly, we have to set up some notations. First of all, as in [18], we denote by ρ_k (resp. σ_i) the k-th power sum polynomial (resp. the i-th elementary symmetric polynomial) in n variables for $k \geq 1$ and $\rho_0 \equiv n$ (resp. $1 \leq i \leq n$ and $\sigma_0 \equiv 1$). For an n by n real symmetric matrix (or self-dual operator) A with n real eigenvalues $(\mu_1, \dots, \mu_n) =: \mu$, we denote by $\rho_k(A) := \mathrm{tr}(A^k) = \rho_k(\mu)$ and $\sigma_i(A) = \sigma_i(\mu)$.

Let M^m be a submanifold of a Riemannian manifold N^{n+1}. For any unit normal vector $\nu \in \mathcal{V}_1 M$ (unit normal bundle of M), denote by S_ν the shape operator of M^m in direction ν. Then for any $k \geq 1$, we define the *k-th order mean curvature* Q_k^ν in direction ν by the k-th power sum polynomial of the shape operator other than the k-th elementary symmetric polynomial as usual, i.e.,

$$Q_k^\nu := \rho_k(S_\nu) = \mathrm{tr}((S_\nu)^k).$$

When M is a hypersurface and ν is a fixed global unit normal vector field, we simply write the k-th order mean curvature Q_k^ν by Q_k. Recall that in [18], we introduced the following notions: For $1 \leq k \leq n$, a non-constant smooth function f on a Riemannian manifold N^{n+1} is called *k-isoparametric*, if $\|\nabla f\|^2$ and $\rho_1(H_f), \rho_2(H_f), \dots, \rho_k(H_f)$ are constant on the level sets of f, [4)] where H_f is the Hessian of f on N^{n+1}; a hypersurface M^n of N^{n+1} is called *k-isoparametric*, if M^n is locally a regular level set of a k-isoparametric function on N^{n+1}; an n-isoparametric function (hypersurface) on N^{n+1} is also called a *totally isoparametric* function (hypersurface). Note that 1-isoparametric functions (hypersurfaces) are just isoparametric functions (hypersurfaces). It was proved there that M^n is a k-isoparametric hypersurface if and only if its nearby parallel hypersurfaces have constant higher order mean curvatures $Q_1, Q_2, \dots, Q_k$. Therefore, the sets of 1-, 2-, …, n-isoparametric functions (hypersurfaces) give a filtration for isoparametric functions (hypersurfaces) on a Riemannian manifold N^{n+1} with the filtered geometrical property that 1-isoparametric hypersurfaces have constant (1st order) mean curvature, 2-isoparametric hypersurfaces have constant 1st and 2nd order mean curvatures, and so on, finally, n-isoparametric hypersurfaces have constant principal curvatures.

Let R be the Riemannian curvature tensor and ∇ the Levi–Civita connection of N^{n+1}. In this paper, we use the curvature convention as the following: for any tangent vectors (vector

4) In [18], we assumed some smoothness of these functions as one-parameter functions of f for some regularity reasons; also see a note given in [17]. Anyway, without confusion, we emphasize the geometrical meaning behind the algebraic definitions.

fields) X, Y, Z, W of N^{n+1},

$$R(X,Y,Z) = R_{XY}Z := (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y])Z,$$

and then the covariant derivative ∇R is also a tensor field and can be written as:

$$\begin{aligned}(\nabla R)(X,Y,Z,W) &= (\nabla_W R)(X,Y,Z)\\ &:= \nabla_W(R_{XY}Z) - R_{(\nabla_W X)Y}Z - R_{X(\nabla_W Y)}Z - R_{XY}(\nabla_W Z).\end{aligned}$$

For any tangent vector $\xi \in \mathcal{T}N$, the *Jacobi operator* $K_\xi : \mathcal{T}N \to \mathcal{T}N$ of N^{n+1} in direction ξ is defined by

$$K_\xi(X) := R_{\xi X}\xi \quad \text{for } X \in \mathcal{T}N. \tag{1.1}$$

Note that by properties of the Riemannian curvature tensor, K_ξ is a self-dual linear operator, $\mathrm{tr}(K_\xi) = \mathrm{Ric}(\xi)$ is just the Ricci curvature in direction ξ, and $K_\xi = K_{-\xi}$, $K_\xi(\xi) \equiv 0$. Then without confusion, we will use the same symbol K_ξ when the Jacobi operator is looked as a self-dual operator on the subspace $\xi^\perp$ normal to ξ in $\mathcal{T}N$. Recall that a submanifold M^m of N^{n+1} is called *curvature-adapted* (or *compatible*), if the direct sum $S_\nu \oplus I_{n-m}$ of the shape operator S_ν and the identity map commutes with the Jacobi operator K_ν, or equivalently, these two self-dual operators are simultaneously diagonalizable, for any unit normal vector ν of M (cf. [1, 19]).

Corresponding to the decomposition of the tangent bundle $\mathcal{T}N$ on M^m, we would like to decompose the Jacobi operator K_ν ($\nu \in \mathcal{V}_1M$) into two self-dual linear operators, say *tangent Jacobi operator* $K_\nu^\top : \mathcal{T}M \to \mathcal{T}M$ and *vertical Jacobi operator* $K_\nu^\perp : \mathcal{V}M \to \mathcal{V}M$ as the following:

$$\begin{aligned}K_\nu^\top(X) &:= \text{projection to } \mathcal{T}M \text{ of } K_\nu(X) \quad \text{for } X \in \mathcal{T}M,\\ K_\nu^\perp(\eta) &:= \text{projection to } \mathcal{V}M \textit{ of } K_\nu(\eta) \quad \text{for } \eta \in \mathcal{V}M.\end{aligned}$$

Obviously, $K_\nu^\top$, $K_\nu^\perp$ are self-dual linear operators and $K_\nu^\top = K_{-\nu}^\top$, $K_\nu^\perp = K_{-\nu}^\perp$, $K_\nu^\perp(\nu) \equiv 0$. Without confusion, we denote by the same symbol $K_\nu^\perp$ when the vertical Jacobi operator $K_\nu^\perp$ is restricted to the subspace $\nu^\perp \cap \mathcal{V}M$ in $\mathcal{V}M$. Then under any orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ of $\mathcal{T}N$ on M with $e_1, \ldots, e_m$ tangent to M and $e_{m+1}, \ldots, e_n, e_{n+1} = \nu$ normal to M, the Jacobi operator K_ν can be expressed as the following symmetric matrix

$$K_\nu = \left(\begin{array}{cc|c} K_\nu^\top & B_\nu & 0 \\ B_\nu^t & K_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{1.2}$$

where $K_\nu^\top$ and $K_\nu^\perp$ are the matrix expressions of the tangent and (restricted) vertical Jacobi operators, B_ν is an m by $(n-m)$ matrix with the property that $B_\nu = B_{-\nu}$.

Finally, for any tangent vector (field) $\xi \in \mathcal{T}N$, we need to introduce another self-dual linear operator, say *covariant Jacobi operator* $\mathcal{K}_\xi : \mathcal{T}N \to \mathcal{T}N$, from covariant derivative of the Riemannian curvature tensor R as the following:

$$\mathcal{K}_\xi(X) := (\nabla R)(\xi, X, \xi, \xi) = (\nabla_\xi R)(\xi, X, \xi) \quad \text{for } X \in \mathcal{T}N. \tag{1.3}$$

Note that when $\nabla_\xi \xi = 0$, $\mathcal{K}_\xi = \nabla_\xi K_\xi$ is just the covariant derivative of the Jacobi operator K_ξ in direction ξ. By properties of the Riemannian curvature tensor and its covariant derivative, it is easily seen that $\mathcal{K}_\xi$ is a self-dual linear operator and $\mathcal{K}_{-\xi} = -\mathcal{K}_\xi$, $\mathcal{K}_\xi(\xi) \equiv 0$. In the same way as the decomposition (1.2) of the Jacobi operator K_ν ($\nu \in \mathcal{V}_1 M$), we also decompose the covariant Jacobi operator $\mathcal{K}_\nu$ into two self dual operators, say *covariant tangent Jacobi operator* $\mathcal{K}_\nu^\top : \mathcal{T}M \to \mathcal{T}M$ and *covariant vertical Jacobi operator* $\mathcal{K}_\nu^\perp : \mathcal{V}M \to \mathcal{V}M$, as the following:

$$\mathcal{K}_\nu^\top(X) := \text{projection to } \mathcal{T}M \text{ of } \mathcal{K}_\nu(X) \quad \text{for } X \in \mathcal{T}M,$$
$$\mathcal{K}_\nu^\perp(\eta) := \text{projection to } \mathcal{V}M \text{ of } \mathcal{K}_\nu(\eta) \quad \text{for } \eta \in \mathcal{V}M.$$

Obviously, $\mathcal{K}_\nu^\top$, $\mathcal{K}_\nu^\perp$ are self-dual linear operators and $\mathcal{K}_{-\nu}^\top = -\mathcal{K}_\nu^\top$, $\mathcal{K}_{-\nu}^\perp = -\mathcal{K}_\nu^\perp$, $\mathcal{K}_\nu^\perp(\nu) \equiv 0$. Without confusion, we denote by the same symbol $\mathcal{K}_\nu^\perp$ when the covariant vertical Jacobi operator $\mathcal{K}_\nu^\perp$ is restricted to the subspace $\nu^\perp \cap \mathcal{V}M$ in $\mathcal{V}M$. Under the same orthonormal frame as in (1.2), the covariant Jacobi operator can be expressed as the following symmetric matrix

$$\mathcal{K}_\nu = \left(\begin{array}{cc|c} \mathcal{K}_\nu^\top & \mathcal{B}_\nu & 0 \\ \mathcal{B}_\nu^t & \mathcal{K}_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{1.4}$$

where $\mathcal{K}_\nu^\top$ and $\mathcal{K}_\nu^\perp$ are the matrix expressions of the covariant tangent and (restricted) covariant vertical Jacobi operators, $\mathcal{B}_\nu$ is an m by $(n-m)$ matrix with the property that $\mathcal{B}_{-\nu} = -\mathcal{B}_\nu$.

Now we are ready to state the theorems. Firstly, for the hypersurface case, by applying the Riccati equation and some algebraic geometry, we obtain the following generalizations of Cartan's theorem (see [5]) and Theorem 1.4 of [18], respectively.

Theorem 1.1 *A hypersurface in a real space form has constant principal curvatures if and only if for some $k \geq 1$, its nearby parallel hypersurfaces have constant k-th order mean curvature Q_k.*

Theorem 1.2 *A curvature-adapted hypersurface in a locally rank one symmetric space has constant principal curvatures if and only if for some $k \geq 1$, its nearby parallel hypersurfaces have constant k-th order mean curvature Q_k.*

Remark 1.3 In these cases, the hypersurface is totally isoparametric. On the other hand, it is still unknown whether a hypersurface of constant principal curvatures in a locally rank one symmetric space other than real space forms is totally isoparametric.

For general submanifolds, by some technical treatment for the Taylor expansion formulae of higher order mean curvatures of the tubular hypersurfaces, we obtain the following generalizations of those results on geometry of the focal submanifolds in the theory of isoparametric hypersurfaces given by [25, 28, 35] and [17].

Theorem 1.4 *Let M^m be a submanifold of a Riemannian manifold N^{n+1}. Suppose that on any nearby tubular hypersurface M_t^n of M^m in N^{n+1} ($t \in (0, \varepsilon)$),*

(a) *for $1 \leq l \leq 4$, the l-th, $(l+1)$-th, …, $(l+[\frac{l}{2}])$-th order mean curvatures $Q_l, Q_{l+1}, \ldots, Q_{l+[\frac{l}{2}]}$ are constant*;

(b) *for $l \geq 5$, the l-th, $(l+1)$-th, …, $(l+[\frac{l}{2}]+1)$-th order mean curvatures $Q_l, Q_{l+1}, \ldots, Q_{l+[\frac{l}{2}]+1}$ are constant.*

Then the l-th order mean curvature Q_l^ν in any direction ν is a constant independent of the choices of the unit normal vector ν and the point of M^m, and so are the curvature invariants:

$\rho_{[\frac{l}{2}]}(K_\nu^\perp)$ *when* l *is even; and* $\mathrm{tr}((K_\nu^\perp)^{[\frac{l}{2}]-1}\mathcal{K}_\nu^\perp)$ *when* $l \geq 3$ *is odd. In particular, when* l *is odd,* $Q_l^\nu \equiv 0$ *and* $\mathrm{tr}((K_\nu^\perp)^{[\frac{l}{2}]-1}\mathcal{K}_\nu^\perp) \equiv 0$*. Furthermore, if in addition we assume the constancy of* Q_{l-1} *in* (a) *and* (b) *for* $l \geq 2$*, then we have a new constant curvature invariant* $\mathrm{tr}((S_\nu)^{l-2}K_\nu^\top)$ *which also vanishes when* l *is odd.*

Remark 1.5 As indicated by the theorem, if we assume more constant higher order mean curvatures on tubular hypersurfaces, we would possibly get more constant curvature invariants such as $\mathrm{tr}((K_\nu^\perp)^i(\mathcal{K}_\nu^\perp)^j)$, $\mathrm{tr}((S_\nu)^i(K_\nu^\top)^j)$ and even $\mathrm{tr}((S_\nu)^i(\mathcal{K}_\nu^\top)^j)$ on the submanifold, though the computations would be rather complicated. See Theorem 4.4 for a more detailed description.

At last, we conclude this section by the following geometrical filtration for the focal submanifolds of isoparametric functions on a complete Riemannian manifold according to the filtration of isoparametric functions by 1-, 2-, …, n-isoparametric functions.

Theorem 1.6 *Let* M^m *be a focal submanifold of an isoparametric function* f *on a complete Riemannian manifold* N^{n+1}*. Suppose* f *is* k*-isoparametric,* $1 \leq k \leq n$*.*

(i) *If* $k = 1$*, then for any unit normal vector* ν *on* M*,*

$$Q_1^\nu \equiv 0, \quad Q_2^\nu + \mathrm{tr}(K_\nu^\top) + \frac{1}{3}\mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const},$$
$$Q_3^\nu + \mathrm{tr}(S_\nu K_\nu^\top) + \frac{1}{2}\mathrm{tr}(\mathcal{K}_\nu^\top) + \frac{1}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

in particular, M *is a minimal submanifold in* N;

(ii) *If* $k = 2$*, then besides the identities in* (i)*, we have further*

$$Q_2^\nu - \frac{2}{3}\mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const}, \quad Q_3^\nu + \mathrm{tr}(S_\nu K_\nu^\top) - \frac{1}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

and thus $\mathrm{tr}(K_\nu) = \mathrm{Ric}(\nu) \equiv \mathrm{Const}, \mathrm{tr}(\mathcal{K}_\nu) \equiv 0$;

(iii) *If* $k = 3$*, then besides the identities in* (i)*–*(ii)*, we have further*

$$\mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const}, \quad Q_3^\nu + \frac{3}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

and thus $Q_2^\nu \equiv \mathrm{Const}, \mathrm{tr}(S_\nu K_\nu^\top) - \mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0$;

(iv) *If* $k = 4$*, then besides the identities in* (i)*–*(iii)*, we have further*

$$\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

and thus $Q_3^\nu \equiv 0, \mathrm{tr}(S_\nu K_\nu^\top) \equiv 0$;

(v) *If* $k = 5$*, then besides the identities in* (i)*–*(iv)*, we have further*

$$Q_4^\nu - \frac{2}{9}\rho_2(K_\nu^\perp) \equiv \mathrm{Const}, \quad 3\mathrm{tr}(S_\nu^2 K_\nu^\top) + \rho_2(K_\nu^\perp) \equiv \mathrm{Const};$$

(vi) *If* $k = 6$*, then besides the identities in* (i)*–*(v)*, we have further*

$$\rho_2(K_\nu^\perp) \equiv \mathrm{Const}, \quad 2\mathrm{tr}(S_\nu^3 K_\nu^\top) + 3Q_5^\nu + \frac{1}{12}\mathrm{tr}(K_\nu^\perp \mathcal{K}_\nu^\perp) \equiv 0;$$

and thus $Q_4^\nu \equiv \mathrm{Const}, \mathrm{tr}(S_\nu^2 K_\nu^\top) \equiv \mathrm{Const}$;

(vii) *If* $k = 3d+1, d \geq 2$*, then besides the identities for* $k \leq 3d$*, we have further*

$$Q_{2d}^\nu \equiv \mathrm{Const}, \quad \rho_d(K_\nu^\perp) \equiv \mathrm{Const}, \quad \mathrm{tr}(S_\nu^{2d-2}K_\nu^\top) \equiv \mathrm{Const},$$
$$Q_{2d+1}^\nu - d\, 3^{-d+1}4^{-1}\mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp) \equiv 0,$$

$$(2d)\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top) + (3d+1)Q_{2d+1}^\nu \equiv 0;$$

(viii) *If $k = 3d+2, d \geq 2$, then besides the identities in* (i)–(vii), *we have further*

$$Q_{2d+1}^\nu \equiv 0, \quad \mathrm{tr}(S_\nu^{2d-1}K_\nu^\top) \equiv 0, \quad \mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp) \equiv 0,$$
$$(2d+1)\mathrm{tr}(S_\nu^{2d}K_\nu^\top) + (3d+2)Q_{2d+2}^\nu - 3^{-d-1}\rho_{d+1}(K_\nu^\perp) \equiv \mathrm{Const};$$

(ix) *If $k = 3d+3, d \geq 2$, then besides the identities in* (i)–(viii), *we have further*

$$Q_{2d+2}^\nu + 3^{-d-1}\rho_{d+1}(K_\nu^\perp) \equiv \mathrm{Const};$$
$$2\mathrm{tr}(S_\nu^{2d+1}K_\nu^\top) + 3Q_{2d+3}^\nu + 3^{-d}4^{-1}\mathrm{tr}((K_\nu^\perp)^d\mathcal{K}_\nu^\perp) \equiv 0.$$

Furthermore,

(a) *if $m = 0$, i.e., M is a point, then the Ricci curvature of N is constant on M*;

(b) *if $m = n$, i.e., M is a hypersurface, then M is a k-isoparametric hypersurface with $Q_1 = Q_3 = \cdots = Q_{2j+1} \equiv 0$ for $2j+1 \leq k$;*

(c) *if $m \leq [\frac{2k+1}{3}]$ for $k \leq 6$, or $m \leq [\frac{2k-1}{3}]$ for $k \geq 7$, or $k = n$, then M is an austere submanifold of constant principal curvatures in N, if in addition $m = 2,\ n \geq 4$; or $m = 3,\ n \geq 5$; or $m = 4,\ n \geq 10$, then M is a totally geodesic submanifold*;

(d) *if $m \geq n - [\frac{k}{3}]$ for $k \leq 6$, or $m \geq n - [\frac{k-1}{3}]$ for $k \geq 7$, or $k = n$, then the vertical Jacobi operator $K_\nu^\perp$ has constant eigenvalues independent of the choices of the unit normal vector ν and the point of M, or equivalently, the restriction of the Riemannian curvature model $(\mathcal{T}N, R)$ of N to the normal bundle of M is an Osserman curvature model.*

Remark 1.7 Recall that [15] introduced $(2j)$-th mean curvature function K_{2j} and $(2j+1)$-th mean curvature vector field H_{2j+1} on a submanifold M^m of a Riemannian manifold N^{n+1} which generalize higher order mean curvature functions on a hypersurface, and showed that K_{2j} (resp. H_{2j+1}) equals, up to a constant factor, the integral of $\sigma_{2j}(S_\nu)$ (resp. $\nu\sigma_{2j+1}(S_\nu)$) over the unit normal sphere of M in N. Consequently, by Newton's identities, $K_l \equiv \mathrm{Const}$ (l even) and $H_l \equiv 0$ (l odd) on the focal submanifold M^m of a k-isoparametric function with $l \leq [\frac{2k+1}{3}]$ for $k \leq 6$, or $l \leq [\frac{2k-1}{3}]$ for $k \geq 7$.

Remark 1.8 An isoparametric function is called a *properly isoparametric* function if the focal submanifolds have codimension greater than 1 (cf. [16]). So f in case (b) is not properly isoparametric and since in this case the focal submanifolds or their normal line bundles could be non-orientable, there may be no global unit normal vector fields, in which case the conclusion in (b) should be considered as local property on M. Note that classically isoparametric hypersurfaces in unit spheres are assumed to be connected and thus the focal submanifolds have codimension greater than one.

Remark 1.9 It was proved by Chi [10] and Nikolayevsky [26, 27] that an Osserman curvature model of dimension $q \neq 16$ is isomorphic to one of the curvature models given by Clifford module structures (see a detailed introduction in [3]). In particular, if q is odd, then the Jacobi operator of an Osserman curvature model has only one constant eigenvalue except the trivial eigenvalue 0. So if $n - m$ is even in (d), then the restricted vertical Jacobi operator $K_\nu^\perp \equiv \mathrm{Const}\cdot\mathrm{id}$ and thus the sectional curvatures of N in normal planes of M are constant.

2 Shape Operators of Tubular Hypersurfaces

In this section, by using the Fermi coordinates, we will mainly derive a Taylor expansion formula up to order 2 about t of the shape operator $S(t)$ of the tubular hypersurface M_t^n of radius $t \in (0, \varepsilon)$ around a submanifold M^m. This Taylor expansion formula up to order 1 has been given in [17].

Let M^m be a submanifold of a Riemannian manifold N^{n+1} and M_t^n be the tubular hypersurface around M of sufficiently small radius $t \in (0, \varepsilon)$. Then the "outward" unit normal vector field ν_t of M_t for $t \in (0, \varepsilon)$ forms a unit vector field, say ξ, on an open subset $\mathcal{N}_\varepsilon M := \bigcup_{t \in (0,\varepsilon)} M_t$ of N^{n+1}. The shape operator $S(t)$ of M_t with respect to ν_t at a point $q \in M_t$ is just the restriction to M_t of the tensorial operator $S : \mathcal{T}(\mathcal{N}_\varepsilon M) \to \mathcal{T}(\mathcal{N}_\varepsilon M)$ defined by

$$S(X) := -\nabla_X \xi \tag{2.1}$$

for $X \in \mathcal{T}_q(\mathcal{N}_\varepsilon M)$, where ∇ denotes the covariant derivative in N. It is easily seen that S is self-dual and $S(\xi) = 0$. Taking covariant derivative of S with respect to ξ gives the well-known Riccati equation (cf. [19]):

$$\nabla_\xi S = S^2 + K_\xi,$$

and its restriction to M_t can be written as

$$S'(t) = S(t)^2 + R(t), \tag{2.2}$$

where $S'(t) := (\nabla_\xi S)|_{\mathcal{T}M_t} = (\nabla_{\nu_t} S)|_{\mathcal{T}M_t}$, K_ξ is the Jacobi operator of N in direction ξ defined in (1.1) and $R(t) := K_\xi|_{\mathcal{T}M_t} = K_{\nu_t}|_{\mathcal{T}M_t}$.

Now we choose a system of Fermi coordinates in a neighborhood $\widetilde{\mathcal{U}}$ of any point $p \in M$ in N as follows (cf. [17]). First we choose normal geodesic coordinates $(y_1, \ldots, y_m)$ centered at p in a neighborhood $\mathcal{U}$ of p in M. Then in $\mathcal{U}$ we fix orthonormal sections $E_{m+1}, \ldots, E_n, E_{n+1}$ of the normal bundle $\mathcal{V}M$ of M in N such that they are parallel with respect to the normal connection along any geodesic ray from p in M and $E_{n+1}|_p = \nu$ for a given unit normal vector ν of M at p. The Fermi coordinates $(x_1, \ldots, x_n, x_{n+1})$ of $(\mathcal{U} \subset M \subset)\ \widetilde{\mathcal{U}} \subset N$ centered at p are defined by

$$x_a\left(\exp_q\left(\sum_{j=m+1}^{n+1} t_j E_j(q)\right)\right) = y_a(q), \quad a = 1, \ldots, m,$$

$$x_i\left(\exp_q\left(\sum_{j=m+1}^{n+1} t_j E_j(q)\right)\right) = t_i, \quad i = m+1, \ldots, n+1$$

for $q \in \mathcal{U}$ and any sufficiently small numbers $t_{m+1}, \ldots, t_{n+1}$ with $\sum_i t_i^2 < \varepsilon^2$. Then the *Generalized Gauss Lemma* shows that in $\widetilde{\mathcal{U}} - M \subset \mathcal{N}_\varepsilon M$,

$$\xi = \sum_{i=m+1}^{n} \frac{x_i}{\sigma} \partial x_i = \nabla \sigma, \tag{2.3}$$

where $\sigma := \sqrt{\sum_i x_i^2}$ is the distance function to M in $\widetilde{\mathcal{U}}$ (cf. [19]). It follows from the definition that along the normal geodesic $\eta_\nu(t) := \exp_p(t\nu)$ in $\widetilde{\mathcal{U}}$,

$$\partial x_{n+1}|_{\eta_\nu(t)} = \eta_\nu'(t) = \nu_t = \xi|_{\eta_v(t)} \quad \text{for } t \in (0, \varepsilon). \tag{2.4}$$

Moreover, the coordinate vector fields $\partial x_1, \ldots, \partial x_{n+1}$ satisfy

$$\begin{aligned} &\nabla_{\partial x_a} \partial x_b|_p \in \mathcal{V}_p M, \quad \nabla_{\partial x_a} \partial x_i|_p \in \mathcal{T}_p M, \quad \nabla_{\partial x_i} \partial x_j|_{\mathcal{U}} = 0, \\ &\langle \partial x_\alpha, \partial x_\beta \rangle|_p = \delta_{\alpha\beta}, \quad \langle \partial x_a, \partial x_i \rangle|_{\mathcal{U}} = 0, \quad \langle \partial x_i, \partial x_j \rangle|_{\mathcal{U}} = \delta_{ij}, \end{aligned} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the metric, and the indices convention is that indices $a, b, \ldots \in \{1, \ldots, m\}$, indices $i, j, \ldots \in \{m+1, \ldots, n+1\}$ and indices $\alpha, \beta, \ldots \in \{1, \ldots, n+1\}$. Then $\partial x_1|_p, \ldots, \partial x_m|_p$ form an orthonormal frame of $\mathcal{T}_p M$ and $\partial x_{m+1}|_p, \ldots, \partial x_{n+1}|_p = \nu$ form an orthonormal frame of $\mathcal{V}_p M$, and under these frames, the Jacobi operator K_ν, the covariant Jacobi operator $\mathcal{K}_\nu$ of N can be written as real symmetric matrices as (1.2) and (1.4) respectively.

Now in $\widetilde{\mathcal{U}} - M \subset \mathcal{N}_\varepsilon M$, we express the self-dual operator S defined by (2.1) as a real matrix $S = (S_{\alpha\beta})$ (not symmetric in general) of order $n+1$ under the coordinate vector fields $\partial x_1, \ldots, \partial x_{n+1}$, i.e., $S(\partial x_\alpha) := \sum_{\beta=1}^{n+1} S_{\alpha\beta} \partial x_\beta$. By properties of the Fermi coordinates, in [17] we obtained the following expansion formula of S.

Proposition 2.1 ([17]) *With notations as above, at the point $\eta_\nu(t) = \exp_p(t\nu) \in M_t$ for any $t \in (0, \varepsilon)$, the following expansion formula holds*

$$S = \begin{pmatrix} S_\nu + t(S_\nu^2 + K_\nu^\top) + \mathcal{O}(t^2) & tB_\nu + \mathcal{O}(t^2) & \mathcal{O}(t^2) \\ \frac{t}{3} B_\nu^t + \mathcal{O}(t^2) & -\frac{1}{t} I + \frac{t}{3} K_\nu^\perp + \mathcal{O}(t^2) & \mathcal{O}(t^2) \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

where $S_\nu := (h_{ab}^\nu)$ is the matrix of the shape operator of M in direction ν under the orthonormal frame $\partial x_1|_p, \ldots, \partial x_m|_p$, $\mathcal{O}(t^2)$ denotes matrices with elements of t's order not less than 2.

Rewrite the expansion formula (2.6) by power series about t as:

$$S = \sum_{r=0}^{\infty} t^{r-1} S_r = \frac{1}{t} S_0 + S_1 + tS_2 + t^2 S_3 + \mathcal{O}(t^3), \tag{2.7}$$

where

$$S_0 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -I & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad S_1 = \left(\begin{array}{cc|c} S_\nu & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad S_2 = \left(\begin{array}{cc|c} S_\nu^2 + K_\nu^\top & B_\nu & 0 \\ \frac{1}{3} B_\nu^t & \frac{1}{3} K_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

and $S_r, r \geq 3$, are matrices independent of t.

To calculate the coefficient matrix S_3 of t^2 for this expansion formula, we need the following lemmas.

Lemma 2.2 ([17, 21]) *Let $g_{\alpha\beta} := \langle \partial x_\alpha, \partial x_\beta \rangle$ and $G := (g_{\alpha\beta})$ be the matrix of the metric. Then at the point $\eta_\nu(t) = \exp_p(t\nu) \in M_t$, we have*

$$\begin{aligned} g_{ab}(t) &= \delta_{ab} - 2h_{ab}^\nu t + \Big(\sum_c h_{ac}^\nu h_{cb}^\nu - \langle R_{\nu \partial x_a} \nu, \partial x_b \rangle \Big) t^2 + O(t^3), \\ g_{ai}(t) &= -\frac{2}{3} \langle R_{\nu \partial x_a} \nu, \partial x_i \rangle t^2 + O(t^3), \\ g_{ij}(t) &= \delta_{ij} - \frac{1}{3} \langle R_{\nu \partial x_i} \nu, \partial x_j \rangle t^2 + O(t^3), \end{aligned}$$

or in matrix form,

$$G(t)=I+t\left(\begin{array}{cc|c}-2S_\nu & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right)+t^2\left(\begin{array}{cc|c}S_\nu^2-K_\nu^\top & -\frac{2}{3}B_\nu & 0\\ -\frac{2}{3}B_\nu^t & -\frac{1}{3}K_\nu^\perp & 0\\ \hline 0 & 0 & 0\end{array}\right)+\mathcal{O}(t^3).$$

Lemma 2.3 ([17]) *Put* $\nabla_{\partial x_\alpha}\partial x_{n+1}:=\sum_\beta w_{\alpha\beta}\partial x_\beta$ *and* $W:=(w_{\alpha\beta})$. *Then at the point* $\eta_\nu(t)=\exp_p(t\nu)\in M_t$, *we have*

$$\begin{aligned}\nabla_{\partial x_a}\partial x_{n+1}&=-\sum_b h_{ab}^\nu\partial x_b-t\sum_b\Big(\sum_c h_{ac}^\nu h_{cb}^\nu+\langle R_{\nu\partial x_a}\nu,\partial x_b\rangle\Big)\partial x_b\\ &\quad -t\sum_k\langle R_{\nu\partial x_a}\nu,\partial x_k\rangle\partial x_k+\sum_\alpha O(t^2)_\alpha\partial x_\alpha,\\ \nabla_{\partial x_i}\partial x_{n+1}&=-\frac{t}{3}\sum_b\langle R_{\nu\partial x_i}\nu,\partial x_b\rangle\partial x_b-\frac{t}{3}\sum_k\langle R_{\nu\partial x_i}\nu,\partial x_k\rangle\partial x_k+\sum_\alpha O(t^2)_\alpha\partial x_\alpha,\end{aligned}$$

or in matrix form,

$$W(t)=\left(\begin{array}{cc|c}-S_\nu & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right)+t\left(\begin{array}{cc|c}-S_\nu^2-K_\nu^\top & -B_\nu & 0\\ -\frac{1}{3}B_\nu^t & -\frac{1}{3}K_\nu^\perp & 0\\ \hline 0 & 0 & 0\end{array}\right)+\mathcal{O}(t^2).$$

Lemma 2.4 *At the point* $p\in M$, *we have*

$$\begin{aligned}\nabla_\nu\nabla_{\partial x_{n+1}}\nabla_{\partial x_{n+1}}\partial x_a&=-\nabla_\nu(R_{\partial x_{n+1}\partial x_a}\partial x_{n+1})=-\mathcal{K}_\nu(\partial x_a)-K_\nu(\nabla_\nu\partial x_a);\\ \nabla_\nu\nabla_{\partial x_{n+1}}\nabla_{\partial x_{n+1}}\partial x_i&=-\frac{1}{2}\nabla_\nu(R_{\partial x_{n+1}\partial x_i}\partial x_{n+1})=-\frac{1}{2}\mathcal{K}_\nu(\partial x_i);\\ \nabla_\nu\nabla_{\partial x_{n+1}}\partial x_a&=-R_{\nu\partial x_a}\partial x_{n+1}=-K_\nu(\partial x_a);\\ \nabla_\nu\nabla_{\partial x_{n+1}}\partial x_i&=-\frac{1}{3}R_{\nu\partial x_i}\partial x_{n+1}=-\frac{1}{3}K_\nu(\partial x_i).\end{aligned}$$

Proof It follows from (2.4) that along the geodesic $\eta_\nu(t)=\exp_p(t\nu)$, $\nabla_{\partial x_{n+1}}\partial x_{n+1}=\nabla_\xi\xi=0$ and thus $\mathcal{K}_\xi=\nabla_\xi K_\xi$ by definition (1.3). Then the first equalities in the identities follow from Lemmas 9.19 and 9.20 in [19] and the second equalities follow immediately from (1.1), (1.3) and (2.5). □

Lemma 2.5 *Let* $W(t):=\sum_{r=0}^\infty t^rW_r$ *be the matrix in Lemma* 2.3 *with* W_r*s independent of* t. *Then*

$$W_2=\left(\begin{array}{cc|c}-\frac{1}{2}\mathcal{K}_\nu^\top-S_\nu^3-K_\nu^\top S_\nu & -\frac{1}{2}\mathcal{B}_\nu & 0\\ -\frac{1}{4}\mathcal{B}_\nu^t-\frac{1}{3}B_\nu^tS_\nu & -\frac{1}{4}\mathcal{K}_\nu^\perp & 0\\ \hline 0 & 0 & 0\end{array}\right).$$

Proof Put $u_{\alpha\beta}:=\langle\nabla_{\partial x_\alpha}\partial x_{n+1},\partial x_\beta\rangle$ and $U(t):=(u_{\alpha\beta})|_{\eta_\nu(t)}$. Then $U(t)=W(t)G(t)$, and thus

$$U''(0)=W''(0)G(0)+2W'(0)G'(0)+W(0)G''(0),$$

where $G(t)$ is the matrix in Lemma 2.2 and so $G(0) = I$,

$$G'(0) = \left(\begin{array}{cc|c} -2S_\nu & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad G''(0) = 2\left(\begin{array}{cc|c} S_\nu^2 - K_\nu^\top & -\frac{2}{3}B_\nu & 0 \\ -\frac{2}{3}B_\nu^t & -\frac{1}{3}K_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{2.8}$$

and by Lemma 2.3,

$$W(0) = \left(\begin{array}{cc|c} -S_\nu & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad W'(0) = \left(\begin{array}{cc|c} -S_\nu^2 - K_\nu^\top & -B_\nu & 0 \\ -\frac{1}{3}B_\nu^t & -\frac{1}{3}K_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{2.9}$$

On the other hand,

$$\begin{aligned} u''_{\alpha\beta}(0) &= \nu\nu_t\langle \nabla_{\partial x_\alpha}\partial x_{n+1}, \partial x_\beta\rangle = \nu\partial x_{n+1}\langle \nabla_{\partial x_\alpha}\partial x_{n+1}, \partial x_\beta\rangle \\ &= \langle \nabla_\nu \partial x_\alpha, \nabla_\nu\nabla_{\partial x_{n+1}}\partial x_\beta\rangle + 2\langle \nabla_\nu\nabla_{\partial x_{n+1}}\partial x_\alpha, \nabla_\nu\partial x_\beta\rangle \\ &\quad + \langle \nabla_\nu\nabla_{\partial x_{n+1}}\nabla_{\partial x_{n+1}}\partial x_\alpha, \partial x_\beta\rangle, \end{aligned}$$

then by Lemma 2.4, we can get

$$U''(0) = \left(\begin{array}{cc|c} -\mathcal{K}_\nu^\top + 2K_\nu^\top S_\nu + 2S_\nu K_\nu^\top & -\mathcal{B}_\nu + \frac{4}{3}S_\nu B_\nu & 0 \\ -\frac{1}{2}\mathcal{B}_\nu^t + \frac{2}{3}B_\nu^t S_\nu & -\frac{1}{2}\mathcal{K}_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Combining the above formulae, we can get the required formula for W_2 by the following

$$W_2 = \frac{1}{2}W''(0) = \frac{1}{2}(U''(0) - 2W'(0)G'(0) - W(0)G''(0)). \qquad \square$$

Corollary 2.6 *Let S_3 be the coefficient matrix of t^2 in* (2.7)*. Then*

$$S_3 = \left(\begin{array}{cc|c} \frac{1}{2}\mathcal{K}_\nu^\top + S_\nu^3 + K_\nu^\top S_\nu & \frac{1}{2}\mathcal{B}_\nu & 0 \\ \frac{1}{4}\mathcal{B}_\nu^t + \frac{1}{3}B_\nu^t S_\nu & \frac{1}{4}\mathcal{K}_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{2.10}$$

Proof It follows from (2.1), (2.3) and (2.4) that at the point $\eta_\nu(t) = \exp_p(t\nu) \in M_t$,

$$-S(\partial x_a) = \nabla_{\partial x_a}\xi = \nabla_{\partial x_a}\Big(\sum_j \frac{x_j}{\sigma}\partial x_j\Big) = \sum_j \frac{x_j}{\sigma}\nabla_{\partial x_a}\partial x_j = \nabla_{\partial x_a}\partial x_{n+1};$$

$$\begin{aligned} -S(\partial x_i) &= \nabla_{\partial x_i}\xi = \nabla_{\partial x_i}\Big(\sum_j \frac{x_j}{\sigma}\partial x_j\Big) = \sum_j \partial x_i\Big(\frac{x_j}{\sigma}\Big)\partial x_j + \sum_j \frac{x_j}{\sigma}\nabla_{\partial x_i}\partial x_j \\ &= \frac{1}{t}\partial x_i - \frac{1}{t}\delta_{i\ n+1}\ \partial x_{n+1} + \nabla_{\partial x_i}\partial x_{n+1}, \end{aligned}$$

which, together with Lemmas 2.3 and 2.5, gives the required formula for S_3 immediately. $\square$

Finally we conclude this section by the following expansion formula for the Jacobi operator K_ξ.

Corollary 2.7 *At the point $\eta_\nu(t) = \exp_p(t\nu) \in M_t$, we have*

$$K_\xi = K_{\nu_t} = K_\nu + t\left(\begin{array}{cc|c} \mathcal{K}_\nu^\top - S_\nu K_\nu^\top + K_\nu^\top S_\nu & \mathcal{B}_\nu - S_\nu B_\nu & 0 \\ \mathcal{B}_\nu^t + B_\nu^t S_\nu & \mathcal{K}_\nu^\perp & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \mathcal{O}(t^2).$$

Proof Obviously, it suffices to verify the coefficient matrix, say K_1, of t in this expansion formula for K_ξ. Firstly by (2.4) and (1.3), we know that $\nabla_\xi \xi \equiv 0$ and thus $\mathcal{K}_\xi = \nabla_\xi K_\xi$. Then by the Taylor expansion formula, we calculate the coefficient matrix K_1 as follows:

$$\begin{aligned} \nu\langle K_\xi(\partial x_\alpha), \partial x_\beta\rangle &= \langle \nabla_\xi(K_\xi(\partial x_\alpha)), \partial x_\beta\rangle|_p + \langle K_\xi(\partial x_\alpha), \nabla_\xi \partial x_\beta\rangle|_p \\ &= \langle (\nabla_\xi K_\xi)(\partial x_\alpha) + K_\xi(\nabla_\xi \partial x_\alpha), \partial x_\beta\rangle|_p + \langle K_\xi(\partial x_\alpha), \nabla_\xi \partial x_\beta\rangle|_p \\ &= \langle \mathcal{K}_\xi(\partial x_\alpha), \partial x_\beta\rangle|_p + \langle \nabla_\xi \partial x_\alpha, K_\xi(\partial x_\beta)\rangle|_p + \langle \nabla_\xi \partial x_\beta, K_\xi(\partial x_\alpha)\rangle|_p, \end{aligned}$$

or in matrix form,

$$(\nu\langle K_\xi(\partial x_\alpha),\ \partial x_\beta\rangle) = \mathcal{K}_\nu + U(0)K_\nu + K_\nu U(0)^t,$$

where $U(0) = W(0)$ as in Lemma 2.5; on the other hand,

$$(\nu\langle K_\xi(\partial x_\alpha),\ \partial x_\beta\rangle) = \frac{d}{dt}\Big|_{t=0}(K_{\nu_t}G(t)) = K_1 + K_\nu G'(0);$$

and therefore,

$$K_1 = \mathcal{K}_\nu + W(0)K_\nu + K_\nu W(0)^t - K_\nu G'(0),$$

which gives the required formula by using (1.2), (1.4), (2.8) and (2.9). □

3 Hypersurface Case

In this section, we deal with the hypersurface case in our subject by proving Theorems 1.1 and 1.2. Throughout this paper, unless stated otherwise, notations will be consistent with those in previous sections.

Firstly, we establish a lemma on algebraic geometry which will be useful in the proof of the theorems.

Lemma 3.1 *For each $m, n \geq 1$, define polynomials $P_k \in \mathbb{C}[x_1, \ldots, x_n]$ by*

$$P_k := \rho_k(x_1, \ldots, x_n) + \widetilde{P}_{k-1}(x_1, \ldots, x_n) \quad \text{for } k = m, m+1, \ldots, m+n-1,$$

where ρ_k is the k-th power sum polynomial, $\widetilde{P}_{k-1}$ is an arbitrary polynomial of degree less than k. Then $P_m, P_{m+1}, \ldots, P_{m+n-1}$ form a regular sequence in $\mathbb{C}[x_1, \ldots, x_n]$. Consequently, the dimension of each variety V_k in $\mathbb{C}^n$ defined by $P_m = P_{m+1} = \cdots = P_{m+k-1} = 0$ is less than or equal to $n-k$ for $k = 1, \ldots, n$. In particular, V_n is a finite subset of $\mathbb{C}^n$.

Proof The proof is similar to that of Lemma 4.4 in [18]. For completeness, we repeat it as follows.

Firstly recall (cf. [14, 22]) that a sequence $r_1, \ldots, r_k$ in a commutative ring $\mathcal{R}$ with identity is called a *regular sequence* if (1) the ideal $(r_1, \ldots, r_k) \neq \mathcal{R}$; (2) r_1 is not a zero divisor in $\mathcal{R}$; and (3) r_{i+1} is not a zero divisor in the quotient ring $\mathcal{R}/(r_1, \ldots, r_i)$ for $i = 1, \ldots, k-1$.

Now we will work on the polynomial ring $\mathcal{R} = \mathbb{C}[x_1, \ldots, x_n]$. Obviously, it is a *Cohen–Macaulay* ring, possessing the property that $\dim(\mathcal{R}/(r_1, \ldots, r_k)) = n-k$ for a regular sequence

$r_1, \ldots, r_k$ in $\mathcal{R}$. Meanwhile, we know that $\dim(V_k) = \dim(\mathcal{R}/I(V_k))$, where $I(V_k) \supset (r_1, \ldots, r_k)$ is the ideal of the variety

$$V_k := \{x \in \mathbb{C}^n | r_1(x) = \cdots = r_k(x) = 0\}.$$

Therefore, when $r_1, \ldots, r_n$ form a regular sequence, $\dim(V_k) \leq n-k$ for $k = 1, \ldots, n$. In particular, $\dim(V_n) = 0$. The last assertion in the lemma is due to the facts that every variety in $\mathbb{C}^n$ can be expressed as a union of finite irreducible varieties and that a zero-dimensional irreducible variety in $\mathbb{C}^n$ is just a point. So it suffices to show that the polynomials $P_m, P_{m+1} \ldots, P_{m+n-1}$ form a regular sequence in $\mathcal{R}$.

Obviously, P_m forms a regular sequence in $\mathcal{R}$. Suppose that $P_m, P_{m+1}, \ldots, P_{m+n-1}$ do not form a regular sequence, there exists some k with $1 \leq k < n$ such that P_{m+k} is a zero divisor modulo $(P_m, \ldots, P_{m+k-1})$ in $\mathcal{R}$. Then we may choose a relation of minimal degree of the form

$$f_m P_m + f_{m+1} P_{m+1} + \cdots + f_{m+k} P_{m+k} = 0, \tag{3.1}$$

where $f_m, \ldots, f_{m+k}$ are polynomials of minimal degrees modulo $(P_m, \ldots, P_{m+k-1})$. Denote by $D(> 0)$ the maximal degree of $f_k P_k$'s. Let $f_{i_1} P_{i_1}, \ldots, f_{i_r} P_{i_r}$ be those of maximal degree D for some $m \leq i_1 < \cdots < i_r \leq m+k$. Then one can pick out the homogeneous components $\tilde{f}_{i_1} \rho_{i_1}, \ldots, \tilde{f}_{i_r} \rho_{i_r}$ of maximal degree from them in (3.1) such that

$$\tilde{f}_{i_1} \rho_{i_1} + \cdots + \tilde{f}_{i_r} \rho_{i_r} = 0, \tag{3.2}$$

where $\tilde{f}_{i_1}, \ldots, \tilde{f}_{i_r}$ are the homogeneous components of maximal degrees of $f_{i_1}, \ldots, f_{i_r}$, respectively. Recall a recent result showed in [12] that the power sum polynomials $\rho_m, \rho_{m+1}, \ldots, \rho_{m+n-1}$ form a regular sequence in $\mathcal{R}$. Then by (3.2), $r > 1$ and $\tilde{f}_{i_r} \in (\rho_m, \rho_{m+1}, \ldots, \rho_{i_r-1})$, which imply that there exist homogeneous polynomials $a_m, a_{m+1}, \ldots, a_{i_r-1}$ such that

$$\tilde{f}_{i_r} = a_m \rho_m + a_{m+1} \rho_{m+1} + \cdots + a_{i_r-1} \rho_{i_r-1},$$

and therefore,

$$f_{i_r} = a_m P_m + a_{m+1} P_{m+1} + \cdots + a_{i_r-1} P_{i_r-1} + \hat{f}_{i_r} \equiv \hat{f}_{i_r} \bmod (P_m, \ldots, P_{m+k-1}),$$

where $\hat{f}_{i_r}$ is a polynomial of degree less than $D - i_r = \deg(f_{i_r})$, which contradicts the original choice of minimal relation (3.1).

The proof is now complete. □

Let M^n be a curvature-adapted hypersurface in a real space form or locally rank one symmetric space N^{n+1}. Denote by M_t, $t \in (-\varepsilon, \varepsilon)$, nearby parallel hypersurfaces of $M_0 = M$ and ν_t the unit normal vector field on M_t. As is well known, a hypersurface in a real space form is always curvature-adapted as mentioned in Introduction, and moreover, its parallel hypersurfaces have common principal eigenvectors up to parallel translations along normal geodesics, which is a nice property also preserved by a curvature-adapted hypersurface in a symmetric space in which case parallel hypersurfaces are still curvature-adapted (cf. [19]). Now in both cases, the Jacobi operator K_ξ of N has constant eigenvalues independent of the choices of the unit tangent vector ξ and the point of N. Therefore, one can choose the principal orthonormal eigenvectors $\{e_i(t) | i = 1, \ldots, n\}$ of M_t such that they are parallel along normal geodesics and simultaneously diagonalize the shape operator $S(t)$ of M_t and the restricted Jacobi operator

$R(t) := K_{\nu_t}|_{\mathcal{T}_{M_t}}$ of N as the following symmetric matrices:

$$S(t) = \mathrm{diag}(\mu_1(t), \ldots, \mu_n(t)), \quad R(t) = \mathrm{diag}(\kappa_1, \ldots, \kappa_n),$$

where $\mu_i(t)$'s are principal curvature functions of M_t, $\kappa_i \equiv c$ for M in a real space form with constant sectional curvature c, or $\kappa_i \in \{c, 4c\}$ for M in a locally rank one symmetric space with non-constant sectional curvature. Moreover, since $\nabla_{\nu_t} e_i(t) = 0$,

$$S'(t) := \nabla_{\nu_t} S(t) = \mathrm{diag}(\mu_1'(t), \ldots, \mu_n'(t)),$$

and thus the Riccati equation (2.2) can be written as

$$\mu_i'(t) = \mu_i(t)^2 + \kappa_i \quad \text{for } i = 1, \ldots, n. \tag{3.3}$$

We are now ready to prove the theorems for the hypersurface case in our subject.

Proof of Theorems 1.1 *and* 1.2 Recall that the k-th order mean curvature $Q_k(t)$ of M_t is defined by the k-th power sum polynomial of the principal curvatures $\mu_1(t), \ldots, \mu_n(t)$ for any $k \geq 1$, i.e.,

$$Q_k(t) = \mathrm{tr}(S(t)^k) = \sum_{i=1}^{n} \mu_i(t)^k = \rho_k(\mu_1(t), \ldots, \mu_n(t)). \tag{3.4}$$

Taking derivative of $Q_k(t)$ with respect to t by applying (3.3), we get

$$\frac{1}{k} Q_k'(t) = \sum_{i=1}^{n} (\mu_i(t)^{k+1} + \kappa_i \mu_i(t)^{k-1}) = Q_{k+1} + \sum_{i=1}^{n} \kappa_i \mu_i(t)^{k-1}. \tag{3.5}$$

Similarly, for any $j \geq 1$, taking the j-th derivative of $Q_k(t)$ with respect to t by applying (3.3), we can get

$$\frac{1}{k(k+1)\cdots(k+j-1)} Q_k^{(j)}(t) = Q_{k+j} + \widehat{P}_{k+j-1}(\mu_1(t), \ldots, \mu_n(t)), \tag{3.6}$$

where $\widehat{P}_{k+j-1}$ is some polynomial of degree less than $k + j$ with constant coefficients in n variables.

Now assume that for some $k \geq 1$, $Q_k(t)$ is constant on M_t for any $t \in (-\varepsilon, \varepsilon)$ and thus it is a smooth function depending only on t, so are the derived functions $Q_k^{(j)}(t)$ for all $j \geq 1$. Then for any fixed $t \in (-\varepsilon, \varepsilon)$, (3.4)–(3.6) show that the principal curvatures $(\mu_1(t), \ldots, \mu_n(t))$ of M_t are solutions of the algebraic equations

$$P_l(x_1, \ldots, x_n) := \rho_l(x_1, \ldots, x_n) + \widetilde{P}_{l-1}(x_1, \ldots, x_n) = 0 \quad \text{for } l = k, k+1 \ldots, \tag{3.7}$$

where ρ_l is the l-th power sum polynomial, $\widetilde{P}_{l-1} = \widehat{P}_{l-1} - \frac{(k-1)!}{(l-1)!} Q_k^{(l-k)}(t)$ is a polynomial of degree less than l with constant coefficients. In particular, $(\mu_1(t), \ldots, \mu_n(t))$ belongs to the variety V_n in $\mathbb{C}^n$ defined by $P_k = P_{k+1} = \cdots = P_{k+n-1} = 0$. Therefore, by Lemma 3.1, we know that $(\mu_1(t), \ldots, \mu_n(t))$ belongs to a finite subset of $\mathbb{C}^n$ and thus $\mu_i(t)$'s are constant on M_t since M_t is connected. It means that M has constant principal curvatures and is totally isoparametric.

Conversely, if M has constant principal curvatures, by the Riccati equation (3.3), we know immediately (cf. [18]) that $\mu_i(t)$'s are constant on M_t and so each order mean curvatures are constant on M_t.

The proof is now complete. □

4 General Submanifold Case

In this section, we deal with the general submanifold case in our subject. Firstly, by using the Taylor expansion formula of the shape operator obtained in Section 2, we derive a power series expansion formula for higher order mean curvatures of tubular hypersurfaces around a submanifold in a general Riemannian manifold. Then through some involved calculations and technical treatments of this formula, we obtain Theorem 4.4 and give a proof of Theorem 1.4.

As in Section 2, let M^m be a submanifold of a Riemannian manifold N^{n+1} and M_t^n be the tubular hypersurface around M of sufficiently small radius $t \in (0,\varepsilon)$. Since the shape operator $S(t)$ of M_t is the restriction of the operator S defined in (2.1) to $\mathcal{T}M_t$ and $S(\nu_t) = S(\xi)|_{M_t} = 0$, it follows that $S(t)$ has the same nonzero eigenvalues as S. Therefore, the k-th order mean curvature $Q_k(t)$ of M_t can be calculated by

$$Q_k(t) = \mathrm{tr}(S(t)^k) = \mathrm{tr}(S^k) = \mathrm{tr}(\widehat{S}(t)^k),$$

where $\widehat{S}(t)$ is the left-up n by n submatrix of S in (2.6) and by (2.7), (2.10), at the point $\eta_\nu(t) = \exp_p(t\nu) \in M_t$ for any $t \in (0,\varepsilon)$, we have the following expansion formula

$$t\widehat{S}(t) = -A_0 + \sum_{r=1}^{\infty} A_r t^r, \tag{4.1}$$

where

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad A_2 = \begin{pmatrix} S_\nu^2 + K_\nu^\top & B_\nu \\ \frac{1}{3}B_\nu^t & \frac{1}{3}K_\nu^\perp \end{pmatrix},$$
$$A_1 = \begin{pmatrix} S_\nu & 0 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \frac{1}{2}\mathcal{K}_\nu^\top + S_\nu^3 + K_\nu^\top S_\nu & \frac{1}{2}\mathcal{B}_\nu \\ \frac{1}{4}\mathcal{B}_\nu^t + \frac{1}{3}B_\nu^t S_\nu & \frac{1}{4}\mathcal{K}_\nu^\perp \end{pmatrix},$$

and A_r, $r \geq 4$, are n by n matrices independent of t.

Put $\Upsilon_i(t) := t^i Q_i(t) = \mathrm{tr}((t\widehat{S}(t))^i) = \sum_{r=0}^{\infty} \Upsilon_{ir} t^r$, $i \geq 1$. Then by comparing the coefficient of t^r in the extended formula for $\Upsilon_i(t)$ with (4.1) substituted, we get

$$\Upsilon_{ir} = \sum_{\sigma\in\mathscr{P}_{ir}} (-1)^{\sigma_0} \sum_{\tau\in\mathscr{S}_i(\sigma)} \mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i}) \quad \text{for } i \geq 1, r \geq 0, \tag{4.2}$$

where

$$\mathscr{P}_{ir} := \left\{ \sigma = (\sigma_0,\sigma_1,\ldots,\sigma_r) \in \mathbb{Z}^{r+1} \,\middle|\, \sum_{s=0}^{r} \sigma_s = i,\ \sum_{s=0}^{r} s\sigma_s = r,\ \sigma_s \geq 0 \right\};$$

and for $\sigma = (\sigma_0,\sigma_1,\ldots,\sigma_r) \in \mathscr{P}_{ir}$, $\Sigma(\sigma) := \{s \mid 0 \leq s \leq r, \sigma_s > 0\}$,

$$\mathscr{S}_i(\sigma) := \{\tau = (\tau_1,\ldots,\tau_i) \in \mathbb{Z}^i \mid \forall\, s \in \Sigma(\sigma), \exists\, \sigma_s \textit{ elements of } \tau \textit{ equal } s.\}.$$

Obviously,

$$\begin{aligned} \Upsilon_{i0} &= (-1)^i \mathrm{tr}(A_0^i) = (-1)^i(n-m), \quad \Upsilon_{i1} = \delta_{i1}\mathrm{tr}(A_1) && \text{for } i \geq 1, \\ \Upsilon_{1r} &= \mathrm{tr}(A_r) && \text{for } r \geq 1. \end{aligned} \tag{4.3}$$

From now on, we assume $i \geq 2$, $r \geq 2$ and put

$$d(i,r) := \min\left\{ \left[\frac{i}{2}\right], \left[\frac{r}{2}\right] \right\}, \quad D(i,r) := \min\left\{ i-1, \left[\frac{r}{2}\right] \right\}.$$

Note that $A_0A_1 = A_1A_0 = 0$ and $\mathrm{tr}(CD) = \mathrm{tr}(DC)$, then $\mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i}) = 0$ if 0 and 1 occur in some successive indices τ_j, τ_{j+1} ($j \mod i$). Since $A_0^2 = A_0$, we can reduce the sequence in non-vanishing $\mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i})$s such that A_0 occurs separately. Moreover, one can see that A_0 occurs separately at most $d(i,r)$ times with some A_s or $A_sA_1^{\iota_s}A_{s'}$ ($s, s' \geq 2$, $\iota_s \geq 1$) between, in fact, each non-vanishing $\mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i})$, $\tau \in \mathscr{S}_i(\sigma)$, $\sigma \in \mathscr{P}_{ir}$ with $\sigma_0 > 0$ can be written as

$$\mathrm{tr}(A_0^{\lambda_0} \cdot \widetilde{A}_{s_1}A_0^{\lambda_1} \cdot \widetilde{A}_{s_2}A_0^{\lambda_2} \cdot \cdots \cdot \widetilde{A}_{s_c}A_0^{\lambda_c}), \tag{4.4}$$

where $\widetilde{A}_s = A_s$ or $A_sA_1^{\iota_s}A_{s'}$ with $s, s' \geq 2$, $\iota_s \geq 1$, the sum of λ_s equals σ_0, the sum of ι_s equals σ_1, and $1 \leq c \leq D(i,r)$. According to these, we will refine the summation (4.2) or actually the index sets $\mathscr{P}_{ir}$ and $\mathscr{S}_i(\sigma)$ as follows.

For $0 \leq a \leq i-1$, denote by $\mathscr{T}_{ir}(a)$ the set of all non-vanishing $\mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i})$s, $\tau \in \mathscr{S}_i(\sigma)$, $\sigma \in \mathscr{P}_{ir}$ with $\sigma_0 = a$, where the elements with different indices are looked as different though they may have the same value. For $1 \leq c \leq D(i,r)$, put

$$\mathcal{A}_{ir}^c := \{a \in \mathbb{Z}^+ \mid \exists \text{ elements in } \mathscr{T}_{ir}(a) \text{ of the form (4.4)}\}.$$

Then by straightforward calculations, we get

$$\mathcal{A}_{ir}^c = \begin{cases} \{1\}, & i = 2,\ c = 1,\ r \geq 2, \\ \{i-c\}, & i \geq 3,\ 2c \leq r \leq 2c+2, \\ \{a \mid i-r+c+1 \leq a \leq i-c-2,\ a \geq 1\} \cup \{i-c\}, & i \geq 3,\ r \geq 2c+3. \end{cases}$$

It follows immediately from the definitions that $\mathscr{T}_{ir}(a)$ is empty if $a > 0$ is not in $\mathcal{A}_{ir}^c$ for any $1 \leq c \leq D(i,r)$. For example, $\mathscr{T}_{33}(1)$ is empty since now each $\mathrm{tr}(A_{\tau_1}A_{\tau_2}A_{\tau_3})$ equals $\mathrm{tr}(A_0A_1A_2) = \mathrm{tr}(A_1A_0A_2) = 0$. On the other hand, putting

$$\Lambda^c(a) := \left\{\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_c) \in \mathbb{Z}^{c+1} \,\middle|\, \sum_{s=0}^{c} \lambda_s = a,\ \lambda_s \geq 0\right\},$$
$$\mathscr{T}_{ir}(\lambda) := \{\omega \in \mathscr{T}_{ir}(a) \mid \omega \text{ is of the form (4.4) with } (\lambda_0, \ldots, \lambda_c) = \lambda\}, \quad \text{for } \lambda \in \Lambda^c(a),$$

we can divide $\mathscr{T}_{ir}(a)$ ($a > 0$) into subsets $\mathscr{T}_{ir}(\lambda)$, $\lambda \in \Lambda^c(a)$, $1 \leq c \leq D(i,r)$. Note that each $\mathscr{T}_{ir}(\lambda)$ should be empty if $\mathscr{T}_{ir}(a)$ is empty. Furthermore, for $1 \leq b \leq c$, putting

$$\Lambda_b^c := \{\mu = (\mu_1, \ldots, \mu_b) \mid 1 \leq \mu_1 < \cdots < \mu_b < c\},$$
$$\overline{\Lambda}_b^c := \{\mu = (\mu_1, \ldots, \mu_b) \mid 1 \leq \mu_1 < \cdots < \mu_{b-1} < \mu_b = c\},$$

and for $\mu \in \Lambda_b^c$,

$$\Lambda^c(a, \mu) := \left\{\lambda \in \Lambda^c(a) \,\middle|\, \sum_{s=1}^{b} \lambda_{\mu_s} = a,\ \lambda_{\mu_s} \geq 1\right\},$$

and for $\mu \in \overline{\Lambda}_b^c$,

$$\Lambda^c(a, \mu) := \left\{\lambda \in \Lambda^c(a) \,\middle|\, \lambda_0 + \lambda_c + \sum_{s=1}^{b-1} \lambda_{\mu_s} = a,\ \lambda_0 + \lambda_c \geq 1,\ \lambda_{\mu_s} \geq 1 \text{ for } s < b\right\},$$

we can divide the index set $\Lambda^c(a)$ into subsets $\Lambda^c(a, \mu)$, $\mu \in \Lambda_b^c$ or $\overline{\Lambda}_b^c$, $1 \leq b \leq c$. Since the number of elements of $\Lambda^c(a, \mu)$ is independent of the choices of $\mu \in \Lambda_b^c$ (resp. $\mu \in \overline{\Lambda}_b^c$) and c, we denote it by $\Theta(a,b)$ (resp. $\overline{\Theta}(a,b)$) which would be zero for $b > a$. In fact, by a detailed study of the definitions, it turns out that the 2-parameter function Θ satisfies the inductive relation

$$\Theta(a+1,b) - \Theta(a,b) = \Theta(a,b-1), \tag{4.5}$$

with initial conditions $\Theta(a,b) = 0$ for $a < b$ or $b < 0$, $\Theta(a,1) = 1$ ($\Theta(a,0) :\equiv 0$) for $a \geq 1$, and so does the 2-parameter function $\overline{\Theta}$ with initial conditions $\overline{\Theta}(a,b) = 0$ for $a < b$ or $b < 0$, $\overline{\Theta}(a,1) = a+1$ ($\overline{\Theta}(a,0) :\equiv 1$) for $a \geq 1$. Notice that for each $\mu \in \Lambda_b^c$ or $\overline{\Lambda}_b^c$, respectively, each element λ of the index subset $\Lambda^c(a,\mu)$ corresponds to the same subset $\mathscr{T}_{ir}(\lambda)$, denoted by $\mathscr{T}_{ir}(a,\mu)$ which is non-empty for $a \in \mathcal{A}_{ir}^c$ [5)], whose elements have values of the following form:

$$\mathrm{tr}((\widetilde{A}_{s_1}\cdots\widetilde{A}_{s_{\mu_1}})A_0\cdot(\widetilde{A}_{s_{\mu_1+1}}\cdots\widetilde{A}_{s_{\mu_2}})A_0\cdots(\widetilde{A}_{s_{\mu_{b-1}+1}}\cdots\widetilde{A}_{s_{\mu_b}})A_0\cdot(\widetilde{A}_{s_{\mu_b+1}}\cdots\widetilde{A}_{s_c}))$$

or, respectively,

$$\mathrm{tr}((\widetilde{A}_{s_1}\cdots\widetilde{A}_{s_{\mu_1}})A_0\cdot(\widetilde{A}_{s_{\mu_1+1}}\cdots\widetilde{A}_{s_{\mu_2}})A_0\cdots(\widetilde{A}_{s_{\mu_{b-1}+1}}\cdots\widetilde{A}_{s_c})A_0). \tag{4.6}$$

Then we can define

$$\Omega_{ir}^c(a,b) := \sum_{\mu\in\Lambda_b^c}\sum_{\omega\in\mathscr{T}_{ir}(a,\mu)}\omega, \quad \overline{\Omega}_{ir}^c(a,b) := \sum_{\mu\in\overline{\Lambda}_b^c}\sum_{\omega\in\mathscr{T}_{ir}(a,\mu)}\omega, \tag{4.7}$$

which are essentially identical when $c > b$ (the first is zero when $c = b$) since, by the symmetry of the *trace* function, both are the sum of all non-vanishing elements $\mathrm{tr}(A_{\tau_1}\cdots A_{\tau_i})$, $\tau \in \mathscr{S}_i(\sigma)$, $\sigma \in \mathscr{P}_{ir}$ with $\sigma_0 = a$ (without counting multiplicities $\Theta, \overline{\Theta}$), of the form (4.6) with b copies of A_0 occurring separately among c number of $\widetilde{A}_s$ where $\widetilde{A}_s = A_s$ or $A_sA_1^{\iota_s}A_{s'}$ $(s,s' \geq 2,\ \iota_s \geq 1)$.

Equipped with these refinements, we are ready to derive a more tractable formula than (4.2) for the coefficients Υ_{ir} in the power series expression of the i-th order mean curvature $Q_i(t)$ of the tubular hypersurface M_t^n.

Proposition 4.1 *With notations as above, we have for $i \geq 2,\ r \geq 2$,*

$$\Upsilon_{ir} = \sum_{\omega\in\mathscr{T}_{ir}(0)}\omega + \sum_{c=1}^{D(i,r)}\sum_{a\in\mathcal{A}_{ir}^c}(-1)^a\sum_{b=1}^{c}(\Theta(a,b)\Omega_{ir}^c(a,b) + \overline{\Theta}(a,b)\overline{\Omega}_{ir}^c(a,b)). \tag{4.8}$$

Remark 4.2 As footnoted before, the index b in the summation actually takes values from 1 to $\min\{c, d(i,r)\}$, though it does not matter for the calculation since when $c \geq b > d(i,r)$, $\mathcal{A}_{ir}^c \ni a \leq i-c \leq d(i,r) < b$ and thus $\Theta(a,b) = \overline{\Theta}(a,b) = 0$.

Remark 4.3 For example, we list some low order cases as follows:

$$\Upsilon_{22} = \mathrm{tr}(A_1^2) - 2\mathrm{tr}(A_2A_0), \quad \Upsilon_{i2} = (-1)^{i-1}i\ \mathrm{tr}(A_2A_0), \quad \text{for } i \geq 3;$$
$$\Upsilon_{23} = 2\mathrm{tr}(A_1A_2) - 2\mathrm{tr}(A_3A_0), \quad \Upsilon_{33} = \mathrm{tr}(A_1^3) + 3\mathrm{tr}(A_3A_0),$$
$$\Upsilon_{i3} = (-1)^{i-1}i\ \mathrm{tr}(A_3A_0), \quad \text{for } i \geq 4;$$
$$\Upsilon_{24} = 2\mathrm{tr}(A_1A_3) + \mathrm{tr}(A_2^2) - 2\mathrm{tr}(A_4A_0), \quad \Upsilon_{34} = 3\mathrm{tr}(A_1^2A_2) + 3\mathrm{tr}(A_4A_0) - 3\mathrm{tr}(A_2^2A_0),$$
$$\Upsilon_{44} = \mathrm{tr}(A_1^4) - 4\mathrm{tr}(A_4A_0) + 4\mathrm{tr}(A_2^2A_0) + 2\mathrm{tr}(A_2A_0A_2A_0),$$
$$\Upsilon_{i4} = (-1)^{i-1}i\ \mathrm{tr}(A_4A_0) + (-1)^{i-2}\Big(i\ \mathrm{tr}(A_2^2A_0) + \frac{i(i-3)}{2}\mathrm{tr}(A_2A_0A_2A_0)\Big), \quad \text{for } i \geq 5.$$

5) Then in this case, $2b \leq b+c \leq a+c \leq i$ and so $b \leq \min\{[\frac{i}{2}],[\frac{r}{2}]\} =: d(i,r)$ is just the number of copies of A_0 occurring separately in non-vanishing $\mathrm{tr}(A_{\tau_1}A_{\tau_2}\cdots A_{\tau_i})$s with some $\widetilde{A}_s$ between.

Proof Based on the refinements above, direct calculations show

$$\begin{aligned}\Upsilon_{ir} &= \sum_{a=0}^{i-1}(-1)^a \sum_{\omega\in\mathscr{T}_{ir}(a)} \omega = \sum_{\omega\in\mathscr{T}_{ir}(0)} \omega + \sum_{a=1}^{i-1}(-1)^a \sum_{c=1}^{D(i,r)} \sum_{\lambda\in\Lambda^c(a)} \sum_{\omega\in\mathscr{T}_{ir}(\lambda)} \omega \\ &= \sum_{\omega\in\mathscr{T}_{ir}(0)} \omega + \sum_{c=1}^{D(i,r)} \sum_{a\in\mathcal{A}^c_{ir}} (-1)^a \sum_{b=1}^{c} \Big(\sum_{\mu\in\Lambda^c_b} + \sum_{\mu\in\overline{\Lambda}^c_b}\Big) \sum_{\lambda\in\Lambda^c(a,\mu)} \sum_{\omega\in\mathscr{T}_{ir}(\lambda)} \omega \\ &= \sum_{\omega\in\mathscr{T}_{ir}(0)} \omega + \sum_{c=1}^{D(i,r)} \sum_{a\in\mathcal{A}^c_{ir}} (-1)^a \sum_{b=1}^{c} \Big(\sum_{\mu\in\Lambda^c_b} + \sum_{\mu\in\overline{\Lambda}^c_b}\Big) \sum_{\lambda\in\Lambda^c(a,\mu)} \sum_{\omega\in\mathscr{T}_{ir}(a,\mu)} \omega \\ &= \sum_{\omega\in\mathscr{T}_{ir}(0)} \omega + \sum_{c=1}^{D(i,r)} \sum_{a\in\mathcal{A}^c_{ir}} (-1)^a \sum_{b=1}^{c} \Big(\sum_{\mu\in\Lambda^c_b} \Theta(a,b) + \sum_{\mu\in\overline{\Lambda}^c_b} \overline{\Theta}(a,b)\Big) \sum_{\omega\in\mathscr{T}_{ir}(a,\mu)} \omega \\ &= \sum_{\omega\in\mathscr{T}_{ir}(0)} \omega + \sum_{c=1}^{D(i,r)} \sum_{a\in\mathcal{A}^c_{ir}} (-1)^a \sum_{b=1}^{c} (\Theta(a,b)\Omega^c_{ir}(a,b) + \overline{\Theta}(a,b)\overline{\Omega}^c_{ir}(a,b)).\end{aligned}$$

The examples in Remark 4.3 can be verified by (4.8) immediately. □

Now we consider the cases when $i \geq r \geq 2$. Obviously, we have now

$$d(i,r) = D(i,r) = \left[\frac{r}{2}\right] =: d.$$

It is easily seen from the definitions that $\mathscr{T}_{rr}(0)$ consists of only one element $\operatorname{tr}(A_1^r)$ and $\mathscr{T}_{ir}(0)$ is empty for $i > r$. Moreover, for each $e \geq 1$, the map from $\mathscr{P}_{rr}$ to $\mathscr{P}_{r+e\ r}$ defined by

$$(\sigma_0, \sigma_1, \ldots, \sigma_r) \mapsto (\sigma_0 + e, \sigma_1, \ldots, \sigma_r)$$

gives a one-to-one correspondence. Consequently, we have

$$\begin{aligned}&\mathcal{A}^c_{r+e\ r} = \{a + e \mid a \in \mathcal{A}^c_{rr}\} \quad \text{for } 1 \leq c \leq d,\\ &\mathscr{T}_{r+e\ r}(a+e,\mu) = \mathscr{T}_{rr}(a,\mu) \quad \text{for } a \in \mathcal{A}^c_{rr}, \mu \in \Lambda^c_b \text{ or } \overline{\Lambda}^c_b,\end{aligned}$$

and thus

$$\Omega^c_{r+e\ r}(a+e,b) = \Omega^c_{rr}(a,b), \quad \overline{\Omega}^c_{r+e\ r}(a+e,b) = \overline{\Omega}^c_{rr}(a,b) \quad \text{for } a \in \mathcal{A}^c_{rr}, 1 \leq b \leq c.$$

Therefore, the formulae for Υ_{rr} and $\Upsilon_{r+e\ r}$ ($r \geq 2,\ e \geq 1$) in the form (4.8) can be rewritten as

$$\Upsilon_{rr} = \operatorname{tr}(A_1^r) + \sum_{c=1}^{d} \sum_{a\in\mathcal{A}^c_{rr}} (-1)^a \sum_{b=1}^{c} (\Theta(a,b)\Omega^c_{rr}(a,b) + \overline{\Theta}(a,b)\overline{\Omega}^c_{rr}(a,b)), \tag{4.9}$$

$$\Upsilon_{r+e\ r} = \sum_{c=1}^{d} \sum_{a\in\mathcal{A}^c_{rr}} (-1)^{a+e} \sum_{b=1}^{c} (\Theta(a+e,b)\Omega^c_{rr}(a,b) + \overline{\Theta}(a+e,b)\overline{\Omega}^c_{rr}(a,b)). \tag{4.10}$$

Similarly, for $r \geq 3$, $\mathscr{T}_{r-1\ r}(0)$ consists of $(r-1)$ copies of $\operatorname{tr}(A_1^{r-2}A_2)$. Moreover, we have

$$\begin{aligned}&\mathcal{A}^c_{r-1\ r} = \{a - 1 \mid a \in \mathcal{A}^c_{rr}\} \quad \text{for } 1 \leq c \leq d,\\ &\mathscr{T}_{r-1\ r}(a-1,\mu) = \mathscr{T}_{rr}(a,\mu) \quad \text{for } a \in \mathcal{A}^c_{rr}, a-1 \geq b, \mu \in \Lambda^c_b \text{ or } \overline{\Lambda}^c_b,\end{aligned}$$

and thus

$$\Omega^c_{r-1\ r}(a-1,b)=\Omega^c_{rr}(a,b),\quad \overline{\Omega}^c_{r-1\ r}(a-1,b)=\overline{\Omega}^c_{rr}(a,b)\quad \text{for } a\in\mathcal{A}^c_{rr}, a-1\geq b, 1\leq b\leq c.$$

Therefore, the formula for $\Upsilon_{r-1\ r}$ $(r\geq 3)$ can be rewritten as

$$\Upsilon_{r-1\ r}=(r-1)\mathrm{tr}(A_1^{r-2}A_2)+\sum_{c=1}^{d}\sum_{a\in\mathcal{A}^c_{rr}}(-1)^{a-1}\sum_{b=1}^{c}(\Theta(a-1,b)\Omega^c_{rr}(a,b)+\overline{\Theta}(a-1,b)\overline{\Omega}^c_{rr}(a,b)),\tag{4.11}$$

which also holds for $r=2$ by (4.3) and the initial conditions of Θ, $\overline{\Theta}$.

Taking iterative sum of (4.5), we obtain the following useful formula for Θ and $\overline{\Theta}$:

$$C^0_e\Theta(a,b)-C^1_e\Theta(a+1,b)+\cdots+(-1)^eC^e_e\Theta(a+e,b)=(-1)^e\Theta(a,b-e)\tag{4.12}$$

for any $a,b\in\mathbb{Z}$ and $e\geq 0$, where $C^j_e=\frac{e!}{j!(e-j)!}$. For $r\geq 2$, $e\geq 0$, put

$$\Phi_r(e):=C^0_e\Upsilon_{rr}+C^1_e\Upsilon_{r+1\ r}+\cdots+C^e_e\Upsilon_{r+e\ r},$$
$$\Psi_r(e):=C^0_e\Upsilon_{r-1\ r}+C^1_e\Upsilon_{rr}+\cdots+C^e_e\Upsilon_{r-1+e\ r}.$$

Then taking sum of (4.9), (4.10) and (4.11) iteratively by using (4.12), we obtain

$$\Phi_r(e)=\mathrm{tr}(A_1^r)+\sum_{c=1}^{d}\sum_{a\in\mathcal{A}^c_{rr}}(-1)^{a+e}\sum_{b=1}^{c}(\Theta(a,b-e)\Omega^c_{rr}(a,b)+\overline{\Theta}(a,b-e)\overline{\Omega}^c_{rr}(a,b)),$$
$$\Psi_r(e)=(r-1)\mathrm{tr}(A_1^{r-2}A_2)+e\ \mathrm{tr}(A_1^r)+\sum_{c=1}^{d}\sum_{a\in\mathcal{A}^c_{rr}}(-1)^{a-1+e}\sum_{b=1}^{c}(\Theta(a-1,b-e)\Omega^c_{rr}(a,b)+\overline{\Theta}(a-1,b-e)\overline{\Omega}^c_{rr}(a,b)).$$

In particular, since $\mathcal{A}^d_{rr}=\{r-d\}$, by the initial conditions of Θ, $\overline{\Theta}$, we have

$$\Phi_r(d)=\mathrm{tr}(A_1^r)+(-1)^r\overline{\Omega}^d_{rr}(r-d,d),$$
$$\Phi_r(d+1)=\mathrm{tr}(A_1^r),$$
$$\Psi_r(d)=(r-1)\mathrm{tr}(A_1^{r-2}A_2)+d\ \mathrm{tr}(A_1^r)+(-1)^{r-1}\overline{\Omega}^d_{rr}(r-d,d),$$
$$\Psi_r(d+1)=(r-1)\mathrm{tr}(A_1^{r-2}A_2)+(d+1)\ \mathrm{tr}(A_1^r),$$

where by (4.7),

$$\overline{\Omega}^d_{rr}(r-d,d)=\begin{cases}\mathrm{tr}((A_2A_0)^d), & \text{for } r=2d;\\ d\,\mathrm{tr}((A_2A_0)^{d-1}A_3A_0), & \text{for } r=2d+1.\end{cases}$$

Recalling the formulae of A_r in (4.1), then the above formulae can be rewritten as $(d\geq 1)$

$$\begin{aligned}
&\Phi_{2d}(d)=\mathrm{tr}(S_\nu^{2d})+3^{-d}\mathrm{tr}((K_\nu^\perp)^d),\\
&\Phi_{2d+1}(d)=\mathrm{tr}(S_\nu^{2d+1})-d\ 3^{-d+1}4^{-1}\mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp),\\
&\Phi_{2d}(d+1)=\mathrm{tr}(S_\nu^{2d}),\\
&\Phi_{2d+1}(d+1)=\mathrm{tr}(S_\nu^{2d+1}),\\
&\Psi_{2d}(d)=(2d-1)\mathrm{tr}(S_\nu^{2d-2}K_\nu^\top)+(3d-1)\ \mathrm{tr}(S_\nu^{2d})-3^{-d}\mathrm{tr}((K_\nu^\perp)^d),\\
&\Psi_{2d+1}(d)=(2d)\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top)+(3d)\ \mathrm{tr}(S_\nu^{2d+1})+d\ 3^{-d+1}4^{-1}\mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp),
\end{aligned}\tag{4.13}$$

$$\Psi_{2d}(d+1) = (2d-1)\mathrm{tr}(S_\nu^{2d-2}K_\nu^\top) + (3d)\ \mathrm{tr}(S_\nu^{2d}),$$
$$\Psi_{2d+1}(d+1) = (2d)\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top) + (3d+1)\ \mathrm{tr}(S_\nu^{2d+1}).$$

In conclusion, we get the following

Theorem 4.4 *Let M^m be a submanifold of a Riemannian manifold N^{n+1} and M_t^n be the tubular hypersurface around M of sufficiently small radius $t \in (0, \varepsilon)$. For any integer $d \geq 1$,*

(i) *if each M_t^n has constant $Q_{2d}, Q_{2d+1}, \ldots, Q_{3d}$, then on M,*

$$Q_{2d}^\nu + 3^{-d}\rho_d(K_\nu^\perp) \equiv \text{Const};$$

(ii) *if each M_t^n has constant $Q_{2d+1}, Q_{2d+2}, \ldots, Q_{3d+1}$, then on M,*

$$Q_{2d+1}^\nu - d\ 3^{-d+1}4^{-1}\mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp) \equiv 0;$$

(iii) *if each M_t^n has constant $Q_{2d}, Q_{2d+1}, \ldots, Q_{3d+1}$, then on M,*

$$Q_{2d}^\nu \equiv \text{Const};$$

(iv) *if each M_t^n has constant $Q_{2d+1}, Q_{2d+2}, \ldots, Q_{3d+2}$, then on M,*

$$Q_{2d+1}^\nu \equiv 0;$$

(v) *if each M_t^n has constant $Q_{2d-1}, Q_{2d}, \ldots, Q_{3d-1}$, then on M,*

$$(2d-1)\mathrm{tr}(S_\nu^{2d-2}K_\nu^\top) + (3d-1)\ Q_{2d}^\nu - 3^{-d}\rho_d(K_\nu^\perp) \equiv \text{Const};$$

(vi) *if each M_t^n has constant $Q_{2d}, Q_{2d+1}, \ldots, Q_{3d}$, then on M,*

$$(2d)\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top) + (3d)\ Q_{2d+1}^\nu + d\ 3^{-d+1}4^{-1}\mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp) \equiv 0;$$

(vii) *if each M_t^n has constant $Q_{2d-1}, Q_{2d}, \ldots, Q_{3d}$, then on M,*

$$(2d-1)\mathrm{tr}(S_\nu^{2d-2}K_\nu^\top) + (3d)\ Q_{2d}^\nu \equiv \text{Const};$$

(viii) *if each M_t^n has constant $Q_{2d}, Q_{2d+1}, \ldots, Q_{3d+1}$, then on M,*

$$(2d)\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top) + (3d+1)\ Q_{2d+1}^\nu \equiv 0.$$

Proof Recall the definition of Υ_{ir} which is the coefficient of t^r defined over the unit normal bundle $\mathcal{V}_1M$ of M in the power series expansion of $\Upsilon_i(t) := t^iQ_i(t)$ with respect to $t \in (0, \varepsilon)$, where $Q_i(t)$ is the i-th order mean curvature of M_t^n. Therefore, if $Q_i(t)$ is constant on M_t^n and thus is a function depending only on t, then Υ_{ir} would be constant on $\mathcal{V}_1M$ for each $r \geq 0$, which implies that each $\Phi_r(e),\ \Psi_r(e)$ $(r = 2d, 2d+1,\ e = d, d+1)$ in the sequence (4.13) would be constant under the corresponding assumptions listed in the proposition. This verifies the identities in the cases where the vanishing assertion in (ii), (iv), (vi), (viii) is because of the anti-symmetry with respect to ν of $Q_{2d+1}^\nu,\ \mathrm{tr}((K_\nu^\perp)^{d-1}\mathcal{K}_\nu^\perp)$ and $\mathrm{tr}(S_\nu^{2d-1}K_\nu^\top)$. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4 The case (b) follows directly from Theorem 4.4 with $d = [\frac{l}{2}]$. It suffices to prove the case (a) for $1 \leq l \leq 4$.

For $l = 1$, it follows from (4.3) that $Q_1^\nu = \mathrm{tr}(A_1) = \Upsilon_{11}$ which is the coefficient of t^1 in the power series expansion of $\Upsilon_1(t) := tQ_1(t)$ with respect to $t \in (0, \varepsilon)$. Therefore, if $Q_1(t)$ is constant on each tubular hypersurface M_t^n, then Q_1^ν is constant on the unit normal bundle of M and thus vanishes since $Q_1^{-\nu} = -Q_1^\nu$.

For $l = 2$, if $Q_2(t), Q_3(t)$ are constant on each tubular hypersurface M_t^n, then as the coefficients of t^2 in the power series expansions of $t^2Q_2(t)$ and $t^3Q_3(t)$, Υ_{22} and Υ_{32} would be constant. So in this case, by Remark 4.3, we get

$$\Upsilon_{22} = Q_2^\nu - \frac{2}{3}\mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const}, \quad \Upsilon_{32} = \mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const},$$

which verify the first assertion. If in addition $Q_1(t)$ is constant on M_t^n, then by (4.3),

$$\Upsilon_{12} = \mathrm{tr}(A_2) = Q_2^\nu + \mathrm{tr}(K_\nu^\top) + \frac{1}{3}\mathrm{tr}(K_\nu^\perp) \equiv \mathrm{Const},$$

which implies $\mathrm{tr}(K_\nu^\top) \equiv \mathrm{Const}$ and thus verifies the second assertion.

For $l = 3$, if $Q_3(t), Q_4(t)$ are constant on each tubular hypersurface M_t^n, then as the coefficients of t^3 in the power series expansions of $t^3Q_3(t)$ and $t^4Q_4(t)$, Υ_{33} and Υ_{43} would be constant. So in this case, by Remark 4.3 and the anti-symmetry of Q_3^ν and $\mathrm{tr}(\mathcal{K}_\nu^\perp)$ with respect to ν, we get

$$\Upsilon_{33} = Q_3^\nu + \frac{3}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0, \quad \Upsilon_{43} = -\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

which verify the first assertion. If in addition $Q_2(t)$ is constant on M_t^n, then by Remark 4.3 and the anti-symmetry of Q_3^ν and $\mathrm{tr}(S_\nu K_\nu^\top)$ with respect to ν,

$$\Upsilon_{23} = 2Q_3^\nu + 2\mathrm{tr}(S_\nu K_\nu^\top) - \frac{1}{2}\mathrm{tr}(\mathcal{K}_\nu^\perp) \equiv 0,$$

which implies $\mathrm{tr}(S_\nu K_\nu^\top) \equiv 0$ and thus verifies the second assertion.

For $l = 4$, if $Q_4(t), Q_5(t), Q_6(t)$ are constant on each tubular hypersurface M_t^n, then as the coefficients of t^4 in the power series expansions of $t^4Q_4(t)$, $t^5Q_5(t)$ and $t^6Q_6(t)$, Υ_{44}, Υ_{54} and Υ_{64} would be constant. So in this case, by Remark 4.3, we get

$$\begin{aligned}
\Upsilon_{44} &= Q_4^\nu - 4(\mathrm{tr}(A_4A_0) - \mathrm{tr}(A_2^2A_0)) + \frac{2}{9}\rho_2(K_\nu^\perp) \equiv \mathrm{Const},\\
\Upsilon_{54} &= 5(\mathrm{tr}(A_4A_0) - \mathrm{tr}(A_2^2A_0)) - \frac{5}{9}\rho_2(K_\nu^\perp) \equiv \mathrm{Const},\\
\Upsilon_{64} &= -6(\mathrm{tr}(A_4A_0) - \mathrm{tr}(A_2^2A_0)) + \rho_2(K_\nu^\perp) \equiv \mathrm{Const},
\end{aligned}$$

which imply that $Q_4^\nu \equiv \mathrm{Const}$, $\rho_2(K_\nu^\perp) \equiv \mathrm{Const}$, $\mathrm{tr}(A_4A_0) - \mathrm{tr}(A_2^2A_0) \equiv \mathrm{Const}$ and thus verify the first assertion. If in addition $Q_3(t)$ is constant on M_t^n, then by Remark 4.3,

$$\Upsilon_{34} = 3Q_4^\nu + 3\mathrm{tr}(S_\nu^2K_\nu^\top) + 3(\mathrm{tr}(A_4A_0) - \mathrm{tr}(A_2^2A_0)) \equiv \mathrm{Const},$$

which implies $\mathrm{tr}(S_\nu^2K_\nu^\top) \equiv \mathrm{Const}$ and thus verifies the second assertion.

The proof is now complete. □

5 Focal Submanifolds

This section is devoted to the proof of Theorem 1.6 which is a geometrical filtration for the focal submanifolds of isoparametric functions on a complete Riemannian manifold. The assertions in (i)–(ix) of this theorem essentially come from Theorem 1.4 and Theorem 4.4, while (a)–(d) of this theorem treat some special cases as corollaries of (i)–(ix) and the following preliminary on austere submanifolds.

Proposition 5.1 *Let M^m be an austere submanifold of constant principal curvatures in a Riemannian manifold N^{n+1}. If $m = 2, n \geq 4$; or $m = 3, n \geq 5$; or $m = 4, n \geq 10$, then M is a totally geodesic submanifold in N.*

Proof Recall that a submanifold M^m is called an austere submanifold of constant principal curvatures if there exist some constants $\lambda_1, \ldots, \lambda_p$ such that the shape operator S_ν of M with respect to any unit normal vector ν at any point has eigenvalues $\lambda_1, -\lambda_1, \ldots, \lambda_p, -\lambda_p, 0, \ldots, 0$ ($(m-2p)$ zeroes). In particular, on such submanifold, we have

$$\|S_\nu\|^2 \equiv \text{Const} =: C,$$

which implies that for any orthonormal frame $\{e_{m+1}, \ldots, e_{n+1}\}$ of the normal bundle $\mathcal{V}M$ of M,

$$\langle S_{e_i}, S_{e_j} \rangle = C\delta_{ij} \quad \text{for } i, j = m+1, \ldots, n+1.$$

Therefore, if M is not totally geodesic and thus $C > 0$, then $S_{e_{m+1}}, \ldots, S_{e_{n+1}}$ are independent self-dual operators on the tangent bundle $\mathcal{T}M$ of M with constant eigenvalues of opposite signs, which means that at any point q of M the space $\mathcal{S}_q$ of shape operators S_ν, $\nu \in \mathcal{V}_qM$, is an $(n+1-m)$-dimensional subspace of self-dual operators on $\mathcal{T}_qM$.

If the shape operators are looked as quadratic functions on $\mathcal{T}_qM$ via the metric, then $\mathcal{S}_q$ is an $(n+1-m)$-dimensional *austere* subspace of quadratic functions on $\mathcal{T}_qM$ in the sense of [4] where, among other things, Bryant solved the classification problem of maximal austere subspaces of quadratic functions on a real vector space of dimension $m = 2, 3$, or 4. In particular, it follows from his classification that each maximal austere subspace is of dimension 2 when $m = 2$ or 3, and not greater than 6 when $m = 4$. Consequently, $n + 1 - m = \dim(\mathcal{S}_q) \leq 2$ when $m = 2$ or 3, and $n + 1 - m = \dim(\mathcal{S}_q) \leq 6$ when $m = 4$, which verifies the assertions by contradiction. □

One can see from the proof above that for fixed m and sufficiently large n, an austere submanifold M^m of constant principal curvatures in N^{n+1} should be totally geodesic. It is interesting to find out an optimal relationship for such pairs of (m, n).

Proof of Theorem 1.6 First of all, by definition f is a k-isoparametric function if and only if each regular level hypersurface M_t^n of f has constant higher order mean curvatures $Q_1(t), \ldots, Q_k(t)$. As showed in section 4, if f is a k-isoparametric function, then the coefficients Υ_{ir} in the power series expansion formula of $t^iQ_i(t)$ are constant for $1 \leq i \leq k$, $r \geq 0$.

Now we come to verify the assertions listed in the theorem case by case.

(i) $k = 1$. By (4.3) and (4.1),

$$\begin{aligned}
\Upsilon_{11} &= \text{tr}(A_1) = \text{tr}(S_\nu) \equiv \text{Const},\\
\Upsilon_{12} &= \text{tr}(A_2) = Q_2^\nu + \text{tr}(K_\nu^\top) + \frac{1}{3}\text{tr}(K_\nu^\perp) \equiv \text{Const},\\
\Upsilon_{13} &= \text{tr}(A_3) = Q_3^\nu + \text{tr}(S_\nu K_\nu^\top) + \frac{1}{2}\text{tr}(\mathcal{K}_\nu^\top) + \frac{1}{4}\text{tr}(\mathcal{K}_\nu^\perp) \equiv \text{Const}.
\end{aligned}$$

Moreover, since $S_{-\nu} = -S_\nu$, $K_{-\nu}^\top = K_\nu^\top$, $\mathcal{K}_{-\nu}^\top = -\mathcal{K}_\nu^\top$ and $\mathcal{K}_{-\nu}^\perp = -\mathcal{K}_\nu^\perp$, we know that Υ_{11} and Υ_{13} are anti-symmetric with respect to ν and thus $\Upsilon_{11} = \Upsilon_{13} \equiv 0$.

(ii) $k = 2$. As in the proof of Theorem 1.4, by Remark 4.3, we can get

$$\Upsilon_{22} = Q_2^\nu - \frac{2}{3}\text{tr}(K_\nu^\perp) \equiv \text{Const},$$

$$\frac{1}{2}\Upsilon_{23}=Q_3^\nu+\mathrm{tr}(S_\nu K_\nu^\top)-\frac{1}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp)\equiv 0.$$

Combining these with (i), we get

$$\begin{aligned}\mathrm{tr}(K_\nu)&=\mathrm{tr}(K_\nu^\top)+\mathrm{tr}(K_\nu^\perp)=\Upsilon_{12}-\Upsilon_{22}\equiv\mathrm{Const},\\ \frac{1}{2}\mathrm{tr}(\mathcal{K}_\nu)&=\frac{1}{2}\mathrm{tr}(\mathcal{K}_\nu^\top)+\frac{1}{2}\mathrm{tr}(\mathcal{K}_\nu^\perp)=\Upsilon_{13}-\frac{1}{2}\Upsilon_{23}\equiv 0.\end{aligned}$$

(iii) $k=3$. Similarly as before, we can get

$$\begin{aligned}\Upsilon_{32}&=\mathrm{tr}(K_\nu^\perp)\equiv\mathrm{Const},\\ \Upsilon_{33}&=Q_3^\nu+\frac{3}{4}\mathrm{tr}(\mathcal{K}_\nu^\perp)\equiv 0.\end{aligned}$$

Combining these with (ii), we get

$$\begin{aligned}Q_2^\nu&=\Upsilon_{22}+\frac{2}{3}\Upsilon_{32}\equiv\mathrm{Const},\\ \mathrm{tr}(S_\nu K_\nu^\top)-\mathrm{tr}(\mathcal{K}_\nu^\perp)&=\frac{1}{2}\Upsilon_{23}-\Upsilon_{33}\equiv 0.\end{aligned}$$

(iv) $k=4$. Similarly as before, we can get

$$\Upsilon_{43}=-\mathrm{tr}(\mathcal{K}_\nu^\perp)\equiv 0,$$

which implies $Q_3^\nu=\Upsilon_{33}+\frac{3}{4}\Upsilon_{43}\equiv 0$, $\mathrm{tr}(S_\nu K_\nu^\top)=\frac{1}{2}\Upsilon_{23}-\Upsilon_{33}-\Upsilon_{43}\equiv 0$.

(v) $k=5$. Similarly as before, we can get

$$\begin{aligned}Q_4^\nu-\frac{2}{9}\rho_2(K_\nu^\perp)&=\Upsilon_{44}+\frac{4}{5}\Upsilon_{54}\equiv\mathrm{Const},\\ 3\mathrm{tr}(S_\nu^2K_\nu^\top)+\rho_2(K_\nu^\perp)&=\Upsilon_{34}-3\Upsilon_{44}-3\Upsilon_{54}\equiv\mathrm{Const}.\end{aligned}$$

(vi) $k=6$. Similarly as before, we can get

$$\rho_2(K_\nu^\perp)=18\left(\frac{1}{5}\Upsilon_{54}+\frac{1}{6}\Upsilon_{64}\right)\equiv\mathrm{Const},$$

which, together with (v), implies $Q_4^\nu\equiv\mathrm{Const}$, $\mathrm{tr}(S_\nu^2K_\nu^\top)\equiv\mathrm{Const}$. By (vi) of Theorem 4.4, we can get $2\mathrm{tr}(S_\nu^3K_\nu^\top)+3Q_5^\nu+\frac{1}{12}\mathrm{tr}(K_\nu^\perp\mathcal{K}_\nu^\perp)\equiv 0$.

(vii) $k=3d+1$, $d\geq 2$. The identities listed in this case successively come from (iii), (i), (v), (ii), (viii) of Theorem 4.4 since now M_t^n has constant $Q_1,\dots,Q_{3d+1}$.

(viii) $k=3d+2$, $d\geq 2$. The identities listed in this case successively come from (iv), (viii), (vi), (v) of Theorem 4.4 since now M_t^n has constant $Q_1,\dots,Q_{3d+2}$.

(ix) $k=3d+3$, $d\geq 2$. The identities listed in this case successively come from (i), (vi) of Theorem 4.4 since now M_t^n has constant $Q_1,\dots,Q_{3d+3}$.

(a) $m=0$. Obviously, $Q_2^\nu=\mathrm{tr}(K_\nu^\top)\equiv 0$, which, together with the second identity in (i), implies $\mathrm{Ric}(\nu)=\mathrm{tr}(K_\nu)=\mathrm{tr}(K_\nu^\perp)\equiv\mathrm{Const}$ for any $k\geq 1$.

(b) $m=n$. By continuity it is easily seen that each higher order mean curvature $Q_i(t)$ of M_t^n converges to Q_i of M when t goes to 0, i.e., $Q_i=\lim_{t\to 0}Q_i(t)$. Therefore, if f is k-isoparametric, then $Q_i(t)$ and hence Q_i are constant functions on M_t^n and M respectively, which shows that M is k-isoparametric since (locally) M_t^n consists of two equidistant parallel hypersurfaces, say M_t^+ and M_t^-, on both "sides" of M. The odd order mean curvatures Q_{2j+1} $(2j+1\leq k)$ vanish on M because of the anti-symmetry of odd order mean curvatures with respect to unit normal vectors and the identically constancy of $Q_{2j+1}(t)$ on M_t^+ and M_t^-.

(c) For $m \leq [\frac{2k+1}{3}]$, $k \leq 6$, (i)–(vi) above show that M^m has constant $Q^\nu_1, \dots, Q^\nu_m$ and thus has constant principal curvatures which occur in opposite signs since all odd order mean curvatures vanish. For $m \leq [\frac{2k-1}{3}]$, $k \geq 7$, (vii)–(ix) derive the same conclusion as above. For $k = n$, i.e., f is totally isoparametric, then each order mean curvature of M^n_t is constant, which by Theorem 1.4 implies that M has constant each order mean curvature and thus is an austere submanifold of constant principal curvatures. The second part of this case has been proved in Proposition 5.1.

(d) Similarly as in (c), it is easily seen that under each assumption of m and k, $\rho_1(K^\perp_\nu), \dots, \rho_{n-m}(K^\perp_\nu)$ are constant on M and thus the (restricted) vertical Jacobi operator $K^\perp_\nu$ has constant eigenvalues since the restricted vertical Jacobi operator $K^\perp_\nu$ is a self-dual operator of order $(n-m)$.

The proof is now complete. $\square$

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