Acta Mathematica Sinica, English Series
Apr., 2017, Vol. 33, No. 4, pp. 509–525
Published online: November 15, 2016
DOI: 10.1007/s10114-016-5184-1
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A Joint Laplace Transform for Pre-exit Diffusion of Occupation Times

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Abstract For $a < r < b$, the approach of Li and Zhou (2014) is adopted to find joint Laplace transforms of occupation times over intervals (a, r) and (r, b) for a time homogeneous diffusion process before it first exits from either a or b. The results are expressed in terms of solutions to the differential equations associated with the diffusions generator. Applying these results, we obtain more explicit expressions on the joint Laplace transforms of occupation times for Brownian motion with drift, Brownian motion with alternating drift and skew Brownian motion, respectively.

Keywords Laplace transform, occupation time, time-homogeneous diffusion, exit time, Brownian motion with alternating drift, skew Brownian motion

MR(2010) Subject Classification 60J60, 60G17

Received March 21, 2015, revised May 26, 2016, accepted July 22, 2016
Supported by NSFC (Grant Nos. 11171101, 11171044, 11571052 and 11671132), Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP), Education Ministry of China, Hu'nan Normal University; Natural Science Foundation of Hu'nan Province (Grant No. 2016JJ4061), Scientific Research Project of Hu'nan University of Arts and Science (Grant No. 15ZD05)
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1 Introduction

Occupation time, the amount of time a stochastic process spends in a certain region, is of theoretical interest and finds many applications in risk theory and finance; see Cai et al. [4], Landriault et al. [8], Gerber et al. [6], Loeffen et al. [13], Yin and Yuen [17] for such applications. Many explicit results on Laplace transforms for occupation times have been obtained for some well known examples of diffusion processes; see e.g. Borodin and Salminen [3] for a collection of such results. But the joint Laplace transforms of general diffusion processes have not been studied systematically.

We quickly review some previous approaches of finding Laplace transforms of occupation times for stochastic processes. Cai et al. [4] studied the jump-diffusion process with two-sided exponential jumps and found a Laplace transform on its occupation by the standard approach of solving the associated integro-partial-differential equation. They also applied the results into several option pricing problems under Kou's double exponential jump diffusion model. Pitman and Yor [14, 15] applied the excursion theory to obtain occupation time related Laplace transforms of one-dimensional diffusions. They found a formula for the joint Laplace transform of the occupation times spent by the process either above or below a level up to a suitable random time.

Landriault et al. [8] found an alternative perturbation approach to study the Laplace transform of occupation time for spectrally negative Lévy processes (in short SNLP). In Landriault et al. [8], the associated approximating occupation times were proposed so that their Laplace transforms can be computed via the solutions to the exit problems for SNLP. Li and Zhou [10] adopted the approach of Landriault et al. [8] to study the joint Laplace transforms of occupation times for diffusion processes up to an independent exponential time. Li et al. [12] used the same strategy to find expressions of double Laplace transforms for diffusion processes. Yin and Yuen [17] considered the SNLP and determined the joint laws for some quantities, which are useful in insurance risk theory. Loeffen et al. [13] also studied the Laplace transforms of occupation times of intervals until the first passage time for SNLP. They found the expressions through approximating the SNLP by an SNLP with sample paths of bounded variation whose Laplace transform for the occupation time can be found directly. The approaches of Landriault et al. [8] and Loeffen et al. [13] aimed at overcoming the difficulty caused by the possible infinite activity for SNLP.

In order to avoid the approximation procedures of Landriault et al. [8] and Loeffen et al. [13], recently Li and Zhou [11] first studied the Laplace transforms of pre-exit joint occupation times for SNLP via a new approach of identifying the Laplace transform with a fluctuation result on the SNLP observed at independent Poisson arrival times. Such fluctuation identities have been investigated in Albrecher et al. [1].

In this paper, we adopt the Poisson approach of Li and Zhou [11] to consider the joint Laplace transform of occupation times for diffusion processes up to the two-sided exit time. The results can be applied to find more explicit Laplace transforms of the occupation times before exiting for Brownian motion with alternating drift and for skew Brownian motion. To our best knowledge, such results have not been known before. Similar to other results on diffusions along this line, our expressions are explicit but quite lengthy in general.

The rest of this paper is arranged as follows. In Section 2, we review the basic facts we need for the time-homogeneous diffusion processes and derive the expression of potential measure which we need later. In Section 3, the desired results on joint Laplace transform of occupation times for diffusion processes are obtained. In Section 4, we find explicit expressions on the joint Laplace transforms of occupation times for several examples of diffusion processes, such as Brownian motion with drift, Brownian motion with alternating drift and skew Brownian motion.

2 Preliminaries

2.1 Time-Homogeneous Diffusion Processes

In this paper, we consider a one-dimensional diffusion process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$. Process X takes values in interval I with end points $-\infty \leq l_1 < l_2 \leq \infty$, which is specified by the following stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{2.1}$$

where $X_0 = x_0$ is the initial value and $W = \{W_t, t \geq 0\}$ is a one-dimensional standard Brownian motion. Throughout the paper, we assume that (2.1) allows a unique weak solution, which is guaranteed if there exists a constant $K > 0$ such that, for all $x, y \in I$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \quad \mu^2(x) + \sigma^2(x) \leq K^2(1 + x^2), \tag{2.2}$$

see Li and Zhou [10].

Two basic characteristics of diffusion processes X, the *speed measure* m and the *scale function* s, are given by

$$m(dx) = m(x)dx := 2\mathrm{e}^{B(x)}/\sigma^2(x)dx \quad \text{and} \quad s(x) := \int^x \mathrm{e}^{-B(y)}dy$$

for $l_1 < x < l_2$, where $B(x) := \int^x 2\mu(y)/\sigma^2(y)dy$. Let $p(\cdot;\cdot,\cdot)$ be the transition density of X with respect to the speed measure for diffusion processes, i.e.,

$$\mathbb{P}_x\{X_t \in dy\} = p(t; x, y)m(dy).$$

For $\lambda > 0$, let $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$ be two independent positive solutions to the (generalized) differential equation associated to the generator of X

$$\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x), \tag{2.3}$$

with $g_{-,\lambda}(\cdot)$ decreasing and $g_{+,\lambda}(\cdot)$ increasing. For many particular diffusion processes, (2.3) yields explicit expressions for $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$, see Borodin and Salminen [3]. Here a solution $g(x)$ to (2.3) satisfies

$$\lambda \int_{[a,b)} g(x)m(dx) = g^-(b) - g^-(a), \tag{2.4}$$

where

$$g^-(x) := \lim_{h\to 0^+} \frac{g(x) - g(x-h)}{s(x) - s(x-h)}.$$

The *Green function* for X is

$$G_\lambda(x, y) := \int_0^\infty \mathrm{e}^{-\lambda t} p(t; x, y)dt.$$

Then

$$G_\lambda(x,y)=\begin{cases}\omega_\lambda^{-1}g_{+,\lambda}(x)g_{-,\lambda}(y), & x\le y,\\ \omega_\lambda^{-1}g_{+,\lambda}(y)g_{-,\lambda}(x), & x\ge y,\end{cases}$$

where

$$\omega_\lambda := g^+_{+,\lambda}(x)g_{-,\lambda}(x)-g_{+,\lambda}(x)g^+_{-,\lambda}(x)=g^-_{+,\lambda}(x)g_{-,\lambda}(x)-g_{+,\lambda}(x)g^-_{-,\lambda}(x)$$

is the so-called *Wronskian* with

$$g^+(x):=\lim_{h\to 0^+}\frac{g(x+h)-g(x)}{s(x+h)-s(x)}.$$

It is known that ω_λ is independent of x.

We refer to Chapter II of Borodin and Salminen [3] for the above facts and more details about diffusion processes.

Furthermore, for $\lambda>0$, define

$$f_\lambda(y,z):=g_{-,\lambda}(y)g_{+,\lambda}(z)-g_{-,\lambda}(z)g_{+,\lambda}(y).$$

We have the following well-known solutions to the exit problems. Let

$$\tau_x:=\inf\{t\ge 0: X_t=x\}$$

be the first passage time of X at level x with the convention $\inf\phi=\infty$. For any $a<x<b$ and $\lambda\ge 0$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_a};\tau_a<\tau_b]=\frac{f_\lambda(x,b)}{f_\lambda(a,b)}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_b};\tau_b<\tau_a]=\frac{f_\lambda(a,x)}{f_\lambda(a,b)};$$

see e.g. Borodin and Salminen [3] and Feller [5].

2.2 Diffusion Potential Measure

The potential measure for diffusion processes is needed for our main results in Section 3. Its expression is given as a lemma in this subsection.

Throughout the paper, we always assume $a,b\in(l_1,l_2)$.

Lemma 2.1 *For $a<x,y<b$ and $\lambda\ge 0$,*

$$\begin{aligned}&\int_0^\infty \mathbb{P}_x\{X_t\in dy, t<\tau_a\wedge\tau_b\}\mathrm{e}^{-\lambda t}dt\\ &\quad=\left[G_\lambda(x,y)-\frac{f_\lambda(x,b)}{f_\lambda(a,b)}G_\lambda(a,y)-\frac{f_\lambda(a,x)}{f_\lambda(a,b)}G_\lambda(b,y)\right]m(dy).\end{aligned}\tag{2.5}$$

Proof Write e_λ for an exponential random variable with rate $\lambda>0$ that is independent of everything else. Because

$$\mathbb{P}_x\{X_{e_\lambda}\in dy, e_\lambda<\tau_a\wedge\tau_b\}=\lambda\int_0^\infty \mathbb{P}_x\{X_t\in dy,t<\tau_a\wedge\tau_b\}\mathrm{e}^{-\lambda t}dt,$$

we have

$$\mathbb{P}_x\{X_{e_\lambda}\in dy, e_\lambda<\tau_a\wedge\tau_b\}=\mathbb{P}_x\{X_{e_\lambda}\in dy\}-\mathbb{P}_x\{X_{e_\lambda}\in dy,\tau_a\wedge\tau_b<e_\lambda\},$$

where

$$\begin{aligned}&\mathbb{P}_x\{X_{e_\lambda}\in dy,\tau_a\wedge\tau_b<e_\lambda\}\\&\quad=\mathbb{P}_x\{X_{e_\lambda}\in dy,\tau_a<\tau_b\wedge e_\lambda\}+\mathbb{P}_x\{X_{e_\lambda}\in dy,\tau_b<\tau_a\wedge e_\lambda\}\\&\quad=\mathbb{P}_x\{\tau_a<\tau_b\wedge e_\lambda\}\mathbb{P}_a\{X_{e_\lambda}\in dy\}+\mathbb{P}_x\{\tau_b<\tau_a\wedge e_\lambda\}\mathbb{P}_b\{X_{e_\lambda}\in dy\}\\&\quad=\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_a};\tau_a<\tau_b]\mathbb{P}_a\{X_{e_\lambda}\in dy\}+\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_b};\tau_b<\tau_a]\mathbb{P}_b\{X_{e_\lambda}\in dy\}\\&\quad=\lambda\frac{f_\lambda(x,b)}{f_\lambda(a,b)}G_\lambda(a,y)m(dy)+\lambda\frac{f_\lambda(a,x)}{f_\lambda(a,b)}G_\lambda(b,y)m(dy).\end{aligned}$$

So,

$$\begin{aligned}&\mathbb{P}_x\{X_{e_\lambda}\in dy,e_\lambda<\tau_a\wedge\tau_b\}\\&\quad=\lambda G_\lambda(x,y)m(dy)-\lambda\frac{f_\lambda(x,b)}{f_\lambda(a,b)}G_\lambda(a,y)m(dy)-\lambda\frac{f_\lambda(a,x)}{f_\lambda(a,b)}G_\lambda(b,y)m(dy).\end{aligned}$$

We thus obtain

$$\begin{aligned}&\int_0^\infty\mathbb{P}_x\{X_t\in dy,t<\tau_a\wedge\tau_b\}\mathrm{e}^{-\lambda t}dt\\&\quad=\left[G_\lambda(x,y)-\frac{f_\lambda(x,b)}{f_\lambda(a,b)}G_\lambda(a,y)-\frac{f_\lambda(a,x)}{f_\lambda(a,b)}G_\lambda(b,y)\right]m(dy).\end{aligned}$$

3 Main Results

In this section, we proceed to find the joint Laplace transform on occupation times of diffusion processes. In the following sections, we always denote λ_-,λ_+ for two nonnegative constants.

Theorem 3.1 *For any $a<x\le r<b$, we have*

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,r)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(r,b)}(X_s)ds};\tau_a<\tau_b]=\frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)}+\frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)}f_-(r);$$

for any $a<r\le x<b$, we have

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,r)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(r,b)}(X_s)ds};\tau_a<\tau_b]=\frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)}f_-(r),$$

where

$$f_-(r):=\frac{\frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)}+\lambda_+\int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x)\frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)}m(dx)}{1-\lambda_+\int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x)\frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)}m(dx)-\lambda_-\int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x)\frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)}m(dx)}$$

and

$$I_\lambda(a,b,r,x):=G_\lambda(r,x)-\frac{f_\lambda(r,b)}{f_\lambda(a,b)}G_\lambda(a,x)-\frac{f_\lambda(a,r)}{f_\lambda(a,b)}G_\lambda(b,x).$$

Proof Define

$$f_-(x):=\begin{cases}\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,r)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(r,b)}(X_s)ds};\tau_a<\tau_b], & a<x<b,\\ 1, & x\le a.\end{cases}$$

Write $0<T_1^-<T_2^-<\cdots$ and $0<T_1^+<T_2^+<\cdots$ for the arrival times of independent Poisson processes with rates λ_- and λ_+, respectively. We also assume that these Poisson processes are independent of process X. By a property of Poisson process, we observe that $f_-(x)=\mathbb{P}_x\{D_-\}$ for event

$$D_-=:\{\{T_i^-\}\cap\{s<\tau_a<\tau_b:a<X_s<r\}=\phi=\{T_i^+\}\cap\{s<\tau_a<\tau_b:r<X_s<b\}\}.$$

Then for independent exponential random variables T_- and T_+ with rate λ_- and λ_+, respectively, we have

$$\begin{aligned}
f_-(r) &= \mathbb{P}_r\{\tau_a < \tau_b \wedge T_+ \wedge T_-\} + \int_a^r \mathbb{P}_r\{T_+ < \tau_a \wedge \tau_b \wedge T_-, X(T_+) \in dx\} f_-(x) \\
&\quad + \int_r^b \mathbb{P}_r\{T_- < \tau_a \wedge \tau_b \wedge T_+, X(T_-) \in dx\} f_-(x) \\
&= \mathbb{E}_r[\mathrm{e}^{-(\lambda_-+\lambda_+)\tau_a}; \tau_a < \tau_b] + \lambda_+ \int_a^r \int_0^\infty \mathbb{P}_r\{t < \tau_a \wedge \tau_b, X(t) \in dx\} \mathrm{e}^{-(\lambda_-+\lambda_+)t} dt f_-(x) \\
&\quad + \lambda_- \int_r^b \int_0^\infty \mathbb{P}_r\{t < \tau_a \wedge \tau_b, X(t) \in dx\} \mathrm{e}^{-(\lambda_-+\lambda_+)t} dt f_-(x) \\
&= \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} + \lambda_+ \int_a^r \bigg[G_{\lambda_-+\lambda_+}(r,x) - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(a,x) \\
&\quad - \frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(b,x) \bigg] m(dx) f_-(x) \\
&\quad + \lambda_- \int_r^b \bigg[G_{\lambda_-+\lambda_+}(r,x) - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(a,x) \\
&\quad - \frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(b,x) \bigg] m(dx) f_-(x). \qquad (3.1)
\end{aligned}$$

For $a < x \le r$,

$$\begin{aligned}
f_-(x) &= \mathbb{E}_x[\mathrm{e}^{-\lambda_- \tau_a}; \tau_a < \tau_r] + \mathbb{E}_x[\mathrm{e}^{-\lambda_- \tau_r}; \tau_r < \tau_a] f_-(r) \\
&= \frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)} + \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} f_-(r); \qquad (3.2)
\end{aligned}$$

for $r \le x < b$,

$$f_-(x) = \mathbb{E}_x[\mathrm{e}^{-\lambda_+ \tau_r}; \tau_r < \tau_b] f_-(r) = \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} f_-(r). \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3), after some algebras, we can obtain $A_- f_-(r) = B_-$, with A_- and B_- given by

$$\begin{aligned}
A_- - 1 &= -\lambda_+ \int_a^r \bigg[G_{\lambda_-+\lambda_+}(r,x) - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(a,x) \\
&\quad - \frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(b,x) \bigg] \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx) \\
&\quad - \lambda_- \int_r^b \bigg[G_{\lambda_-+\lambda_+}(r,x) - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(a,x) \\
&\quad - \frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(b,x) \bigg] \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} m(dx) \\
&= -\lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx) \\
&\quad - \lambda_- \int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} m(dx) \qquad (3.4)
\end{aligned}$$

and

$$B_- - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} = \lambda_+ \int_a^r \bigg[G_{\lambda_-+\lambda_+}(r,x) - \frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(a,x)$$

$$- \frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} G_{\lambda_-+\lambda_+}(b,x)\Big] \frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)} m(dx)$$
$$= \lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx). \tag{3.5}$$

So,

$$f_-(r) = \frac{B_-}{A_-}$$
$$= \frac{\frac{f_{\lambda_-+\lambda_+}(r,b)}{f_{\lambda_-+\lambda_+}(a,b)} + \lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)} m(dx)}{1 - \lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx) - \lambda_- \int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} m(dx)}.$$

Combining (3.2) and (3.3), we can get the results of Theorem 3.1.

Throughout the paper, we denote $I_{\lambda_-+\lambda_+}(a,b,r,x)$ as in Theorem 3.1.

Theorem 3.2 *For $a < x \le r < b$,*

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(r,b)}(X_s)ds}; \tau_b < \tau_a] = \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} f_+(r);$$

for $a < r \le x < b$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(r,b)}(X_s)ds}; \tau_b < \tau_a] = \frac{f_{\lambda_+}(r,x)}{f_{\lambda_+}(r,b)} + \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} f_+(r),$$

where

$$f_+(r) = \frac{\frac{f_{\lambda_-+\lambda_+}(a,r)}{f_{\lambda_-+\lambda_+}(a,b)} + \lambda_- \int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_+}(r,x)}{f_{\lambda_+}(r,b)} m(dx)}{1 - \lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx) - \lambda_- \int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} m(dx)}.$$

Proof Define

$$f_+(x) = \begin{cases} \mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(r,b)}(X_s)ds}; \tau_b < \tau_a], & a < x < b; \\ 1, & x \le a. \end{cases}$$

Observe that $f_+(x) = \mathbb{P}_x\{D_+\}$ for event

$$D_+ := \{\{T_i^-\} \cap \{s < \tau_b < \tau_a : a < X_s < r\} = \phi = \{T_i^+\} \cap \{s < \tau_b < \tau_a : r < X_s < b\}\}$$

and for independent exponential random variables T_- and T_+ with rate λ_- and λ_+, respectively. Following similar arguments in the proof of Theorem 3.1, we have the results of Theorem 3.2.

Combining Theorems 3.1 and 3.2, we have the following result.

Theorem 3.3 *For $a < x \le r < b$,*

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(r,b)}(X_s)ds}] = \frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)} + \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} [f_-(r) + f_+(r)];$$

for $a < r \le x < b$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(r,b)}(X_s)ds}] = \frac{f_{\lambda_+}(r,x)}{f_{\lambda_+}(r,b)} + \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} [f_-(r) + f_+(r)],$$

where $f_-(r)$ and $f_+(r)$ have been defined in Theorem 3.1 and Theorem 3.2, respectively.

4 Examples

In the following Subsection 4.1, we apply the results in Section 3 to Brownian motion with drift to compare with the known results. In Subsection 4.2, the explicit expressions of joint Laplace transform for occupation times of Brownian motion with alternating drift are found. Subsection 4.3 concerns skew Brownain motion.

4.1 Brownian Motion with Drift

Let $X_t = \mu t + W_t$ be a Brownian motion with drift. The corresponding differential equation (2.3) is

$$\frac{1}{2}g''(x) + \mu g'(x) = \lambda g(x), \quad \lambda > 0.$$

With two independent solutions

$$g_{+,\lambda}(x) = \mathrm{e}^{(-\mu+\sqrt{\mu^2+2\lambda})x} \quad \text{and} \quad g_{-,\lambda}(x) = \mathrm{e}^{(-\mu-\sqrt{\mu^2+2\lambda})x};$$

see Borodin and Salminen [3, pp. 127–128], we also have

$$m(dx) = 2\mathrm{e}^{2\mu x}dx, \quad \omega_\lambda = 2\sqrt{2\lambda+\mu^2}$$

and

$$G_\lambda(x,y) = \begin{cases} \omega_\lambda^{-1}\mathrm{e}^{-(\mu+\sqrt{\mu^2+2\lambda})x}\mathrm{e}^{(-\mu+\sqrt{\mu^2+2\lambda})y}, & x \geq y, \\ \omega_\lambda^{-1}\mathrm{e}^{-(\mu+\sqrt{\mu^2+2\lambda})y}\mathrm{e}^{(-\mu+\sqrt{\mu^2+2\lambda})x}, & y \geq x. \end{cases}$$

Furthermore,

$$f_\lambda(y,z) = \mathrm{e}^{-\mu(y+z)}(\mathrm{e}^{\sqrt{\mu^2+2\lambda}(z-y)} - \mathrm{e}^{-\sqrt{\mu^2+2\lambda}(z-y)}),$$

denote $\Upsilon_\lambda := \sqrt{\mu^2+2\lambda}$, we can get

$$\begin{aligned} I_\lambda(a,b,r,x) =& \frac{\mathrm{e}^{-\mu(r+x)-\Upsilon_\lambda|r-x|}}{2\Upsilon_\lambda} - \frac{\mathrm{e}^{-\mu r}\mathrm{sh}(\Upsilon_\lambda(b-r))}{\mathrm{sh}(\Upsilon_\lambda(b-a))} \times \frac{\mathrm{e}^{-\mu x-\Upsilon_\lambda(x-a)}}{2\Upsilon_\lambda} \\ & - \frac{\mathrm{e}^{-\mu r}\mathrm{sh}(\Upsilon_\lambda(r-a))}{\mathrm{sh}(\Upsilon_\lambda(b-a))} \times \frac{\mathrm{e}^{-\mu x-\Upsilon_\lambda(b-x)}}{2\Upsilon_\lambda} \end{aligned}$$

and

$$\begin{aligned} &\lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x)\frac{f_{\lambda_-}(x,r)}{f_{\lambda_-}(a,r)}m(dx) \\ &= \lambda_+ \int_a^r \Bigg[\frac{\mathrm{e}^{-\mu(r+x)-\Upsilon_{\lambda_-+\lambda_+}(r-x)}}{2\Upsilon_{\lambda_-+\lambda_+}} - \frac{\mathrm{e}^{-\mu r}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r))}{\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))} \times \frac{\mathrm{e}^{-\mu x-\Upsilon_{\lambda_-+\lambda_+}(x-a)}}{2\Upsilon_{\lambda_-+\lambda_+}} \\ &\quad - \frac{\mathrm{e}^{-ur}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(r-a))}{\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))} \times \frac{\mathrm{e}^{-\mu x-\Upsilon_{\lambda_-+\lambda_+}(b-x)}}{2\Upsilon_{\lambda_-+\lambda_+}}\Bigg] \\ &\quad \times \frac{\mathrm{e}^{\mu(x+a)}(\mathrm{e}^{\Upsilon_{\lambda_-}(r-x)} - \mathrm{e}^{-\Upsilon_{\lambda_-}(r-x)})}{\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))}dx \\ &= \frac{\mathrm{e}^{\mu(a-r)}N}{2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))}, \end{aligned}$$

where

$$\begin{aligned} N =& \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) + \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r))\mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\ & - \Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \end{aligned}$$

$$
\begin{aligned}
&+ \Upsilon_{\lambda_-} \mathrm{e}^{\Upsilon_{\lambda_-+\lambda_+}(a-b)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(r-a)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-} \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(r-a)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-+\lambda_+} \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(r-a)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-} \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(r-a)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \\
&- \Upsilon_{\lambda_-} \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b-r)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(r-a)) \\
&+ \Upsilon_{\lambda_-+\lambda_+} \mathrm{e}^{\Upsilon_{\lambda_-+\lambda_+}(a-b)} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(r-a)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
= {}& \Upsilon_{\lambda_-} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) + \Upsilon_{\lambda_-} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&+ \frac{1}{2} \Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) (\mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b-r)} - \mathrm{e}^{\Upsilon_{\lambda_-+\lambda_+}(b-r)}) \\
&+ \frac{1}{2} \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) (\mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b-r)} - \mathrm{e}^{\Upsilon_{\lambda_-+\lambda_+}(b-r)}) \\
&- \frac{1}{2} \Upsilon_{\lambda_-} (\mathrm{e}^{\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)} + \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)} - 2\mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b-a)}) \\
= {}& \Upsilon_{\lambda_-} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) + \Upsilon_{\lambda_-} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&- \Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \\
&+ \Upsilon_{\lambda_-} \mathrm{e}^{-\Upsilon_{\lambda_-+\lambda_+}(b-a)} - \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \\
&- \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)) \\
= {}& -2\Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&+ \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) - \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)).
\end{aligned}
$$

With similar computations, we have

$$
\begin{aligned}
&\lambda_+ \int_a^r I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,r)} m(dx) \\
&\quad = \frac{1}{2\Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a))} \\
&\qquad \times [\Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)) \mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&\qquad + \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a)) \\
&\qquad - \Upsilon_{\lambda_-} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{ch}(\Upsilon_{\lambda_-}(r-a))] + \frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda_- \int_r^b I_{\lambda_-+\lambda_+}(a,b,r,x) \frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(r,b)} m(dx) \\
&\quad = \frac{1}{2\Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{sh}(\Upsilon_{\lambda_+}(b-r))} \\
&\qquad \times [-\Upsilon_{\lambda_-+\lambda_+} \mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)) \mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) \\
&\qquad + \Upsilon_{\lambda_+} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)) \mathrm{ch}(\Upsilon_{\lambda_+}(b-r)) \\
&\qquad - \Upsilon_{\lambda_+} \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) \mathrm{ch}(\Upsilon_{\lambda_+}(b-r))] + \frac{1}{2}.
\end{aligned}
$$

By (3.4) and (3.5), we obtain

$$
\begin{aligned}
B_- &= \frac{\mathrm{e}^{\mu(a-r)}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r))}{\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))} \\
&\quad + \frac{\mathrm{e}^{\mu(a-r)}}{2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))} \\
&\quad \times [-2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&\quad + \Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) - \Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r))] \\
&= \frac{\mathrm{e}^{\mu(a-r)}\Upsilon_{\lambda_-}(\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) - \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r)))}{2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))}
\end{aligned}
$$

and

$$
\begin{aligned}
A_- &= \frac{-1}{2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))} \\
&\quad \times [\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r))\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) \\
&\quad + \Upsilon_{\lambda_+}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) \\
&\quad - \Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) \\
&\quad - \Upsilon_{\lambda_+}\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))] \\
&= \frac{1}{2\Upsilon_{\lambda_-+\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}(b-a))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))} \\
&\quad \times [(\mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b-a)) - \mathrm{ch}(\Upsilon_{\lambda_-+\lambda_+}(b+a-2r))) \\
&\quad \times (\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r)))].
\end{aligned}
$$

So,

$$
f_-(r) = \frac{B_-}{A_-} = \frac{\Upsilon_{\lambda_-}\mathrm{e}^{\mu(a-r)}\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))}{\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))}.
$$

By Theorem 3.1, we have for $a < x \le r < b$,

$$
\begin{aligned}
&\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,r)}(X_s)ds - \lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(r,b)}(X_s)ds}; \tau_a < \tau_b] \\
&\quad = \frac{\mathrm{e}^{-\mu(x+r)}(\mathrm{e}^{\Upsilon_{\lambda_-}(r-x)} - \mathrm{e}^{-\Upsilon_{\lambda_-}(r-x)})}{\mathrm{e}^{-\mu(a+r)}(\mathrm{e}^{\Upsilon_{\lambda_-}(r-a)} - \mathrm{e}^{-\Upsilon_{\lambda_-}(r-a)})} + \frac{\mathrm{e}^{-\mu(a+x)}(\mathrm{e}^{\Upsilon_{\lambda_-}(x-a)} - \mathrm{e}^{-\Upsilon_{\lambda_-}(x-a)})}{\mathrm{e}^{-\mu(a+r)}(\mathrm{e}^{\Upsilon_{\lambda_-}(r-a)} - \mathrm{e}^{-\Upsilon_{\lambda_-}(r-a)})}f_-(r) \\
&\quad = \frac{\mathrm{e}^{\mu(a-x)}\mathrm{sh}(\Upsilon_{\lambda_-}(r-x))}{\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))} + \frac{\mathrm{e}^{\mu(r-x)}\mathrm{sh}(\Upsilon_{\lambda_-}(x-a))}{\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))}f_-(r) \\
&\quad = \frac{1}{\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))} \\
&\qquad \times \frac{\mathrm{e}^{\mu(a-x)}}{\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))} \\
&\qquad \times [\Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-x))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r)) \\
&\qquad + \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_-}(r-x))\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) \\
&\qquad + \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_-}(x-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))], \\
&\quad = \frac{1}{\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{e^{\mu(a-x)}}{\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))}[\Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-x))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r)) \\
&+ \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_-}(r-x))] \\
= &\frac{e^{\mu(a-x)}}{\Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))} \\
&\times [\Upsilon_{\lambda_+}\mathrm{sh}(\Upsilon_{\lambda_-}(r-x))\mathrm{ch}(\Upsilon_{\lambda_+}(b-r)) + \Upsilon_{\lambda_-}\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))\mathrm{ch}(\Upsilon_{\lambda_-}(r-x))], \qquad (4.1)
\end{aligned}
$$

where in order to obtain the fourth equation, we use the following property of hyperbolic functions.

$$
\begin{aligned}
\mathrm{sh}(\Upsilon_{\lambda_-}(x-a)) &= \mathrm{sh}\big(\Upsilon_{\lambda_-}((r-a)-(r-x))\big) \\
&= \mathrm{sh}(\Upsilon_{\lambda_-}(r-a))\mathrm{ch}(\Upsilon_{\lambda_-}(r-x)) - \mathrm{sh}(\Upsilon_{\lambda_-}(r-x))\mathrm{ch}(\Upsilon_{\lambda_-}(r-a)).
\end{aligned}
$$

And for $r \le x < b$, we have

$$
\begin{aligned}
&\mathbb{E}_x[e^{-\lambda_- \int_0^{\tau_a \wedge \tau_b} \mathbf{1}_{(a,r)}(X_s)ds - \lambda_+ \int_0^{\tau_a \wedge \tau_b} \mathbf{1}_{(r,b)}(X_s)ds}; \tau_a < \tau_b] \\
&= \frac{e^{-\mu(x+b)}(e^{\Upsilon_{\lambda_+}(b-x)} - e^{-\Upsilon_{\lambda_+}(b-x)})}{e^{-\mu(r+b)}(e^{\Upsilon_{\lambda_+}(b-r)} - e^{-\Upsilon_{\lambda_+}(b-r)})} f_-(r) \\
&= \frac{e^{\mu(r-x)}\mathrm{sh}(\Upsilon_{\lambda_+}(b-x))}{\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))} f_-(r) \\
&= \frac{\Upsilon_{\lambda_-} e^{\mu(a-x)}\mathrm{sh}(\Upsilon_{\lambda_+}(b-x))}{\Upsilon_{\lambda_+}\mathrm{ch}(\Upsilon_{\lambda_+}(b-r))\mathrm{sh}(\Upsilon_{\lambda_-}(r-a)) + \Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}(r-a))\mathrm{sh}(\Upsilon_{\lambda_+}(b-r))}. \qquad (4.2)
\end{aligned}
$$

Both (4.1) and (4.2) agree with formula 3.6.5 (a) on p. 316 of Borodin and Salminen [3].

4.2 Brownian Motion with Alternating Drift

Let X be a Brownian motion with alternating drift, specified by the following stochastic differential equation:

$$
dX_t = (\mu \mathbf{1}_{(-\infty,0)}(X_t) - \mu \mathbf{1}_{(0,\infty)}(X_t))dt + dW_t, \qquad (4.3)
$$

where $\mu \in \mathbb{R}$ and W_t is the standard one-dimensional Brownian motion. Although the Lipschitz assumption (2.2) for drift function $\mu(\cdot) = \mu 1_{(-\infty,0)}(\cdot) - \mu 1_{(0,\infty)}(\cdot)$ fails, (4.3) still has a unique strong solution, see Prokhorov and Shiryaev [16].

Referring to Li and Zhou [10] and Li et al. [12], for $\lambda > 0$ the two independent positive solutions of the differential equation

$$
\frac{1}{2}g''(x) + (\mu \mathbf{1}_{(-\infty,0)}(x) - \mu \mathbf{1}_{(0,\infty)}(x))g'(x) = \lambda g(x)
$$

are given by

$$
\begin{aligned}
g_{-,\lambda}(x) = &\Bigg[\frac{\mu}{\sqrt{\mu^2+2\lambda}} e^{(-\mu+\sqrt{\mu^2+2\lambda})x} \\
&+ \left(1 - \frac{\mu}{\sqrt{\mu^2+2\lambda}}\right) e^{(-\mu-\sqrt{\mu^2+2\lambda})x}\Bigg] \mathbf{1}_{x<0} + e^{(\mu-\sqrt{\mu^2+2\lambda})x}\mathbf{1}_{x>0}
\end{aligned}
$$

and

$$g_{+,\lambda}(x) = \left[\left(1 - \frac{\mu}{\sqrt{\mu^2+2\lambda}}\right)\mathrm{e}^{(\mu+\sqrt{\mu^2+2\lambda})x} + \frac{\mu}{\sqrt{\mu^2+2\lambda}}\mathrm{e}^{(\mu-\sqrt{\mu^2+2\lambda})x}\right]\mathbf{1}_{x>0} + \mathrm{e}^{(-\mu+\sqrt{\mu^2+2\lambda})x}\mathbf{1}_{x<0}.$$

In addition, we have

$$m(dx) = 2\mathrm{e}^{2\mu|x|}dx$$

and

$$G_\lambda(x,y) = \frac{1}{2\sqrt{2\lambda+\mu^2}}\mathrm{e}^{-\mu(|x|+|y|)}\left(\mathrm{e}^{-|x-y|\sqrt{2\lambda+\mu^2}} - \frac{2\mu}{\omega_\lambda}\mathrm{e}^{-(|x|+|y|)\sqrt{2\lambda+\mu^2}}\right),$$

where $\omega_\lambda = 2(\sqrt{2\lambda+\mu^2}+\mu)$; see pp. 128–129 of Borodin and Salminen [3].

We consider the case $a < 0$, $r = 0$, $b > 0$. Similarly, denote $\Upsilon_\lambda := \sqrt{\mu^2+2\lambda}$. Then for $x, y > 0$,

$$f_\lambda(x,y) = \left(1 - \frac{\mu}{\Upsilon_\lambda}\right)(\mathrm{e}^{(\mu-\Upsilon_\lambda)x+(\mu+\Upsilon_\lambda)y} - \mathrm{e}^{(\mu+\Upsilon_\lambda)x+(\mu-\Upsilon_\lambda)y});$$

for $x < 0, y > 0$,

$$f_\lambda(x,y) = \mathrm{e}^{\mu(y-x)}\left[\frac{2\mu}{\Upsilon_\lambda}\mathrm{ch}(\Upsilon_\lambda(x+y)) + 2\mathrm{sh}(\Upsilon_\lambda(y-x)) - \frac{4\mu^2}{\Upsilon_\lambda^2}\mathrm{sh}(\Upsilon_\lambda x)\mathrm{sh}(\Upsilon_\lambda y)\right];$$

and for $x < 0, y < 0$,

$$f_\lambda(x,y) = \left(1 - \frac{\mu}{\Upsilon_\lambda}\right)(\mathrm{e}^{(-\mu-\Upsilon_\lambda)x+(-\mu+\Upsilon_\lambda)y} - \mathrm{e}^{(-\mu+\Upsilon_\lambda)x+(-\mu-\Upsilon_\lambda)y}).$$

After some calculations, we have for $x \le 0$,

$$I_\lambda(a,b,0,x) = \frac{\mathrm{e}^{(\mu+\Upsilon_\lambda)x}}{2(\mu+\Upsilon_\lambda)} - \frac{(\Upsilon_\lambda-\mu)\mathrm{e}^{\mu b+(\mu+\Upsilon_\lambda)a}\mathrm{sh}(\Upsilon_\lambda b)}{\Upsilon_\lambda^2 f_\lambda(a,b)}\left(\mathrm{e}^{(\mu-\Upsilon_\lambda)x} - \frac{\mu}{\mu+\Upsilon_\lambda}\mathrm{e}^{(\mu+\Upsilon_\lambda)x}\right) + \frac{(\Upsilon_\lambda-\mu)\mathrm{e}^{-\mu a-(\mu+\Upsilon_\lambda)b}\mathrm{e}^{(\mu+\Upsilon_\lambda)x}\mathrm{sh}(\Upsilon_\lambda a)}{(\mu+\Upsilon_\lambda)\Upsilon_\lambda f_\lambda(a,b)},$$

and for $x \ge 0$,

$$I_\lambda(a,b,0,x) = \frac{\mathrm{e}^{-(\mu+\Upsilon_\lambda)x}}{2(\mu+\Upsilon_\lambda)} - \frac{(\Upsilon_\lambda-\mu)\mathrm{e}^{\mu b+(\mu+\Upsilon_\lambda)a}\mathrm{e}^{-(\mu+\Upsilon_\lambda)x}\mathrm{sh}(\Upsilon_\lambda b)}{(\mu+\Upsilon_\lambda)\Upsilon_\lambda f_\lambda(a,b)} + \frac{(\Upsilon_\lambda-\mu)\mathrm{e}^{-\mu a-(\mu+\Upsilon_\lambda)b}\mathrm{sh}(\Upsilon_\lambda a)}{\Upsilon_\lambda^2 f_\lambda(a,b)}\left(\mathrm{e}^{(-\mu+\Upsilon_\lambda)x} - \frac{\mu}{\mu+\Upsilon_\lambda}\mathrm{e}^{-(\mu+\Upsilon_\lambda)x}\right).$$

Let

$$c_\pm := \frac{1}{2\mu^2+\lambda_+ \pm 2\mu\Upsilon_{\lambda_-+\lambda_+}},$$
$$l_\pm := \frac{1}{2\mu^2+\lambda_- \pm 2\mu\Upsilon_{\lambda_-+\lambda_+}},$$
$$k_{\pm,a} := -\Upsilon_{\lambda_-} + (2\mu \pm \Upsilon_{\lambda_-+\lambda_+})\mathrm{e}^{-(2\mu\pm\Upsilon_{\lambda_-+\lambda_+})a}\mathrm{sh}(\Upsilon_{\lambda_-}a) + \Upsilon_{\lambda_-}\mathrm{e}^{-(2\mu\pm\Upsilon_{\lambda_-+\lambda_+})a}\mathrm{ch}(\Upsilon_{\lambda_-}a),$$
$$k_{\pm,b} := \Upsilon_{\lambda_+} + (2\mu \pm \Upsilon_{\lambda_-+\lambda_+})\mathrm{e}^{(2\mu\pm\Upsilon_{\lambda_-+\lambda_+})b}\mathrm{sh}(\Upsilon_{\lambda_+}b) - \Upsilon_{\lambda_+}\mathrm{e}^{(2\mu\pm\Upsilon_{\lambda_-+\lambda_+})b}\mathrm{ch}(\Upsilon_{\lambda_+}b),$$
$$d_{\pm,a} := (-2\mu \pm \Upsilon_{\lambda_-+\lambda_+})\mathrm{sh}(\Upsilon_{\lambda_-}a) + \Upsilon_{\lambda_-}\mathrm{ch}(\Upsilon_{\lambda_-}a) - \Upsilon_{\lambda_-}\mathrm{e}^{(-2\mu\pm\Upsilon_{\lambda_-+\lambda_+})a},$$
$$d_{\pm,b} := (-2\mu \pm \Upsilon_{\lambda_-+\lambda_+})\mathrm{sh}(\Upsilon_{\lambda_+}b) - \Upsilon_{\lambda_+}\mathrm{ch}(\Upsilon_{\lambda_+}b) + \Upsilon_{\lambda_+}\mathrm{e}^{-(-2\mu\pm\Upsilon_{\lambda_-+\lambda_+})b}.$$

Further define

$$M(x,y) := \frac{(\Upsilon_{\lambda_-+\lambda_+}-\mu)\mathrm{e}^{\mu x+(\mu+\Upsilon_{\lambda_-+\lambda_+})y}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}x)}{f_{\lambda_-+\lambda_+}(a,b)},$$

$$A_1 := \frac{2(1-\frac{\mu}{\Upsilon_{\lambda_-+\lambda_+}})\mathrm{e}^{\mu b}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}b)}{f_{\lambda_-+\lambda_+}(a,b)} + \frac{\lambda_+}{\mathrm{e}^{-\mu a}\mathrm{sh}(\Upsilon_{\lambda_-}a)} \times \Bigg[c_-k_{-,a}\Bigg(\frac{1}{2(\mu+\Upsilon_{\lambda_-+\lambda_+})}+\frac{M(-a,-b)}{\Upsilon_{\lambda_-+\lambda_+}(\mu+\Upsilon_{\lambda_-+\lambda_+})}\Bigg) - \frac{M(b,a)}{\Upsilon^2_{\lambda_-+\lambda_+}}\Bigg(c_+k_{+,a}-\frac{c_-k_{-,a}\mu}{\mu+\Upsilon_{\lambda_-+\lambda_+}}\Bigg)\Bigg],$$

$$A_2 := \frac{-2(1-\frac{\mu}{\Upsilon_{\lambda_-+\lambda_+}})\mathrm{e}^{-\mu a}\mathrm{sh}(\Upsilon_{\lambda_-+\lambda_+}a)}{f_{\lambda_-+\lambda_+}(a,b)} + \frac{\lambda_-}{\mathrm{e}^{\mu b}\mathrm{sh}(\Upsilon_{\lambda_+}b)} \times \Bigg[l_-k_{-,b}\Bigg(\frac{1}{2(\mu+\Upsilon_{\lambda_-+\lambda_+})}-\frac{M(b,a)}{\Upsilon_{\lambda_-+\lambda_+}(\mu+\Upsilon_{\lambda_-+\lambda_+})}\Bigg) + \frac{M(-a.-b)}{\Upsilon^2_{\lambda_-+\lambda_+}}\Bigg(l_+k_{+,b}-\frac{l_-k_{-,b}\mu}{\mu+\Upsilon_{\lambda_-+\lambda_+}}\Bigg)\Bigg]$$

and

$$B-1 := -\frac{\lambda_+}{\mathrm{sh}(\Upsilon_{\lambda_-}a)}\Bigg[c_-d_{+,a}\Bigg(\frac{1}{2(\mu+\Upsilon_{\lambda_-+\lambda_+})}\frac{M(-a,-b)}{\Upsilon_{\lambda_-+\lambda_+}(\mu+\Upsilon_{\lambda_-+\lambda_+})}\Bigg) - \frac{M(b,a)}{\Upsilon^2_{\lambda_-+\lambda_+}}\Bigg(c_+d_{-,a}-\frac{c_-d_{+,a}\mu}{\mu+\Upsilon_{\lambda_-+\lambda_+}}\Bigg)\Bigg] - \frac{\lambda_-}{\mathrm{sh}(\Upsilon_{\lambda_+}b)}\Bigg[l_-d_{+,b}\Bigg(\frac{1}{2(\mu+\Upsilon_{\lambda_-+\lambda_+})}-\frac{M(b,a)}{\Upsilon_{\lambda_-+\lambda_+}(\mu+\Upsilon_{\lambda_-+\lambda_+})}\Bigg) + \frac{M(-a,-b)}{\Upsilon^2_{\lambda_-+\lambda_+}}\Bigg(l_+d_{-,b}-\frac{l_-d_{+,b}\mu}{\mu+\Upsilon_{\lambda_-+\lambda_+}}\Bigg)\Bigg].$$

Combining Theorems 3.1, 3.2 and 3.3, we have for $a < x \le 0$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_a<\tau_b] = \frac{\mathrm{e}^{\mu(a-x)}\mathrm{sh}(\Upsilon_{\lambda_-}x)}{\mathrm{sh}(\Upsilon_{\lambda_-}a)} + \frac{\mathrm{e}^{-\mu x}\mathrm{sh}(\Upsilon_{\lambda_-}(a-x))A_1}{\mathrm{sh}(\Upsilon_{\lambda_-}a)B},$$

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_b<\tau_a] = \frac{\mathrm{e}^{-\mu x}\mathrm{sh}(\Upsilon_{\lambda_-}(a-x))A_2}{\mathrm{sh}(\Upsilon_{\lambda_-}a)B}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds}] = \frac{\mathrm{e}^{\mu(a-x)}\mathrm{sh}(\Upsilon_{\lambda_-}x)}{\mathrm{sh}(\Upsilon_{\lambda_-}a)} + \frac{\mathrm{e}^{-\mu x}\mathrm{sh}(\Upsilon_{\lambda_-}(a-x))(A_1+A_2)}{\mathrm{sh}(\Upsilon_{\lambda_-}a)B};$$

for $0 \le x < b$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_a<\tau_b] = \frac{\mathrm{e}^{\mu x}\mathrm{sh}(\Upsilon_{\lambda_+}(b-x))A_1}{\mathrm{sh}(\Upsilon_{\lambda_+}b)B},$$

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_b<\tau_a] = \frac{\mathrm{e}^{\mu(x-b)}\mathrm{sh}(\Upsilon_{\lambda_+}x)}{\mathrm{sh}(\Upsilon_{\lambda_+}b)} + \frac{\mathrm{e}^{\mu x}\mathrm{sh}(\Upsilon_{\lambda_+}(b-x))A_2}{\mathrm{sh}(\Upsilon_{\lambda_+}b)B}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-\lambda_- \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(a,0)}(X_s)ds-\lambda_+ \int_0^{\tau_a\wedge\tau_b} \mathbf{1}_{(0,b)}(X_s)ds}]$$
$$= \frac{\mathrm{e}^{\mu(x-b)}\mathrm{sh}(\Upsilon_{\lambda_+}x)}{\mathrm{sh}(\Upsilon_{\lambda_+}b)} + \frac{\mathrm{e}^{\mu x}\mathrm{sh}(\Upsilon_{\lambda_+}(b-x))(A_1+A_2)}{\mathrm{sh}(\Upsilon_{\lambda_+}b)B}.$$

Letting $\mu = 0$, we can recover the results of Brownian motion.

4.3 Skew Brownian Motion

The skew Brownian motion is a natural generalization of the Brownian motion, which is proposed in Itô and McKean [7]. We now briefly introduce the skew Brownian motion. Let X be a skew Brownian motion of parameter β with $\beta \in (0,1)$. Process X is specified by the following stochastic differential equation:

$$\mathrm{d}X_t = \mathrm{d}W_t + (2\beta - 1)\mathrm{d}L_t^0(X), \tag{4.4}$$

where W_t is a one-dimensional standard Brownian motion and $L_t^0(X)$ is the local time at 0 for X. (4.4) has a strong unique solution, see Lejay [9]. In addition, from Borodin and Salminen [3, p.126], we have

$$m(\mathrm{d}x) = \begin{cases} 2\beta\mathrm{d}x, & x > 0, \\ 2(1-\beta)\mathrm{d}x, & x < 0; \end{cases}$$
$$s(x) = \begin{cases} \frac{x}{\beta}, & x \geq 0, \\ \frac{x}{1-\beta}, & x < 0; \end{cases}$$

and

$$G_\lambda(x,y) = \frac{\mathrm{e}^{-|x-y|\sqrt{2\lambda}} - \mathrm{e}^{-(|x|+|y|)\sqrt{2\lambda}}}{\sqrt{2\lambda}(1+(2\beta-1)\mathrm{sgn}(x\wedge y))} + \frac{\mathrm{e}^{-(|x|+|y|)\sqrt{2\lambda}}}{\sqrt{2\lambda}}.$$

In addition, $\omega_\lambda = \sqrt{2\lambda}$. We refer to Borodin and Salminen [3] and Lejay [9] for more details about skew Brownian motion and Appuhamillage et al. [2] for an occupation time related results on skew Brownian motion.

Then for $a < 0 < b$, the corresponding differential equation (2.4) of skew BM is

$$2\lambda(1-\beta)\int_{[a,0)} g(x)dx + 2\lambda\beta\int_{[0,b)} g(x)dx = \beta g'_-(b) - (1-\beta)g'_-(a),$$

where g'_- denotes the usual left derivative. We can obtain the two independent positive solutions

$$g_{-,\lambda}(x) = \left[\frac{1-2\beta}{1-\beta}\mathrm{sh}(x\sqrt{2\lambda}) + \mathrm{e}^{-x\sqrt{2\lambda}}\right]\mathbf{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x\sqrt{2\lambda}}\mathbf{1}_{[0,\infty)}(x),$$

and

$$g_{+,\lambda}(x) = \mathrm{e}^{x\sqrt{2\lambda}}\mathbf{1}_{(-\infty,0]}(x) + \left[\frac{1-2\beta}{\beta}\mathrm{sh}(x\sqrt{2\lambda}) + \mathrm{e}^{x\sqrt{2\lambda}}\right]\mathbf{1}_{(0,\infty)}(x).$$

For $x, y > 0$,

$$f_\lambda(x,y) = \frac{\mathrm{e}^{\sqrt{2\lambda}(y-x)} - \mathrm{e}^{-\sqrt{2\lambda}(y-x)}}{2\beta};$$

for $x < 0, y > 0$,

$$f_\lambda(x,y) = \frac{(1-2\beta)^2}{(1-\beta)\beta}\mathrm{sh}(x\sqrt{2\lambda})\mathrm{sh}(y\sqrt{2\lambda}) + \frac{1-2\beta}{1-\beta}\mathrm{sh}(x\sqrt{2\lambda})\mathrm{e}^{y\sqrt{2\lambda}} + \frac{1-2\beta}{\beta}\mathrm{sh}(y\sqrt{2\lambda})\mathrm{e}^{-x\sqrt{2\lambda}} + \mathrm{e}^{\sqrt{2\lambda}(y-x)} - \mathrm{e}^{-\sqrt{2\lambda}(y-x)};$$

and for $x < 0, y < 0$,

$$f_\lambda(x,y) = \frac{1}{2(1-\beta)}(\mathrm{e}^{\sqrt{2\lambda}(y-x)} - \mathrm{e}^{-\sqrt{2\lambda}(y-x)}).$$

We consider the case of $a < 0, r = 0$ and $b > 0$. After some computations, we have for $x \le 0$,

$$I_\lambda(a,b,0,x) = \frac{\mathrm{e}^{x\sqrt{2\lambda}}}{\sqrt{2\lambda}} - \frac{\mathrm{sh}(b\sqrt{2\lambda})(\mathrm{e}^{-\sqrt{2\lambda}(x-a)} + (1-2\beta)\mathrm{e}^{\sqrt{2\lambda}(x+a)})}{2\sqrt{2\lambda}(1-\beta)\beta f_\lambda(a,b)} + \frac{\mathrm{e}^{\sqrt{2\lambda}(x-b)}\mathrm{sh}(a\sqrt{2\lambda})}{\sqrt{2\lambda}(1-\beta)f_\lambda(a,b)},$$

and for $x \ge 0$,

$$I_\lambda(a,b,0,x) = \frac{\mathrm{e}^{-x\sqrt{2\lambda}}}{\sqrt{2\lambda}} - \frac{\mathrm{e}^{-\sqrt{2\lambda}(x-a)}\mathrm{sh}(b\sqrt{2\lambda})}{\beta\sqrt{2\lambda}f_\lambda(a,b)} + \frac{\mathrm{sh}(a\sqrt{2\lambda})(\mathrm{e}^{\sqrt{2\lambda}(x-b)} + (2\beta-1)\mathrm{e}^{-\sqrt{2\lambda}(x+b)})}{2\sqrt{2\lambda}\beta(1-\beta)f_\lambda(a,b)}.$$

Define functions

$$F_1(x,y,z) := -\sqrt{2\lambda_-}\mathrm{e}^{x\sqrt{2(\lambda_-+\lambda_+)}} + \sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(z\sqrt{2\lambda_-})\mathrm{e}^{y\sqrt{2(\lambda_-+\lambda_+)}} + \sqrt{2\lambda_-}\mathrm{ch}(z\sqrt{2\lambda_-})\mathrm{e}^{y\sqrt{2(\lambda_-+\lambda_+)}}$$

and

$$F_2(x,y,z) := \sqrt{2\lambda_+}\mathrm{e}^{x\sqrt{2(\lambda_-+\lambda_+)}} + \sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(z\sqrt{2\lambda_+})\mathrm{e}^{y\sqrt{2(\lambda_-+\lambda_+)}} - \sqrt{2\lambda_+}\mathrm{ch}(z\sqrt{2\lambda_+})\mathrm{e}^{y\sqrt{2(\lambda_-+\lambda_+)}}.$$

In addition, put

$$\begin{aligned} A_1^* &:= \lambda_+\int_a^0 I_{\lambda_-+\lambda_+}(a,b,0,x)\frac{f_{\lambda_-}(x,0)}{f_{\lambda_-}(a,0)}m(dx) \\ &= \frac{1}{\sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(a\sqrt{2\lambda_-})} \\ &\quad \times\left[(1-\beta)F_1(0,a,-a) - \frac{\mathrm{sh}(b\sqrt{2(\lambda_-+\lambda_+)})(F_1(a,0,a)+(1-2\beta)F_1(a,2a,-a))}{2\beta f_{\lambda_-+\lambda_+}(a,b)}\right. \\ &\quad \left. + \frac{\mathrm{sh}(a\sqrt{2(\lambda_-+\lambda_+)})F_1(-b,a-b,-a)}{f_{\lambda_-+\lambda_+}(a,b)}\right], \end{aligned}$$

$$\begin{aligned} A_2^* &:= \lambda_+\int_a^0 I_{\lambda_-+\lambda_+}(a,b,0,x)\frac{f_{\lambda_-}(a,x)}{f_{\lambda_-}(a,0)}m(dx) \\ &= \frac{1}{\sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(a\sqrt{2\lambda_-})} \\ &\quad \times\left[(1-\beta)F_1(a,0,a) - \frac{\mathrm{sh}(b\sqrt{2(\lambda_-+\lambda_+)})(F_1(0,a,-a)+(1-2\beta)F_1(2a,a,a))}{2\beta f_{\lambda_-+\lambda_+}(a,b)}\right. \end{aligned}$$

$$+\frac{\mathrm{sh}(a\sqrt{2(\lambda_-+\lambda_+)})F_1(a-b,-b,a)}{f_{\lambda_-+\lambda_+}(a,b)}\Big],$$

$$\begin{aligned}A_3^* &:= \lambda_-\int_0^b I_{\lambda_-+\lambda_+}(a,b,0,x)\frac{f_{\lambda_+}(x,b)}{f_{\lambda_+}(0,b)}m(dx)\\ &= \frac{1}{\sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(b\sqrt{2\lambda_+})}\\ &\quad\times\Big[\beta F_2(-b,0,b)-\frac{\mathrm{sh}(a\sqrt{2(\lambda_-+\lambda_+)})(F_2(0,-b,-b)+(1-2\beta)F_2(-2b,-b,b))}{2(\beta-1)f_{\lambda_-+\lambda_+}(a,b)}\\ &\quad-\frac{\mathrm{sh}(b\sqrt{2(\lambda_-+\lambda_+)})F_2(a-b,a,b)}{f_{\lambda_-+\lambda_+}(a,b)}\Big],\\ A_4^* &:= \lambda_-\int_0^b I_{\lambda_-+\lambda_+}(a,b,0,x)\frac{f_{\lambda_+}(0,x)}{f_{\lambda_+}(0,b)}m(dx)\\ &= \frac{1}{\sqrt{2(\lambda_-+\lambda_+)}\mathrm{sh}(b\sqrt{2\lambda_+})}\\ &\quad\times\Big[\beta F_2(0,-b,-b)-\frac{\mathrm{sh}(a\sqrt{2(\lambda_-+\lambda_+)})(F_2(-b,0,b)+(1-2\beta)F_2(-b,-2b,-b))}{2(\beta-1)f_{\lambda_-+\lambda_+}(a,b)}\\ &\quad-\frac{\mathrm{sh}(b\sqrt{2(\lambda_-+\lambda_+)})F_2(a,a-b,-b)}{f_{\lambda_-+\lambda_+}(a,b)}\Big].\end{aligned}$$

Combining Theorems 3.1, 3.2 and 3.3 we have for $a<x\le 0$,

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_a<\tau_b]\\ &\quad=\frac{\mathrm{sh}(x\sqrt{2\lambda_-})}{\mathrm{sh}(a\sqrt{2\lambda_-})}-\frac{\mathrm{sh}(\sqrt{2\lambda_-}(x-a))}{\mathrm{sh}(a\sqrt{2\lambda_-})}f_-(0),\\ &\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_b<\tau_a]\\ &\quad=-\frac{\mathrm{sh}(\sqrt{2\lambda_-}(x-a))}{\mathrm{sh}(a\sqrt{2\lambda_-})}f_+(0),\\ &\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds}]\\ &\quad=\frac{\mathrm{sh}(x\sqrt{2\lambda_-})}{\mathrm{sh}(a\sqrt{2\lambda_-})}-\frac{\mathrm{sh}(\sqrt{2\lambda_-}(x-a))}{\mathrm{sh}(a\sqrt{2\lambda_-})}(f_-(0)+f_+(0));\end{aligned}$$

and for $0\le x<b$,

$$\begin{aligned}&\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_a<\tau_b]\\ &\quad=\frac{\mathrm{sh}(\sqrt{2\lambda_+}(b-x))}{\mathrm{sh}(b\sqrt{2\lambda_+})}f_-(0),\\ &\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds};\tau_b<\tau_a]\\ &\quad=\frac{\mathrm{sh}(x\sqrt{2\lambda_+})}{\mathrm{sh}(b\sqrt{2\lambda_+})}+\frac{\mathrm{sh}(\sqrt{2\lambda_+}(b-x))}{\mathrm{sh}(b\sqrt{2\lambda_+})}f_+(0),\\ &\mathbb{E}_x[\mathrm{e}^{-\lambda_-\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(a,0)}(X_s)ds-\lambda_+\int_0^{\tau_a\wedge\tau_b}\mathbf{1}_{(0,b)}(X_s)ds}]\\ &\quad=\frac{\mathrm{sh}(x\sqrt{2\lambda_+})}{\mathrm{sh}(b\sqrt{2\lambda_+})}+\frac{\mathrm{sh}(\sqrt{2\lambda_+}(b-x))}{\mathrm{sh}(b\sqrt{2\lambda_+})}(f_-(0)+f_+(0)),\end{aligned}$$

where

$$f_-(0) = \frac{\frac{\mathrm{sh}(b\sqrt{2(\lambda_-+\lambda_+)})}{\beta f_{\lambda_-+\lambda_+}(a,b)} + A_1^*}{1 - A_2^* - A_3^*}$$

and

$$f_+(0) = \frac{\frac{\mathrm{sh}(a\sqrt{2(\lambda_-+\lambda_+)})}{(\beta-1) f_{\lambda_-+\lambda_+}(a,b)} + A_4^*}{1 - A_2^* - A_3^*}.$$

Letting $\beta = \frac{1}{2}$, one can recover the well-known results for Brownian motion.

Acknowledgements We thank the referee for helpful suggestions and comments.

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