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Judicious Bisection of Hypergraphs

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Abstract Judicious bisection of hypergraphs asks for a balanced bipartition of the vertex set that optimizes several quantities simultaneously. In this paper, we prove that if G is a hypergraph with n vertices and m_i edges of size i for $i = 1, 2, \dots, k$, then G admits a bisection in which each vertex class spans at most $\frac{m_1}{2} + \frac{1}{4}m_2 + \cdots + (\frac{1}{2^k})m_k + o(m_1 + \cdots + m_k)$ edges, where G is dense enough or $\Delta(G) = o(n)$ but has no isolated vertex, which turns out to be a bisection version of a conjecture proposed by Bollobás and Scott.

Keywords Partition, judicious bisection, hypergraph

MR(2010) Subject Classification 05B05, 05B25, 20B25

1 Introduction

Some notations we use in this paper is in order. Let G be a hypergraph with n vertices and m edges and let S, V_1, V_2 be subsets of $V(G)$, respectively. We denote by $e(S)$ the number of edges of which all end points are in S, $d(S)$ the number of edges adjacent to at least one vertex in S, and $e(V_1, V_2)$ the number of edges adjacent to both V_1 and V_2. Especially for hypergraphs, we denote by $E_i(G)$ the set of edges of size i, and each edge in $E_i(G)$ is called an i-edge, and $d_i(v)$ the number of edges adjacent to v in $E_i(G)$. Thus, the degree of v is $d(v) = \sum_{i=1}^{k} d_i(v)$. As in graphs, we denote the maximum degree of G by $\Delta = \max_{v \in V(G)} d(v)$.

Graph partitioning problems often seek a partition of a graph or hypergraph which optimizes a single or several quantities. Max Cut [6] and the Bottleneck Bipartition Problem [7] are two examples.

Unlike classical partitioning problems, judicious partitioning problems seek optimize more than one parameters simultaneously. In the judicious bipartition of hypergraphs problems, an early conjecture of Bollobás and Scott [3] was stated as follows.

Conjecture 1.1 *Any hypergraph with m_i edges of size i, $1 \le i \le k$ admits a bipartition of vertex set into V_1, V_2, such that for $i = 1, 2$,*

$$e(V_i) \le \frac{m_1}{2} + \frac{1}{4}m_2 + \cdots + \left(\frac{1}{2^k}\right)m_k + o(m_1 + \cdots + m_k). \tag{1.1}$$

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Ma et al. [5] proved that a hypergraph G with m_i edges of size i, $1 \le i \le 2$, admits a bipartition V_1, V_2 such that $d(V_i) \ge \frac{m_1}{2} + \frac{3}{4}m_2 + o(m_2)$. Note that for bipartition, we have $d(V_1) + e(V_2) = m_1 + m_2$. So this result partially established this conjecture for the case when $k = 2$.

Judicious k-partition asks for a required partition of vertex set into k parts, where k is an integer. For uniform hypergraphs, Bollobás and Scott [2] proved that every 3-uniform hypergraph with m edges has a vertex-partition into k sets V_i, such that $e(V_i) \le \frac{(1+o(1))}{k^3}m$ for each $i = 1, 2, \ldots, k$. For mixed hypergraphs, Ma et al. [5] showed that a hypergraph G with m_i edges of size i, for $i = 1, 2$, admits k-partition $V_1, \ldots, V_k$, such that for each $j = 1, 2, \ldots, k$, $e(V_j) \le \frac{m_1}{k} + \frac{m_2}{k^2} + o(m_2)$. For more results and problems in k-partition problems, the readers are referred to [1, 3, 8].

In this paper, we focus on judicious bisection of hypergraphs, which is a bipartition V_1, V_2 of vertex set with $||V_1| - |V_2|| \le 1$. A bisection is also called a balanced bipartition, but the extremal problems for balanced partitions have been relatively little investigated [3]. Compared to judicious bipartitions, judicious bisections are more intricate to analyze. For instance, there even are not any result or conjecture on judicious bisection of hypergraphs.

About hypergraphs, we define a "star" S by adding a new vertex v to a $(k-1)$-uniform complete hypergraph G of order n, such that v is contained in every edge of G. It is easy to check that S has $m = \binom{n}{k-1}$ edges, and in every bisection of S, $e(V_i) = \binom{\frac{n}{2}}{k-1}$ when n is even. In fact, for large n, $\frac{e(V_i)}{m}$ is far beyond $\frac{1}{2^k}$. In this paper, we show that the following theorem is true.

Theorem 1.2 *Let ε be a fixed positive constant and let G be a hypergraph, with n vertices and m_i edges of size i for $i = 1, 2, \ldots, k$, such that* (i) $m_i \ge \varepsilon^{-2} \cdot \frac{i(k-1)}{2^{i-1}}\binom{n}{i-1}$, *or* (ii) $\Delta \le \frac{\varepsilon^2}{(k-1)^3 k}n$ *and has no isolated vertex. If n is sufficiently large, then G admits a bisection $V = V_1 \cup V_2$ such that*

$$e(V_i) \le \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \varepsilon(m_1 + m_2 + \cdots + m_k). \tag{1.2}$$

In Section 2, Theorem 1.2 is proved, and then we study judicious bisection of k-uniform hypergraphs. In Section 3, we present some discussion and propose several problems.

2 Hypergraphs with Edges of Size at Most k

To prove Theorem 1.2, we first design a basic random bisection algorithm to locate the 1-edges equally in each vertex class, then we analyze the variance of the number of edges within the two vertex sets. It turns out that a second-moment argument would be useful to prove Theorem 1.2.

We begin by pairing the vertices, which itself is a 1-edge. Let T be the set of vertices in which each vertex is an edge in G, and then run the following algorithm:

Step 1 To randomly pair the vertices within T and within $V(G)\backslash T$ respectively as many as possible;

Step 2 To split each pair across $V_1 \cup V_2$ independently and uniformly at random.

Note that under this basic algorithm, each V_i contains exactly $\frac{m_1}{2}$ of the 1-edges if m_1 is even, or $\frac{m_1 \pm 1}{2}$ of the 1-edges if m_1 is odd. In the remained part of this paper, we can always assume that m_1 is even. Otherwise, we let any of the vertex in $V(G)\backslash T$ be a new 1-edge, and this affect the number of edges in each V_i by at most 1, which can be absorbed in the term εm_1. We have the following lemma.

Lemma 2.1 *Let G be an n-vertex hypergraph with m_t edges of size t, $t = 1, 2, \ldots, k$. Then G admits a bisection such that*

$$e(V_i) \le \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \sum_{j=2}^{k} \sqrt{2(k-1)\Lambda_j}, \quad i = 1, 2, \tag{2.1}$$

where

$$\Lambda_j = \frac{1}{2^j}\bigg(\Big(1 - \frac{1}{2^j}\Big)m_j + \Big(\frac{1}{2} - \frac{1}{2^j}\Big)\sum_{v\in V} d_j(v)^2\bigg).$$

Proof Let X_i be the number of edges contained in each V_i, and X_i^j be the number of j-edges lied in V_i after running the basic bisection algorithm as described at the beginning of Section 2 for $i = 1, 2$, $j = 1, 2, \ldots, k$. It is clear that $X_i = \sum_{j=2}^k X_i^j + \frac{m_1}{2}$, and $\mathbb{E}[X_i^j] = \frac{m_j}{2^j}$. We need to prove the following inequality:

$$\mathbb{P}\bigg(X_i^j \ge \frac{m_j}{2^j} + \sqrt{2(k-1)\Lambda_j}\bigg) < \frac{1}{2(k-1)} \quad \text{for } i = 1, 2,\ j = 2, 3, \ldots, k. \tag{2.2}$$

That is, with positive probability, $X_i^j < \frac{m_j}{2^j} + \sqrt{2(k-1)\Lambda_j}$ holds for all $i = 1, 2$, $j = 2, 3, \ldots, k$ simultaneously. It implies that there exists a bisection such that

$$\begin{aligned} e(V_i) = X_i &= \sum_{j=2}^{k} X_i^j + \frac{m_1}{2} \\ &\le \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \sum_{j=2}^{k} \sqrt{2(k-1)\Lambda_j}. \end{aligned} \tag{2.3}$$

By Chebyshev's inequality, we have

$$\mathbb{P}\big(X_i^j \ge \mathbb{E}[X_i^j] + \sqrt{2(k-1)\Lambda_j}\big) \le \frac{\mathrm{Var}[X_i^j]}{2(k-1)\Lambda_j}. \tag{2.4}$$

Thus it suffices to show that $\mathrm{Var}[X_i^j] < \Lambda_j$. For fixed j, we define an indicator variable I_e for each j-edge e as follows:

$$I_e = \begin{cases} 1, & \text{if all the } j \text{ endpoints of } e \text{ fall in } V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} \mathrm{Var}[X_1^j] &= \mathbb{E}[(X_1^j - \mathbb{E}[X_1^j])^2] \\ &= \mathbb{E}\bigg[\bigg(\sum_{e\in E_j(G)} (I_e - \mathbb{E}[I_e])\bigg)^2\bigg] \\ &= \mathbb{E}\bigg[\sum_{e\in E_j(G)} (I_e - \mathbb{E}[I_e])^2 + \sum_{e\ne f\in E_j(G)} (I_e - \mathbb{E}[I_e])(I_f - \mathbb{E}[I_f])\bigg] \\ &= \sum_{e\in E_j(G)} (\mathbb{E}[I_e^2] - \mathbb{E}[I_e]^2) + \sum_{e\ne f\in E_j(G)} (\mathbb{E}[I_e I_f] - \mathbb{E}[I_e]\mathbb{E}[I_f]) \end{aligned}$$

$$\leq \left(\frac{1}{2^j} - \frac{1}{2^{2j}}\right)m_j + \sum_{e\neq f\in E_j(G)} (\mathbb{E}[I_eI_f] - \mathbb{E}[I_e]\mathbb{E}[I_f]).$$

The last inequality holds because $\mathbb{E}[I_e^2] - \mathbb{E}[I_e]^2 = 0$ if at least two of the endpoints of e were initially paired, otherwise it is exactly $\frac{1}{2^j} - (\frac{1}{2^j})^2$.

We now bound the second sum in the above inequality. Note that only when all the j endpoints of e and f are not paired could the term in the second sum be none zero. In addition, if e and f share no common endpoint, then I_e, I_f are independent random variables, from which comes $\mathbb{E}[I_eI_f] - \mathbb{E}[I_e]\mathbb{E}[I_f] = 0$. So we only need to consider the edges e and f sharing at least 1 endpoint and of whose no two endpoints are paired initially. In those cases, $\mathbb{E}[I_e]\mathbb{E}[I_f] = \frac{1}{2^{2j}}$, and $\mathbb{E}[I_eI_f] \leq \frac{1}{2^{j+1}}$. This is true because for two j-edges $e \neq f$ sharing at least 1 endpoint, they contains at least $j+1$ distinct endpoints and only when all of the endpoints lie in V_1 could ensure $\mathbb{E}[I_eI_f] > 0$, observing that the probability that all unpaired vertices lie in V_1 is at most $\frac{1}{2^{j+1}}$. Hence, we have

$$\begin{aligned}\sum_{e\neq f\in E_j(G)} (\mathbb{E}[I_eI_f] - \mathbb{E}[I_e]\mathbb{E}[I_f]) &\leq \sum_{e\neq f,\text{adjacent}} \left(\frac{1}{2^{j+1}} - \frac{1}{2^{2j}}\right)\\ &\leq \left(\frac{1}{2^{j+1}} - \frac{1}{2^{2j}}\right)\sum_{v\in V} d_j(v)(d_j(v)-1)\\ &< \left(\frac{1}{2^{j+1}} - \frac{1}{2^{2j}}\right)\sum_{v\in V} d_j(v)^2. \end{aligned} \tag{2.5}$$

Therefore,

$$\mathrm{Var}[X_1^j] \leq \left(\frac{1}{2^j} - \frac{1}{2^{2j}}\right)m_j + \left(\frac{1}{2^{j+1}} - \frac{1}{2^{2j}}\right)\sum_{v\in V} d_j(v)^2 = \Lambda_j. \tag{2.6}$$

By symmetry, the same inequality holds for $\mathrm{Var}[X_2^j]$, and this completes the proof. □

Proof of Theorem 1.2 Let $\varepsilon > 0$ be given.

(i) First we suppose that $m_i \geq \varepsilon^{-2} \cdot \frac{i(k-1)}{2^{i-1}}\binom{n}{i-1}$ for $i = 2, 3, \ldots, k$.

First, we have $\sum_{v\in V} d_j(v) = jm_j$. Then, since the number of j-edge that contains v is at most $\binom{n}{j-1}$, thus $d_j(v) \leq \binom{n}{j-1}$. Finally, using this bound and letting n be sufficiently large, we have

$$\begin{aligned}\Lambda_j &= \frac{1}{2^j}\left(\left(1 - \frac{1}{2^j}\right)m_j + \left(\frac{1}{2} - \frac{1}{2^j}\right)\sum_{v\in V} d_j(v)^2\right)\\ &\leq \frac{1}{2^j}\left(\left(1 - \frac{1}{2^j}\right)m_j + \left(\frac{1}{2} - \frac{1}{2^j}\right)\binom{n}{j-1}\sum_{v\in V} d_j(v)\right)\\ &= \frac{1}{2^j}\left(\left(1 - \frac{1}{2^j}\right)m_j + \left(\frac{1}{2} - \frac{1}{2^j}\right)\binom{n}{j-1}\cdot jm_j\right),\end{aligned}$$

which is less than $\frac{1}{2^j}\binom{n}{j-1}jm_j$ for large n. Combined with the condition (i), we have

$$\Lambda_j \leq \frac{1}{2^j}\binom{n}{j-1}jm_j \leq \frac{\varepsilon^2 m_j^2}{2(k-1)}.$$

This together with Lemma 3.1 shows that there exists a bisection such that

$$e(V_i) \le \frac{1}{2}m_1 + \frac{1}{4}m_2 + \cdots + \frac{m_k}{2^k} + \sum_{i=2}^{k} \sqrt{2(k-1) \cdot \frac{\varepsilon^2 m_i^2}{2(k-1)}}$$
$$= \frac{1}{2}m_1 + \frac{1}{4}m_2 + \cdots + \frac{m_k}{2^k} + \varepsilon(m_2 + \cdots + m_k).$$

(ii) By the requirement of the theorem, we may now suppose that $\Delta_j \le \frac{\varepsilon^2}{(k-1)^3 k}n$ for $j = 2, 3, \ldots, k$, and G has no isolated vertex.

By applying the bound $d_j(v) \le \frac{\varepsilon^2}{(k-1)^3 k}n$ to Λ_j and assuming large n, we have

$$\Lambda_j \le \frac{1}{2^j}\left(\left(1 - \frac{1}{2^j}\right)m_j + \left(\frac{1}{2} - \frac{1}{2^j}\right)\frac{\varepsilon^2}{(k-1)^3 k}n \sum_{v \in V} d_j(v)\right)$$
$$= \frac{1}{2^j}\left(\left(1 - \frac{1}{2^j}\right)m_j + \left(\frac{1}{2} - \frac{1}{2^j}\right)\frac{\varepsilon^2}{(k-1)^3 k}n \cdot j m_j\right)$$
$$\le \frac{1}{2^j} \cdot \frac{\varepsilon^2}{(k-1)^3 k}n \cdot j m_j$$
$$= \frac{\varepsilon^2}{2(k-1)^3} \cdot \frac{j}{2^{j-1}k} n m_j.$$

Because G has no isolated vertex and the size of edge is at most k, G has at least $\frac{n}{k}$ edges. Thus, we have

$$\Lambda_j < \frac{\varepsilon^2}{2(k-1)^3} \cdot \frac{j}{2^{j-1}k} k(m_1 + m_2 + \cdots + m_k)m_j$$
$$\le \frac{\varepsilon^2}{2(k-1)^3}(m_1 + m_2 + \cdots + m_k)^2,$$

which together with Lemma 3.1 implies that G admits a bisection such that

$$e(V_i) \le \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \sum_{j=2}^{k} \sqrt{2(k-1)\frac{\varepsilon^2}{2(k-1)^3}(m_1 + m_2 + \cdots + m_k)^2}$$
$$= \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \sum_{j=2}^{k} \frac{\varepsilon}{k-1}(m_1 + m_2 + \cdots + m_k)$$
$$= \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + \varepsilon(m_1 + m_2 + \cdots + m_k), \quad i = 1, 2.$$

This completes the proof. □

Based on the above result, we immediately get the following corollary about r-uniform hypergraphs.

Corollary 2.2 *Let ε be a fixed positive constant and let G be an r-uniform hypergraph, with n vertices and m edges, such that* (i) $m \ge \varepsilon^{-2} \cdot \frac{r(r-1)}{2^{r-1}}\binom{n}{r-1}$ *or* (ii) $\Delta \le \frac{\varepsilon^2}{(r-1)^3 r}n$ *and has no isolated vertex. If n is sufficiently large, then G admits a bisection $V = V_1 \cup V_2$ in which each V_i spans at most $(\frac{1}{2^r} + \varepsilon)m$ edges.*

3 Discussion

Note that when $r = 2$, Corollary 2.2 shows a weaker result compared to [4, Theorem 1.7]. Because the condition (ii) in Theorem 1.2 additionally requires that there are no isolated vertex in G, we ask whether one can remove this from (ii)?

Conjecture 3.1 *Let G be a hypergraph, with n vertices and m_i edges of size i for $i = 1, 2, \ldots, k$, such that $\Delta = o(n)$. Then G admits a bisection $V = V_1 \cup V_2$ such that*

$$e(V_i) \leq \frac{m_1}{2} + \frac{m_2}{4} + \cdots + \frac{m_k}{2^k} + o(m_1 + m_2 + \cdots + m_k). \tag{3.1}$$

Bollobás and Scott [3] asked what is the smallest $c(k)$ such that every graph G with m edges and minimal degree k has a bisection with at most $c(k)m$ edges in each vertex class. Lee et al. [4] asymptotically answered this question by showing that graphs with m edges and minimum degree at least δ has a bisection in which for each i, the following holds:

$$e(V_i) \leq \left(\frac{\delta + 2}{4(\delta + 1)} + o(1)\right)m.$$

It is interesting to know the answer of the same question in an r-uniform hypergraphs version.

Problem What is the smallest $c_r(k)$ such that every r-uniform hypergraph G with m edges and minimal degree as least k has a bisection with at most $c_r(k)m$ edges in each vertex class?

The discussion in this section has already shown $c_r(\binom{n-2}{r-2}) = \frac{1}{2^{r-1}} + o(1)$.

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