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Existence of Periodic Solutions for Neutral Functional Differential Equations with Nonlinear Difference Operator

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Abstract In this paper, the authors consider the problem of existence of periodic solutions for a second order neutral functional differential system with nonlinear difference D-operator. For such a system, since the possible periodic solutions may not be differentiable, our method is based on topological degree theory of condensing field, not based on Leray Schauder topological degree theory associated to completely continuous field.

Keywords Neutral functional differential equation, condensing operator, topological degree, periodic solution

MR(2010) Subject Classification 34C25, 34K40, 34K37

1 Introduction

In the past fifty years, researchers have given a lot of attentions to the study of existence of periodic solutions for the following neutral functional differential equations:

$$\frac{dDx_t}{dt} = f(t, x_t), \tag{1.1}$$

where $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$, $\tau > 0$ is a constant; $D : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is linear, continuous and atomic at zero; $f \in C(\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$ with $f(t+\omega, \varphi) \equiv f(t, \varphi)$ for all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, and for any bounded set $\Omega \subset C([-\tau, 0], \mathbb{R}^n)$, $f([0, \omega] \times \Omega)$ is bounded in $\mathbb{R}^n$. This equation arises in the study of two or more simple oscillatory systems with some interconnections between them [10, 11], and in modeling physical problems such as the lossless transmission line networks and, the vibration of masses attached to an elastic bar [4, 24]. Compared with retarded functional differential equations, the study of existence of periodic solutions for (1.1) is more difficult. This due to the fact that for any ω-periodic solution $u(t)$ of (1.1), it is only required that $D(u_t)$ is continuously differentiable in t, but, generally, $u(t)$ may not be differentiable in t [1, 10, 11]. Just for this reason, the compact property of some solution operator associated to (1.1), which is crucial for solving periodic solutions, is far away from clear. Under the condition that the linear difference operator D is stable, Jack Hale obtained that any ω-periodic solution to (1.1) has a continuous first derivative [10]. By using this result, many researchers [2, 8, 13, 23] studied the existence of periodic solutions for (1.1) by means of

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some fixed point theorems. Recently, for the special case of the linear autonomous D-operator defined by $D : C([-\tau, 0], \mathbb{R}), D(\varphi) = \varphi(0) - k\varphi(-\tau)$, under the condition of $|k| > 1$, that is the linear D-operator is un-stable [10], the authors in [16] obtained that any ω-periodic solution to the difference equation $D(x_t) = h(t)$ has a continuous first derivative, where $h \in C^1(\mathbb{R}, \mathbb{R})$ with $h(t+\omega) \equiv h(t)$; and then, the existence of periodic solutions for some kinds of neutral function differential equations is studied in [12, 17–19] by using Mawhin's continuation theorem. For neutral functional differential equation with nonlinear difference D-operator, Corduneanu [5, 6] studied the existence of solution to an initial value problem, and Burton [3] investigated the problem of Perron-type stability. But the works to study the existence of periodic solutions for neutral functional differential equations with nonlinear difference D-operator rarely appeared [7, 20]. For example, in [7], by using a new topological degree theorem associated to condensing operator, Erbe et al. studied neutral functional differential equations with nonlinear difference D-operator in the following form:

$$\frac{d}{dt}(x(t) - \bar{b}(t, x_t)) = f(t, x_t), \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top \in \mathbb{R}^n$, $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$, f and $\bar{b} \in C(\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$ with $f(t+\omega, \varphi) \equiv f(t, \varphi)$ and $\bar{b}(t+\omega, \varphi) \equiv \bar{b}(t, \varphi)$ for all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, and for any bounded set $\Omega \subset C([-\tau, 0], \mathbb{R}^n)$, $f([0, \omega] \times \Omega)$ is bounded in $\mathbb{R}^n$; τ and ω are two positive constants. Under the crucial condition that $\overline{B} : P_\omega \to P_\omega, (\overline{B}x)(t) = \bar{b}(t, x_t)$ is condensing, where $P_\omega = \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t+\omega) \equiv x(t)\}$, some results on the existence of periodic solutions are obtained. In [20], the authors studied the problem of periodic solution for a neutral functional differential equation with nonlinear difference D-operator defined by

$$D : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n, \quad D(\varphi) = B\varphi(-\tau) - h(\varphi),$$

where $B = [b_{ij}]_{n \times n}$ is a real matrix and $h : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous. Clearly, the nonlinear difference D-operator in [20] is autonomous type. Through the discussion that the operator $F : \mathbb{R}^n \to \mathbb{R}^n, F(a) = (I - B)a - h(a)$ has its inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, and by calculating the Brouwer's degree

$$\deg(\Delta_1, F^{-1}(B_\rho), 0) \neq 0,$$

the main result on the existence of periodic solutions was obtained. However, if the difference D-operator is non-autonomous type, the degree $\deg(\Delta_1, F^{-1}(B_\rho), 0)$ is difficult to calculate. This is due to the fact that $F^{-1}(B_\rho)$ is in P_ω but not in $\mathbb{R}^n$.

In present paper, we consider the existence of periodic solutions for the following neutral functional differential equations with non-autonomous difference D-operator:

$$\frac{d^2}{dt^2}(x(t) - Cx(t-\tau) - b(t, x_t)) = f(t, x_t), \tag{1.3}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^\top \in \mathbb{R}^n$, $C = [c_{ij}]_{n \times n}$ is a real matrix and, $x_t \in C_\tau := C([-\tau, 0], \mathbb{R}^n)$ and functions f and b is defined as same as f and $\bar{b}$ in (1.2), respectively. By using the degree theory associated to condensing field, some new results on the existence of ω-periodic solutions to (1.3) are obtained.

The significance of our paper lies in the following two respects. Firstly, the crucial condition in our paper is that $BA^{-1} : P_\omega \to P_\omega$ is a condensing operator, while Erbe et al. in [7] require that $\overline{B}: P_\omega \to P_\omega$ is condensing, where $B: P_\omega \to P_\omega, (Bx)(t) = b(t, x_t)$ and $A: P_\omega \to P_\omega, (Ax)(t) = x(t) - Cx(t-\tau)$. The second is that the main result of our paper is related to the value of delay τ.

In [20], $\bar{b}(t,\varphi)$, which is defined in (1.2), can be regarded as $\bar{b}(t,\varphi) = C\varphi(-\tau) + b(t,\varphi)$ for all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$. Clearly, if $C = O$, then $\bar{b}(t,\varphi) \equiv b(t,\varphi)$ and the linear operator $A = I$. This implies that the nonlinear difference operator $\overline{B}: P_\omega \to P_\omega$ in [7] is equal to the operator $BA^{-1}: P_\omega \to P_\omega$. So the crucial condition in present paper generalizes the corresponding ones of [7].

2 Main Lemmas

Throughout this paper, unless otherwise specified, we use the following notation. Let E and Y be two Banach spaces with $Y \subset E$. By Θ we denote the set of all bounded subsets of E and, $\mu: \Theta \to \mathbb{R}^+$ be a measure of non-compactness. Suppose that $F: Y \to E$ is a continuous map with sending bounded sets into bounded sets. F is called a μ-Lipschitzian map with a constant $k \geq 0$, if $\mu(F(A)) \leq k\mu(A)$ for all bounded $A \subset Y$. A μ-Lipschitzian map with a constant $k \in [0,1)$ is called a k-set contractive map or a Darbo map and, if $\mu(F(A)) < \mu(A)$ for all bounded $A \subset Y$ with $\mu(A) > 0$, then F is called a condensing map. Clearly, a completely continuous map must be condensing. But, generally, the converse is not true.

Now, let us recall the definition of topological degree associated to condensing field [15, 22]. Let $\Omega \subset E$ be open bounded, F be a k-set contractive map and $0 \notin (I-F)(\partial\Omega)$. Set $D_1 = \overline{\text{co}}(F(\overline{\Omega}))$, $D_n = \overline{\text{co}}F(D_{n-1} \cap \overline{\Omega}), n = 2, 3, \ldots$.

(1) If there is an integer n_0 such that $D_{n_0} \cap \overline{\Omega} = \phi$, define

$$\deg(I - F, \Omega, 0) = 0.$$

(2) If $D_n \cap \overline{\Omega} \neq \phi$ for $n = 1, 2, \ldots$, let $D = \bigcap_{n=1}^{+\infty} D_n$. Clearly, D is a bounded closed subset of D_n $(n = 1, 2, \ldots, n)$ and $D \neq \phi$. Furthermore, $\mu(D_n) \leq k^n \mu(\overline{\Omega}) \to 0$ as $n \to \infty$, which implies that $D \cap \overline{\Omega} = \bigcap_{n=1}^{\infty}(D_n \cap \overline{\Omega})$ is compact in E. Since $F(D \cap \overline{\Omega}) \subset \bigcap_{n=1}^{\infty} F(D_n \cap \overline{\Omega}) \subset D$, $F: D \cap \overline{\Omega} \to D$ is completely continuous. By using the continuation theorem of completely continuous operator, we see that there is a completely continuous operator $F_1: \overline{\Omega} \to D$ such that $F_1(x) = F(x)$ for all $x \in D \cap \overline{\Omega}$ and $0 \notin (I-F)(\partial\Omega)$, and then the degree of k-set contractive field $I - F: \Omega \to E$ is well defined in the following form:

$$\deg(I - F, \Omega, 0) := \deg(I - F_1, \Omega, 0).$$

Suppose that $\Omega \subset E$ is open bounded and $F: \overline{\Omega} \to E$ is a a condensing map such that $0 \in E \backslash (I-F)(\partial\Omega)$. Since $(I-F)(\partial\Omega)$ is a closed bounded subset of E, $\tau = \text{dist}(0, (I-F)(\partial\Omega)) > 0$. Taking an arbitrary k-set contractive map $f: \overline{\Omega} \to E$ such that

$$\|Fx - fx\|_E < \frac{\tau}{3}, \quad \forall x \in \overline{\Omega},$$

then

$$\|(I-f)(x)\|_E \geq \|(I-F)(x)\|_E - \|Fx - fx\|_E > \frac{2}{3}\tau,$$

which implies that the degree $\deg(I - f, \Omega, 0)$ is valid. Therefore, the degree of condensing field $I - F: \Omega \to E$ can be defined as follows:

$$\deg(I - F, \Omega, 0) = \deg(I - f, \Omega, 0).$$

This degree has the following basic properties of k-set contractive field degree to which it reduces when F is a k-set contractive map [22]:

(1) *Normalization* Let $F:\overline{\Omega}\to\{*\}\subset\Omega$ be a constant map. Then

$$\deg(I-F,\Omega,0)=1.$$

(2) *Existence* If $\deg(I-F,\Omega,0)\neq 0$, then $0\in(I-F)(\Omega)$.

(3) *Additivity* If Ω_1 and Ω_2 are disjoint open subsets of Ω such that

$$0\notin(I-F)[\overline{\Omega}\setminus(\Omega_1\cup\Omega_1)],$$

then

$$\deg(I-F,\Omega,0)=\deg(I-F,\Omega_1,0)+\deg(I-F,\Omega_2,0).$$

(4) *Homotopy Invariance* Suppose that $H:[0,1]\times\overline{\Omega}\to E$ is continuous. For each $t\in[0,1]$, $H(t,\cdot):\overline{\Omega}\to E$ is a condensing operator and $H(t,x)$ is continuous with respect to t uniformly for $x\in\overline{\Omega}$. Let $F_t(x)=x-H(t,x)$. If $0\notin F_t(\partial\Omega)$ for all $t\in[0,1]$, then

$$\deg(I-F_t,\Omega,0)$$

is independent of $t\in[0,1]$.

Let $L:\mathrm{D}(L)\subset E\to Y$ be a Fredholm operator with index zero. By [9], this means that L is linear and $\mathrm{Im}L$ is closed in Y, $\dim\ker L=\dim\frac{Y}{\mathrm{Im}L}<+\infty$, and there are continuous linear projectors $P:E\to E$ and $Q:Y\to Y$ such that $\mathrm{Im}P=\ker L$, $\ker Q=\mathrm{Im}L$, $E=\ker L\oplus\ker P$ and $Y=\mathrm{Im}L\oplus\mathrm{Im}Q$. Obviously, $L:\mathrm{D}(L)\cap\ker P\to\mathrm{Im}L$ has its right inverse. Let $K_P:\mathrm{Im}L\to\mathrm{D}(L)\cap\ker P$ be the right inverse of $L:\mathrm{D}(L)\cap\ker P\to\mathrm{Im}L$ and let $J:\mathrm{Im}Q\to\ker L$ be an isomorphism, then $L+J^{-1}P:\mathrm{D}(L)\to Y$ is a bijection. Set $T_L=(L+J^{-1}P)^{-1}:Y\to\mathrm{D}(L)$. Then

$$T_LQy=JQy,\quad T_L(I-Q)y=K_P(I-Q)y\quad\text{for all } y\in Y \tag{2.1}$$

and

$$T_LJ^{-1}Px=Px\quad\text{for all } x\in E. \tag{2.2}$$

Definition 2.1 *Let $L:\mathrm{D}(L)\subset E\to Y$ be a Fredholm operator with index zero, Ω be an open bounded subset of E with $\mathrm{D}(L)\cap\Omega\neq 0$, $G:\overline{\Omega}\subset E\to Y$ be a continuous map. If $QG:\overline{\Omega}\to Y$ and $K_P(I-Q)G:\overline{\Omega}\to Y$ are all condensing, then the map G is called L-condensing in $\overline{\Omega}$.*

Lemma 2.2 *Suppose that $L:\mathrm{D}(L)\subset E\to Y$ is a Fredholm operator with index zero, $B:E\to E$ and $G:E\to Y$ are two maps and, $\Gamma:E\to E$ is a continuous invertiable map with $\Gamma^{-1}:E\to E$ being bounded. Then, the existence of solution to the equation*

$$L(\Gamma(x)-\lambda B(x))=\lambda G(x),\quad\lambda\in(0,1), \tag{2.3}$$

is equivalent to the existence of solution to the following equation:

$$\begin{aligned}x&=\lambda B(\Gamma^{-1}(x))+JQG(\Gamma^{-1}(x))+\lambda K_P(I-Q)G(\Gamma^{-1}(x))\\&\quad+P(x-\lambda B(\Gamma^{-1}(x))),\quad\lambda\in(0,1).\end{aligned} \tag{2.4}$$

Proof Suppose that $u \in E$ is a solution of (2.3). Then for all $\lambda \in (0,1)$, $\Gamma(u) - \lambda B(u) \in \mathrm{D}(L)$ and

$$L(\Gamma(u) - \lambda B(u)) = \lambda G(u),$$

which implies $G(u) \in \mathrm{Im}L$. So $QG(u) = 0$, and then

$$L(\Gamma(u) - \lambda B(u)) = QG(u) + \lambda(I - Q)G(u),$$

i.e.,

$$(L + J^{-1}P)(\Gamma(u) - \lambda B(u)) = QG(u) + \lambda(I - Q)G(u) + J^{-1}P(\Gamma(u) - \lambda B(u)),$$

which implies

$$\Gamma(u) = \lambda B(u) + T_L QG(u) + \lambda T_L(I - Q)G(u) + T_L J^{-1}P(\Gamma(u) - \lambda B(u)).$$

It follows from (2.1) and (2.2) that

$$\Gamma(u) = \lambda B(u) + JQG(u) + \lambda K_P(I - Q)G(u) + P(\Gamma(u) - \lambda B(u)).$$

Since $\Gamma : E \to E$ is a bijective continuous linear operator, it follows from Banach–Schauder theorem, we know that the linear map $\Gamma^{-1} : E \to E$ is also continuous. Let $\Gamma(u) = v$. Then $v \in E$ and

$$v = \lambda B(\Gamma^{-1}(v)) + JQG(\Gamma^{-1}(v)) + \lambda K_P(I - Q)G(\Gamma^{-1}(v)) + P(v - \lambda B(\Gamma^{-1}(v))).$$

This implies that v is a solution to (2.4).

On the other hand, suppose that $v \in E$ is a solution to (2.4), and let $u = \Gamma^{-1}(v)$. Then

$$\Gamma(u) - \lambda B(u) = JQG(u) + \lambda K_P(I - Q)G(u) + P(\Gamma(u) - \lambda B(u)),$$

which leads to

$$L(\Gamma(u) - \lambda B(u)) = \lambda G(u), \quad \lambda \in (0,1).$$

So u is a solution to (2.3).

Lemma 2.3 *Suppose that* $\Gamma : E \to E$ *is continuous and has its inverse* $\Gamma^{-1} : E \to E$ *with*

$$|\Gamma^{-1}(x_1) - \Gamma^{-1}(x_2)|_E \le k_1|x_1 - x_2|_E \quad \textit{for all } x_1, x_2 \in E, \tag{2.5}$$

where $k_1 > 0$ *is a constant. Furthermore, there is a constant* $k_2 > 0$ *such that* $k_1k_2 < 1$ *and*

$$|B(x_1) - B(x_2)|_E \le k_2|x_1 - x_2|_E \quad \textit{for all } x_1, x_2 \in E. \tag{2.6}$$

Let $L : \mathrm{D} \subset E \to Y$ *be a Fredholm operator with index zero,* $\Omega \subset E$ *is open bounded and* $G\Gamma^{-1} : \overline{\Omega} \to Y$ *is* L*-condensing in* $\overline{\Omega}$ *such that the following conditions hold*:

(1) *for every* $\lambda \in (0,1)$, *the equation*

$$L(\Gamma(x) - \lambda B(x)) = \lambda G(x)$$

has no solution in $\partial\Omega$;

(2) $QG(\Gamma^{-1}(x)) \neq 0$ *for every* $x \in \ker L \cap \partial\Omega$;

(3) $d_B\{JQG\Gamma^{-1}, \Omega \cap \ker L, 0\} \neq 0$,

where the last number denotes the Brouwer degree at $0 \in \mathrm{Im}Q$. *Then the equation*

$$L(\Gamma(x) - B(x)) = G(x)$$

has a solution in $\overline{\Omega}$.

Proof By Lemma 2.2, we see that the existence of solution to the equation

$$L(\Gamma(x)-\lambda B(x))=\lambda G(x),\quad \lambda\in(0,1)$$

is equivalent to the existence of solution to the following equation:

$$\begin{aligned}x=\Psi(x,\lambda)&:=\lambda B(\Gamma^{-1}(x))+JQG(\Gamma^{-1}(x))+\lambda K_P(I-Q)G(\Gamma^{-1}(x))\\&\quad+P(x-\lambda B(\Gamma^{-1}(x)),\quad \lambda\in(0,1).\end{aligned}\tag{2.7}$$

From (2.5)–(2.6), we see

$$|B(\Gamma^{-1}(x_1))-B(\Gamma^{-1}(x_2))|_E\le k_1k_2|x_1-x_2|_E\quad \text{for all } x_1,x_2\in E,$$

which together with $k_1k_2<1$ implies that $B\Gamma^{-1}:E\to E$ is condensing. Furthermore, since $G\Gamma^{-1}:\overline{\Omega}\to Y$ is L-condensing in $\overline{\Omega}$, we see that $QG\Gamma^{-1}$ and $K_P(I-Q)G\Gamma^{-1}$ are all condensing in $\overline{\Omega}$. It follows from (2.7) that the operator $\Psi:\overline{\Omega}\times[0,1]\to E$ is continuous, and for each $\lambda\in[0,1]$, $\Psi(\lambda,\cdot):\overline{\Omega}\to E$ is a condensing operator; also, $\Psi(\lambda,x)$ is continuous with respect to λ uniformly for $x\in\overline{\Omega}$. Bellow, we will show that for every $\lambda\in[0,1]$, the equation

$$x=\Psi(x,\lambda)$$

has no solution in $\partial\Omega$. In fact, if there is a $u\in E$ such that $u=\Psi(u,0)$, then from (2.7), we have

$$u=JQG(\Gamma^{-1}(u))+P(u),$$

which gives $u\in\ker L$. Therefore, by the assumption (2), $u\notin\partial\Omega$.

If there is a $u\in\partial\Omega$ such that

$$u=\Psi(u,1),$$

then the equation $L(\Gamma(x)-B(x))=G(x)$ has a solution in $\overline{\Omega}$. So, we may assume that the equation

$$x=\Psi(x,\lambda)$$

has no solution $u\in\partial\Omega$ for $\lambda=1$.

For $\lambda\in(0,1)$, it is easy to see from the assumption (1) that the equation

$$x=\Psi(x,\lambda)$$

has no solution $u\in\partial\Omega$. Thus, the operator $\Psi(x,\lambda)$ is homotopy in $\overline{\Omega}$ and, by homotopy invariance, we have

$$\begin{aligned}&\deg(I-\Psi(\cdot,1),\Omega,0)=\deg(I-\Psi(\cdot,0),\Omega,0)=\deg(I-JQG\Gamma^{-1}-P,\Omega,0)\\&\quad=\deg(JQG\Gamma^{-1},\ker L\cap\Omega,0).\end{aligned}$$

From the assumption (3), we see that

$$\deg(I-\Psi(\cdot,1),\Omega,0)\ne 0.$$

By using the normalization of topology degree associated to condensing field, we see that the equation $x=\Psi(x,1)$ has a solution in $\overline{\Omega}$.

In order to investigate the existence of periodic solutions to (1.3), we should give some definitions. For $a = (a_1, a_2, \ldots, a_n)^\top \in \mathbb{C}^n$ being a complex vector, $|a| = (\sum_{i=1}^n |a_i|^2)^{1/2}$, and for a complex matrix $H = [h_{ij}]_{n\times n}$, $|H| = (\sum_{i=1}^n \sum_{j=1}^n |h_{ij}|^2)^{1/2}$. Let $P_\omega = \{x : x \in C(\mathbb{R}, \mathbb{R}^n), x(t+\omega) \equiv x(t)$ for all $t \in \mathbb{R}\}$ with the norm $|\varphi|_{P_\omega} = \max_{t\in[0,\omega]} |\varphi(t)|$. Clearly, P_ω is a Banach space.

In order to study the existence of periodic solutions for (1.3), we should study some properties of the following operator:

$$A : P_\omega \to P_\omega, \quad [Ax](t) := x(t) - Cx(t-\tau), \tag{2.8}$$

where C and τ are defined as same as the ones in (1.3).

Since C is an n-order real matrix, it is easy to see that there must be a complex matrix U such that

$$UCU^{-1} = E_\lambda = \mathrm{diag}(J_1, J_2, \ldots, J_m), \tag{2.9}$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}_{n_i\times n_i}$$

with $\sum_{i=1}^m n_i = n$, $\{\lambda_i : i = 1, 2, \ldots, m\}$ is the set of eigenvalues of matrix C.

Lemma 2.4 ([18]) *Suppose that the matrix U and the operator A is defined by* (2.8) *and* (2.9)*, respectively, and for all $i = 1, 2, \ldots, m$, $|\lambda_i| \neq 1$. Then A has its inverse $A^{-1} : P_\omega \to P_\omega$ with the following properties*:

(1) $\|A^{-1}\| \le |U^{-1}\|U|\sigma_0$, *where* $\sigma_0 = \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{1}{|1-|\lambda_i\|^k}$;

(2) *For all* $e \in C_\omega$, $\int_0^\omega |[A^{-1}e](s)|^p ds \le \sigma_1 |U^{-1}|^p |U|^p \int_0^\omega |e(s)|^p ds, p \in [1, +\infty)$, *where*

$$\sigma_1 = \begin{cases} \left[\displaystyle\sum_{i=1}^m \sum_{j=1}^{n_i} \left(\sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}\right)^2\right]^{1/2}, & p = 1, \\ n^{\frac{2-p}{2}} \left[\displaystyle\sum_{i=1}^m \sum_{j=1}^{n_i} \left(\sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}\right)^q\right]^{\frac{p}{q}}, & p \in (1,2), \\ \displaystyle\sum_{i=1}^m \sum_{j=1}^{n_i} \left(\sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}\right)^2, & p = 2, \\ \left[\displaystyle\sum_{i=1}^m \sum_{j=1}^{n_i} \left(\sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}\right)^q\right]^{\frac{p}{q}}, & p \in (2, +\infty) \end{cases}$$

and $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$;

(3) $A^{-1}(\mathbb{R}^n) = \mathbb{R}^n$.

3 Main Results

In this section, we will apply Lemmas 2.3 and 2.4 to study the existence of periodic solutions for (1.3).

Theorem 3.1 *Suppose that the operator A and the matrix U is defined by* (2.8) *and* (2.9)*, respectively, and for all $i = 1, 2, \ldots, m$, $|\lambda_i| \neq 1$. Furthermore, $b(t, \varphi)$ is satisfied with*

$$|b(t,\varphi_1) - b(t,\varphi_2)| \\ \le l \max_{\theta\in[-\tau,0]} |\varphi_1(\theta) - \varphi_2(\theta)| \quad \text{for all } t \in [0,\omega] \text{ and } \varphi_1, \varphi_2 \in C([-\tau,0],\mathbb{R}^n), \tag{3.1}$$

where $l \in (0, +\infty)$ is a constant with

$$l|U^{-1}||U| \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{j} \frac{1}{|1-|\lambda_i||^k} < 1. \tag{3.2}$$

If there is a constant $\rho > 0$ such that all the following conditions hold:

(D1) *For each $\lambda \in (0, 1)$, the equation*

$$\frac{d^2}{dt^2}(x(t) - Cx(t-\tau) - \lambda b(t, x_t)) = \lambda f(t, x_t) \tag{3.3}$$

has no solution on $\partial\Omega_\rho$, where $\Omega_\rho = \{x \in P_\omega : \|x\|_{P_\omega} < \rho\}$;

(D2) *The equation*

$$\Delta(a) := \frac{1}{\omega}\int_0^\omega f(s, A^{-1}a)ds = 0$$

has no solution on ∂B_ρ, where $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$;

(D3) *The Brouwer degree*

$$\deg\{\Delta, B_\rho, 0\} \neq 0.$$

Then (1.3) *has at least one ω-periodic solution in $\overline{\Omega}_\rho$.*

Proof In order to use Lemma 2.3, set $E = Y = P_\omega$,

$$L : \mathrm{D}(L) \subset P_\omega \to P_\omega, \quad (Lx)(t) = x''(t),$$

where $D(L) = P_\omega^2$,

$$B : P_\omega \to P_\omega, \quad (Bx)(t) = b(t, x_t),$$
$$\Gamma : P_\omega \to P_\omega, \quad [\Gamma x](t) \equiv (Ax)(t) = x(t) - Cx(t-\tau),$$

and

$$G : P_\omega \to P_\omega, \quad (Gx)(t) = f(t, x_t). \tag{3.4}$$

Clearly, (3.1) is converted to

$$L[(Ax)(t) - \lambda(Bx)(t)] = \lambda(Gx)(t). \tag{3.5}$$

It is easy to see that $\ker L = \mathbb{R}^n$ and $\mathrm{Im}L = \{y \in P_\omega : \int_0^\omega y(s)ds = 0\}$. Thus, $\dim\ker L = \dim \frac{P_\omega}{\mathrm{Im}L} = n < +\infty$, which implies that L is a Fredholm operator with index zero. Furthermore, by using Lemma 2.4, we have that A has its inverse $A^{-1} : P_\omega \to P_\omega$ with

$$\|A^{-1}\| \le |U^{-1}||U| \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{j} \frac{1}{|1-|\lambda_i||^k},$$

i.e.,

$$\|A^{-1}(x_1) - A^{-1}(x_2)\|_{P_\omega}$$

$$= \|A^{-1}(x_1 - x_2)\|_{P_\omega}$$
$$\leq |U^{-1}\|U| \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{1}{|1-|\lambda_i\|^k} \|x_1 - x_2\|_{P_\omega} \quad \text{for all } x_1, x_2 \in P_\omega;$$

and by using (3.1), we see

$$\begin{aligned}
\|B(x) - B(y)\|_{P_\omega} &= \|b(t, x_t) - b(t, y_t)\|_{P_\omega} \\
&= \max_{t\in[0,\omega]} |b(t, x_t) - b(t, y_t)| \\
&\leq l \max_{t\in[0,\omega]} \max_{\theta\in[-\tau,0]} |x_t(\theta) - y_t(\theta)| = l \max_{t\in[0,\omega]} \max_{\theta\in[-\tau,0]} |x(t+\theta) - y(t+\theta)| \\
&= l\|x - y\|_{P_\omega} \text{ for all } x, y \in P_\omega,
\end{aligned}$$

which together with (3.2) yields that the conditions (2.5) and (2.6) in Lemma 2.3 hold. This implies that $BA^{-1} : P_\omega \to P_\omega$ is condensing.

Define $P : P_\omega \to P_\omega, [Px](t) = x(0)$ for all $t \in \mathbb{R}$ and $Q : P_\omega \to P_\omega, [Qy](t) = \frac{1}{\omega}\int_0^\omega y(s)ds$. Obviously, P and Q are projections and

$$P_\omega = \ker P \oplus \ker L, \quad P_\omega = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Clearly, if set $J : \mathrm{Im}Q \to \ker L, J(x) = x$, i.e., $J : \mathbb{R}^n \to \mathbb{R}^n, J(x) = x$, then $L + J^{-1}P : D(L) \subset P_\omega \to P_\omega$ has it's inverse $(L + J^{-1}P)^{-1} : P_\omega \to D(L)$. In view of $J^{-1}Px \equiv 0$ for all $x \in \ker P$, we see $T_L = (L + J^{-1}P)^{-1} = K_P : \mathrm{Im}L \to \ker P \cap D(L)$. By calculating, we have

$$K_P : \mathrm{Im}L \to \ker P \cap D(L), \quad [K_P y](t) = \int_0^\omega R(t,s)y(s)ds, \tag{3.6}$$

where

$$R(t,s) = \begin{cases} \dfrac{(t-\omega)s}{\omega}, & 0 \leq s < t \leq \omega, \\ \dfrac{(s-\omega)t}{\omega}, & 0 \leq t < s \leq \omega. \end{cases}$$

By (3.1), it is easy to see that $K_P(I-Q)GA^{-1} : \overline{\Omega}_\rho \subset P_\omega \to \ker P \cap D(L)$ and $QGA^{-1} : \overline{\Omega}_\rho \subset P_\omega \to \mathrm{Im}\ Q \subset P_\omega$ are all completely continuous operators. So GA^{-1} is L-condensing on $\overline{\Omega}_\rho$.

In view of

$$JQG\Gamma^{-1} : P_\omega \to \mathbb{R}^n, \quad (JQG\Gamma^{-1})(x) = \frac{1}{\omega}\int_0^\omega f(s, (A^{-1}x)_s)ds,$$

which together with conclusion (3) of Lemma 2.4 ($A^{-1}(\mathbb{R}^n) = \mathbb{R}^n$) yields that

$$\Delta(a) := \frac{1}{\omega}\int_0^\omega f(s, A^{-1}a)ds = (JQG\Gamma^{-1})(a) \quad \text{for all } a \in \mathbb{R}^n$$

and

$$\deg(\Delta, B_\rho, 0\} = \deg(JQG\Gamma^{-1}, \ker L \cap \Omega_\rho, 0).$$

It follows from the assumption (D3) that the assumptions (3) in Lemma 2.3 is satisfied. Furthermore, from the fact that the assumptions (D1)–(D2) are satisfied, we can verify that the assumptions (1)–(2) in Lemma 2.3 hold. Thus, by using Lemma 2.3, we see that (3.1) has at least one ω-periodic solution in Ω_ρ.

For illustrating the application of Theorem 3.1, now, we consider the following equation:

$$\frac{d^2}{dt^2}(x(t) - Cx(t-\tau) - s(x(t-\mu))) = g(x(t-\gamma)) + e(t), \tag{3.7}$$

where τ, μ and γ are positive constants, $C = [c_{ij}]_{n\times n}$ is a real matrix, $g, s \in C(\mathbb{R}^n, \mathbb{R}^n)$, and $e \in C(\mathbb{R}, \mathbb{R}^n)$ with $e(t+\omega) \equiv e(t)$. Let $\mu_M = \max\{\sqrt{\mu_1}, \sqrt{\mu_2}, \dots, \sqrt{\mu_n}\}$, where $\{\mu_i : i = 1, 2, \dots, n\}$ is the set of eigenvalues of matrix $CC^\top$. Clearly, (3.7) is a special type of (3.1) for $b(t, x_t) = s(x_t(-\mu)) = s(x(t-\mu))$ and $f(t, x_t) = g(x_t(-\gamma)) + e(t) = g(x(t-\gamma)) + e(t)$.

For estimating *a prior bounds* of periodic solutions to (3.7), we give the following results.

Lemma 3.2 ([21]) *Let $\tau \in C(\mathbb{R}, \mathbb{R})$ with $\tau(t+\omega) \equiv \tau(t)$ and $\tau(t) \in [0, \omega]$ or $\tau(t) \in [-\omega, 0]$ for all $t \in \mathbb{R}$. Suppose $\alpha_0 = \max_{t\in[0,\omega]} |\tau(t)|$ and $u \in C^1(R, \mathbb{R}^n)$ with $u(t+\omega) \equiv u(t)$. Then*

$$\int_0^\omega |u(t) - u(t-\tau(t))|^p dt \le \alpha_0^p \int_0^\omega |u'(t)|^p dt, \quad p \in (1, +\infty).$$

Lemma 3.3 ([20]) *Let*

$$\Gamma_\lambda : P_\omega \to P_\omega, \quad [\Gamma_\lambda x](t) := x(t) - Cx(t-\tau) - \lambda s(x(t-\mu), \quad \lambda \in (0, 1). \tag{3.8}$$

Suppose that the following conditions are satisfied:

(A1) *the nonlinear function s is satisfied with $s(0) = 0$ and $|s(x_1) - s(x_2)| \le l|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^n$, where $l \in (0, +\infty)$ is a constant;*

(A2) *for all $i = 1, 2, \dots, m$, $|\lambda_i| \ne 1$, where $\lambda_i, i = 1, 2, \dots, m$ is determined by* (2.9);

(A3) $l|U^{-1}\|U| \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k} < 1$.

Then for each $\lambda \in (0, 1)$, operator Γ_λ has a unique inverse $\Gamma_\lambda^{-1} : P_\omega \to P_\omega$ with the following properties:

(1) $\|\Gamma_\lambda^{-1} e\|_{P_\omega} \le \frac{|U^{-1}\|U| \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}}{1-l|U^{-1}\|U| \sum_{i=1}^m \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i\|^k}} \|e\|_{P_\omega}$ *for all $e \in P_\omega$ and $\lambda \in (0, 1)$;*

(2) *for each $\lambda \in (0, 1)$, Γ_λ^{-1} is continuous in P_ω.*

Lemma 3.4 *Suppose that the assumptions* (A1)–(A3) *in Lemma* 3.3 *hold, and $\Gamma_\lambda x \in C_\omega^1$ for all $\lambda \in (0, 1)$. If $|U\|U^{-1}|l\sigma_1^{1/p} < 1$, then*

$$\left(\int_0^\omega |x(t) - x(t-\gamma)|^p dt\right)^{1/p} \le \frac{|\gamma\|U\|U^{-1}|\sigma_1^{1/p}}{1 - l|U\|U^{-1}|\sigma_1^{1/p}} \left(\int_0^\omega |(\Gamma_\lambda x)'(t)|^p dt\right)^{1/p}, \quad \lambda \in (0, 1),$$

where constant σ_1 and matrix U are determined by Lemma 2.4, γ, p are constants with $|\gamma| \in (0, \omega)$ and $p \in (1, +\infty)$.

Proof From (3.8), we see for each $\lambda \in (0, 1)$,

$$x(t) - Cx(t-\tau) - \lambda s(x(t-\mu)) = (\Gamma_\lambda x)(t)$$

and

$$x(t-\gamma) - Cx(t-\tau-\gamma) - \lambda s(x(t-\mu-\gamma)) = (\Gamma_\lambda x)(t-\gamma),$$

which results in

$$\begin{aligned} &x(t) - x(t-\gamma) - C(x(t-\tau) - x(t-\tau-\gamma)) \\ &\quad = (\Gamma_\lambda x)(t) - (\Gamma_\lambda x)(t-\gamma) + \lambda[s(x(t-\mu)) - s(x(t-\mu-\gamma))]. \end{aligned}$$

By using Lemma 2.4, we have

$$\begin{aligned}&\Big(\int_0^\omega |x(t)-x(t-\gamma)|^p dt\Big)^{1/p}\\ &\le |U|\|U^{-1}|\sigma_1^{1/p}\bigg\{\bigg[\int_0^\omega |(\Gamma_\lambda x)(t)-(\Gamma_\lambda x)(t-\gamma)|^p dt\bigg]^{1/p}\\ &\quad+\lambda\bigg[\int_0^\omega |s(x(t-\mu))-s(x(t-\mu-\gamma))|^p dt\bigg]^{1/p}\bigg\}\\ &\le |U|\|U^{-1}|\sigma_1^{1/p}\bigg[\int_0^\omega |(\Gamma_\lambda x)(t)-(\Gamma_\lambda x)(t-\gamma)|^p dt\bigg]^{1/p}\\ &\quad+\lambda l|U|\|U^{-1}|\sigma_1^{1/p}\Big(\int_0^\omega |x(t)-x(t-\gamma)|^p dt\Big)^{1/p}.\end{aligned}$$

So

$$\begin{aligned}&\Big(\int_0^\omega |x(t)-x(t-\gamma)|^p dt\Big)^{1/p}\\ &\le \frac{|U|\|U^{-1}|\sigma_1^{1/p}}{1-|U|\|U^{-1}|\sigma_1^{1/p}}\bigg[\int_0^\omega |(\Gamma_\lambda x)(t)-(\Gamma_\lambda x)(t-\gamma)|^p dt\bigg]^{1/p}.\end{aligned}$$

Since $\Gamma_\lambda x\in P_\omega^1$, by using Lemma 3.2, we see

$$\Big(\int_0^\omega |x(t)-x(t-\gamma)|^p dt\Big)^{1/p}\le \frac{|\gamma\|U\|U^{-1}|\sigma_1^{1/p}}{1-l|U\|U^{-1}|\sigma_1^{1/p}}\Big(\int_0^\omega |(\Gamma_\lambda x)'(t)|^p dt\Big)^{1/p}\quad \text{for all } \lambda\in(0,1).$$

Lemma 3.5 ([14]) *If $q:\mathbb{R}\to\mathbb{R}^n$ is continuous differentiable on $\mathbb{R}$ and $a>0$ is a constant, then for every $t\in\mathbb{R}$ the following inequality holds*:

$$|q(t)|\le (2a)^{-\frac12}\Big(\int_{t-a}^{t+a}|q(s)|^2ds\Big)^{1/2}+\Big(\frac{a}{2}\Big)^{\frac12}\Big(\int_{t-a}^{t+a}|q'(s)|^2ds\Big)^{1/2}.$$

Theorem 3.6 *Suppose that $|\tau-\gamma|<\omega$ and the assumptions* (A1)–(A3) *in Lemma* 3.3 *hold. Furthermore, there are positive constants l_0 and l_1 such that the following conditions are satisfied*:

(S$_1$) $x^\top C^\top g(x)\le -l_0|x|^2$ *and* $|g(x)|\le l_1|x|$ *for all* $x\in\mathbb{R}^n$;

(S$_2$) $l_0>l_1+ll_1$, $|U\|U^{-1}|l\sigma_1^{1/2}<1$ *and*

$$\frac{\mu_M^2 l_1^2|U|^2|U^{-1}|^2\sigma_1|\tau-\gamma|^2}{(l_0-l_1-ll_1)(1-l|U\|U^{-1}|\sigma_1^{1/2})^2}<1,$$

where $l>0$ is determined by the assumption (A1) *in Lemma* 3.3 *and σ_1 is determined in Lemma* 2.4 *for the case of $p=2$.*

Then (3.7) *has at least one ω-periodic solution.*

Proof Suppose that $x\in P_\omega$ is an arbitrary solution to the equation

$$\frac{d^2[\Gamma_\lambda y](t)}{dt^2}=\lambda g(y(t-\gamma))+\lambda e(t),\quad \lambda\in(0,1),$$

i.e.,

$$\frac{d^2[\Gamma_\lambda x](t)}{dt^2}=\lambda g(x(t-\gamma))+\lambda e(t),\quad \lambda\in(0,1). \tag{3.9}$$

Multiplying both sides of (3.9) with $[\Gamma_\lambda x]^\top(t)$ and integrating it on the interval $[0,\omega]$, we have

$$\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt = \lambda\int_0^\omega (x^\top(t-\tau)C^\top - x^\top(t)+\lambda s^\top(x(t-\tau)))g(x(t-\gamma))dt$$
$$-\lambda\int_0^\omega (x^\top(t) - x^\top(t-\tau)C^\top - \lambda s^\top(x(t-\tau)))e(t)dt. \qquad (3.10)$$

From the assumption (S_2), we have

$$\begin{aligned}
&\int_0^\omega x^\top(t-\tau)C^\top g(x(t-\gamma))dt\\
&= \int_0^\omega x^\top(t-\gamma)C^\top g(x(t-\gamma))dt + \int_0^\omega (x^\top(t-\tau)-x^\top(t-\gamma))C^\top g(x(t-\gamma))dt\\
&\le -l_0\int_0^\omega |x(t-\gamma)|^2dt + \left(\int_0^\omega |C^\top g(x(t-\gamma)|^2dt\right)^{1/2}\left(\int_0^\omega |x(t-\tau)-x(t-\gamma)|^2dt\right)^{1/2}\\
&\le -l_0\int_0^\omega |x(t)|^2dt + \mu_M l_1\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\left(\int_0^\omega |x(t)-x(t+\tau-\gamma)|^2dt\right)^{1/2}.
\end{aligned}$$

It follows from Lemma 3.4

$$\begin{aligned}
&\int_0^\omega x(t-\tau)C^\top g(x(t-\gamma))dt\\
&\le -l_0\int_0^\omega |x(t)|^2dt + \frac{\mu_M l_1|\tau-\gamma|\|U\||U^{-1}|\sigma_1^{1/2}}{1-l|U\||U^{-1}|\sigma_1^{1/2}}\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\left(\int_0^\omega |(\Gamma_\lambda x)'(t)|^2dt\right)^{1/2}.
\end{aligned}$$

Moreover, by using the assumptions (A1) and (S_1), we have

$$\int_0^\omega |x^\top(t)g(x(t-\gamma))|dt \le l_1\int_0^\omega |x(t)|^2dt$$

and

$$\lambda\int_0^\omega |s^\top(x(t-\mu))g(x(t-\gamma))|dt = \lambda\int_0^\omega |s^\top(x(t-\mu))-s(0)\|g(x(t-\gamma))|dt \le l_1 l\int_0^\omega |x(t)|^2dt.$$

Substituting the above three formulas into (3.10), we get

$$\begin{aligned}
&\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt + \lambda(l_0-l_1-l_1 l)\int_0^\omega |x(t)|^2dt\\
&\quad\le \frac{\lambda\mu_M l_1|\tau-\gamma|\|U\||U^{-1}|\sigma_1^{1/2}}{1-l|U\||U^{-1}|\sigma_1^{1/2}}\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\left(\int_0^\omega |(\Gamma_\lambda x)'(t)|^2dt\right)^{1/2}\\
&\qquad + \lambda\int_0^\omega |(x^\top(t)-x^\top(t-\tau)C^\top-\lambda s^\top(x(t-\tau)))e(t)|dt\\
&\quad\le \frac{\lambda\mu_M l_1|\tau-\gamma|\|U\||U^{-1}|\sigma_1^{1/2}}{1-l|U\||U^{-1}|\sigma_1^{1/2}}\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\left(\int_0^\omega |(\Gamma_\lambda x)'(t)|^2dt\right)^{1/2}\\
&\qquad + \lambda(1+\mu_M+l)\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\left(\int_0^\omega |e(t)|^2dt\right)^{1/2}, \qquad (3.11)
\end{aligned}$$

which results in

$$\begin{aligned}
&\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\\
&\quad\le \frac{\mu_M l_1|\tau-\gamma|\|U\||U^{-1}|\sigma_1^{1/2}}{(1-l|U\||U^{-1}|\sigma_1^{1/2})(l_0-l_1-ll_1)}\left(\int_0^\omega |(\Gamma_\lambda x)'(t)|^2dt\right)^{1/2}
\end{aligned}$$

$$+\frac{1+\mu_M+l}{l_0-l_1-ll_1}\left(\int_0^\omega |e(t)|^2dt\right)^{1/2}. \tag{3.12}$$

It follows from (3.11) that

$$\begin{aligned}&\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt\\ &\quad\le \frac{\mu_M^2 l_1^2|\tau-\gamma|^2|U|^2|U^{-1}|^2\sigma_1}{(1-l|U\|U^{-1}|\sigma_1^{1/2})^2(l_0-l_1-ll_1)}\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt\\ &\qquad+\frac{2\mu_M l_1|\tau-\gamma\|U\|U^{-1}|\sigma_1^{1/2}(1+\mu_M+l)}{(1-l|U\|U^{-1}|\sigma_1^{1/2})(l_0-l_1-ll_1)}\left(\int_0^\omega |e(t)|^2dt\right)^{1/2}\left(\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt\right)^{1/2}\\ &\qquad+\frac{(1+\mu_M+l)^2}{l_0-l_1-ll_1}\int_0^\omega |e(t)|^2dt,\end{aligned}$$

which together with the condition of

$$\frac{\mu_M^2 l_1^2|\tau-\gamma|^2|U|^2|U^{-1}|^2\sigma_1}{(1-l|U\|U^{-1}|\sigma_1^{1/2})^2(l_0-l_1-ll_1)}<1$$

yields that there is a constant $M_0>0$ (independent of λ and x) such that

$$\left(\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt\right)^{1/2}\le M_0 \quad \text{for all } \lambda\in(0,1). \tag{3.13}$$

Substituting (3.13) into (3.12), we have

$$\left(\int_0^\omega |x(t)|^2dt\right)^{1/2}\le M_1,$$

where $M_1>0$ is a constant independent of λ and x; and by(3.8), we have

$$\begin{aligned}&\left(\int_0^\omega |[\Gamma_\lambda x](t)|^2dt\right)^{1/2}\\ &\quad=\left(\int_0^\omega |(x(t)-Cx(t-\tau)-s(x(t-\gamma)))^\top(x(t)-Cx(t-\tau)-s(x(t-\gamma)))|dt\right)^{1/2}\\ &\quad\le(1+\mu_M^{1/2}+l^{1/2})^2M_1 \quad \text{for all } \lambda\in(0,1).\end{aligned} \tag{3.14}$$

From (3.13) and (3.14), and by using Lemma 3.5, we see that there is a constant $\widetilde{M_2}>0$ (independent of λ and x) such that

$$\|\Gamma_\lambda x\|_{P_\omega}\le\omega^{-\frac12}\left(\int_0^\omega |[\Gamma_\lambda x](t)|^2dt\right)^{\frac12}+\left(\frac{\omega}{4}\right)^{\frac12}\left(\int_0^\omega |[\Gamma_\lambda x]'(t)|^2dt\right)^{\frac12}\le\widetilde{M_2} \quad \text{for all } \lambda\in(0,1).$$

By using Lemma 3.3 again, we have

$$\begin{aligned}\|x\|_{P_\omega}&=\|\Gamma_\lambda^{-1}\Gamma_\lambda x\|_{P_\omega}\\ &\le\frac{|U^{-1}\|U|\sum_{i=1}^m\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i\|^k}}{1-l|U^{-1}\|U|\sum_{i=1}^m\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i\|^k}}\|\Gamma_\lambda x\|_{P_\omega}\\ &:=M_2,\end{aligned} \tag{3.15}$$

where M_2 is a constant independent of λ and x.

Now, let $a \in \mathbb{R}^n$ such that $\Delta(a) = \frac{1}{\omega}\int_0^\omega [g(A^{-1}(a)) + e(t)]dt = 0$. Then

$$g(A^{-1}(a)) + \bar{e} = 0.$$

So

$$(A^{-1}(a))^\top C^\top g(A^{-1}(a)) = -(A^{-1}(a))^\top C^\top \bar{e},$$

where $\bar{e} = \frac{1}{\omega}\int_0^\omega e(s)ds$. By using the assumption (S_1), we see

$$l_0|A^{-1}(a)|^2 \le -(A^{-1}(a))^\top C^\top g(A^{-1}(a)) = (A^{-1}(a))^\top C^\top \bar{e} \le \mu_M|A^{-1}(a)||\bar{e}|,$$

which leads to

$$|A^{-1}(a)| \le \frac{\mu_M}{l_0}|\bar{e}|, \tag{3.16}$$

i.e.,

$$|a| = |[AA^{-1}](a)| = |(I-C)A^{-1}](a)| \le \frac{1+\mu_M}{l_0}\mu_M|\bar{e}|.$$

If set $\rho = \max\{\frac{1+\mu_M}{l_0}\mu_M|\bar{e}|, M_2\} + 1$, then it follows from (3.15)–(3.16) that the assumptions (D1) and (D2) in Theorem 3.1 are satisfied. The reminder is to verify the assumption (D3) of Theorem 3.1. From the assumption (S_1), we see $x^\top C^\top g(x) \le -l_0|x|^2$, i.e., $l_0|x|^2 \le |x^\top C^\top g(x)| \le l_1|Cx||x|$. So

$$|Cx| \ge \frac{l_0}{l_1}|x|,$$

which implies that matrix C is nonsingular. Take

$$H: B_\rho \times [0,1] \to \mathbb{R}^n, \quad H(x,\mu) = -\mu C^\top A^{-1}x + (1-\mu)g(A^{-1}(x)) + \bar{e}.$$

Then

$$\begin{aligned}
&(A^{-1}x)^\top CH(x,\mu)\\
&\quad= -\mu(A^{-1}x)^\top CC^\top A^{-1}x + (1-\mu)(A^{-1}x)^\top C^\top g(A^{-1}x) + (1-\mu)(A^{-1}x)^\top C^\top \bar{e}\\
&\quad\le (1-\mu)(A^{-1}x)^\top C^\top g(A^{-1}x) + (1-\mu)(A^{-1}x)^\top C^\top \bar{e}\\
&\quad\le -(1-\mu)l_0|A^{-1}x|^2 + (1-\mu)\mu_M|\bar{e}||A^{-1}x|
\end{aligned}$$

for all $(x,\mu) \in B_\rho \times [0,1]$. Clearly, if $x \in \partial B_\rho$, then from (2.8), we have $A^{-1}x = (I-C)^{-1}x$, i.e., $|A^{-1}x| + |CA^{-1}x| \ge |x| = \rho$, which results in

$$|A^{-1}x| \ge \frac{\rho}{1+\mu_M}.$$

In view of $\rho = \max\{\frac{1+\mu_M}{l_0}\mu_M|\bar{e}|, M_2\} + 1$, we see that

$$\begin{aligned}
&(A^{-1}x)^\top CH(x,\mu)\\
&\quad< -(1-\mu)l_0|A^{-1}x|^2 + (1-\mu)\mu_M|\bar{e}||A^{-1}x|\\
&\quad< 0
\end{aligned}$$

for all $(x,\mu) \in B_\rho \times [0,1]$. This together with the fact that C and $(I-C)$ are nonsingular implies that

$$\begin{aligned}
\deg\{\Delta, B_\rho, 0\} &= \deg\{H(\cdot,0), B_\rho, 0\} = \deg\{H(\cdot,1), B_\rho, 0\} = \deg\{-CA^{-1}, B_\rho, 0\}\\
&= \deg\{-C^\top(I-C)^{-1}, B_\rho, 0\} \ne 0,
\end{aligned}$$

which follows that the assumption (D3) holds. Thus, by using Theorem 3.1, we see that (3.7) has a ω-periodic solution $u_0(t)$.

For example, we consider the following equation

$$\frac{d^2}{dt^2}(x(t) - 4x(t-\tau) - s(x(t-\tau))) = g(x(t-\gamma)) + e(t), \tag{3.17}$$

where $x(t) \in \mathbb{R}$, τ, γ are positive constants, $s(x) = \frac{|x|}{1+|x|}$, $g(x) = -\frac{x^3}{1+x^2}$. $e(t) = \sin t$. Clearly, we can chose $\lambda_1 = 4$ and $l = 1$ such that the assumptions (A1)–(A3) in Lemma 3.3 hold. Furthermore, we can chose $l_0 = 4$ and $l_1 = 1$ such that $x^\top C^\top g(x) \leq -l_0|x|^2$ and $|g(x)| \leq l_1|x|$ for all $x \in \mathbb{R}$. This implies that the assumption (S_1) of Theorem 3.6 holds. Since $\sigma_1 = \frac{1}{9}$, $\mu_M = 4$, $l_0 - l_1 - ll_1 = 2 > 0$ and $1 - |U||U^{-1}l\sigma_1^{1/2} = \frac{2}{3} > 0$, we see that if $|\tau - \gamma| < \frac{\sqrt{2}}{2}$, then

$$\frac{\mu_M^2 l_1^2 |U|^2 |U^{-1}|^2 \sigma_1 |\tau - \gamma|^2}{(l_0 - l_1 - ll_1)(1 - l|U||U^{-1}|\sigma_1^{1/2})^2} < 1.$$

Thus, by using Theorem 3.2, we obtain that (3.17) has a 2π-periodic solution $u_0(t)$.

Remark 3.7 Since $s(x) = \frac{|x|}{1+|x|}$ is nonlinear and may not be differentiable in t, the result of above example cannot be obtained by [2, 8, 12, 16–19, 23].

Remark 3.8 From (3.17), we see that the operator $\overline{B} : P_{2\pi} \to P_{2\pi}, [\overline{B}x](t) = 4x(t-\tau) - \frac{|x(t-\tau)|}{1+|x(t-\tau)|}$ is not condensing. So (3.17) cannot be dealt with by methods employed in [7].

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