Acta Mathematica Sinica, English Series
Jan., 2016, Vol. 32, No. 1, pp. 83–114
Published online: February 15, 2015
DOI: 10.1007/s10114-015-4574-0
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Recent Development of Chaos Theory in Topological Dynamics

Jian LI
Department of Mathematics, Shantou University, Shantou 515063*, P. R. China*
E-mail: *lijian09@mail.ustc.edu.cn*

Xiang Dong YE
Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences and School of Mathematics, University of Science and Technology of China, Hefei 230026*, P. R. China*
E-mail: *yexd@ustc.edu.cn*

Abstract We give a summary on the recent development of chaos theory in topological dynamics, focusing on Li–Yorke chaos, Devaney chaos, distributional chaos, positive topological entropy, weakly mixing sets and so on, and their relationships.

Keywords Li–Yorke chaos, Devaney chaos, sensitive dependence on initial conditions, distributional chaos, weak mixing, topological entropy, Furstenberg family

MR(2010) Subject Classification 54H20, 37B05, 37B40

1 Introduction

Topological dynamics is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied, where a dynamical system is a phase (or state) space X endowed with an evolution map T from X to itself. In this survey, we require that the phase space X is a compact metric space and the evolution map $T\colon X \to X$ is continuous.

The mathematical term *chaos* was first introduced by Li and Yorke in 1975 [69], where the authors established a simple criterion on the existence of chaos for interval maps, known as "period three implies chaos". Since then, the study of chaos theory has played a big role in dynamical systems, even in nonlinear science.

In common usage, "chaos" means "a state of disorder". However, in chaos theory, the term is defined more precisely. Various alternative, but closely related definitions of chaos have been proposed after Li–Yorke chaos.

Although there is still no definitive, universally accepted mathematical definition of chaos (in our opinion it is also impossible), most definitions of chaos are based on one of the following aspects:

1. complex trajectory behavior of points, such as Li–Yorke chaos and distributional chaos;

Received September 30, 2014, accepted November 2, 2014
Supported by NNSF of China (Grant Nos. 11371339, 11431012, 11401362, 11471125) and NSF of Guangdong province (Grant No. S2013040014084)

2. sensitivity dependence on initial conditions, such as Devaney chaos, Auslander–Yorke chaos and Li–Yorke sensitivity;

3. fast growth of different orbits of length n, such as having positive topological entropy, or its variants;

4. strong recurrence property, such as weakly mixing property and weakly mixing sets.

Since there are so many papers dealing with the chaos theory in topological dynamics, we are not able to give a survey on all of them. We only can select those which we are familiar with, and are closely related to our interest, knowledge and ability. As we said before it is impossible to give a universe definition of chaos which covers all the features of the complex behaviors of a dynamical system, it is thus important to know the relationships among the various definitions of chaos. So we will focus on Li–Yorke chaos, Devaney chaos, distributional chaos, positive topological entropy, weakly mixing sets, sensitivity and so on, and their relationships in this survey. See [14, 85] for related surveys.

Now we outline the development of chaos theory for general topological dynamical systems briefly. The notions of topological entropy and weak mixing were introduced by Adler et al. [1] in 1965 and Furstenberg [25] in 1967 respectively. After the mathematical term of chaos by Li–Yorke appearing in 1975, Devaney [19] defined a kind of chaos, known as Devaney chaos today, in 1989 based on the notion of sensitivity introduced by Guckenheimer [35]. The implication among them has attracted a lot of attention. In 1991, Iwanik [50] showed that weak mixing implies Li–Yorke chaos. A breakthrough concerning the relationships among positive entropy, Li–Yorke chaos and Devaney chaos came in 2002. In that year, it was shown by Huang and Ye [46] that Devaney chaos implies Li–Yorke one by proving that a non-periodic transitive system with a periodic point is Li–Yorke chaotic; Blanchard et al. [15] proved that positive entropy also implies Li–Yorke chaos (we remark that the authors obtained this result using ergodic method, and for a combinatorial proof see [53]). Moreover, the result also holds for sofic group actions by Kerr and Li [54, Corollary 8.4].

In 1991, Xiong and Yang [109] showed that in a weakly mixing system there are considerably many points in the domain whose orbits display highly erratic time dependence. It is known that a dynamical system with positive entropy may not contain any weakly mixing subsystem. Capturing a common feature of positive entropy and weak mixing, Blanchard and Huang [17] in 2008 defined the notion of weakly mixing set and showed that positive entropy implies the existence of weakly mixing sets which also implies Li–Yorke chaos. A further discussion along the line will be appeared in a forthcoming paper by Huang et al. [41].

Distributional chaos was introduced in 1994 by Schweizer and Smítal [92], and there are at least three versions of distributional chaos in the literature (DC1, DC2 and DC3). It is known that positive entropy does not imply DC1 chaos [86] and Smítal conjectured that positive entropy implies DC2 chaos. Observing that DC2 chaos is equivalent to so called mean Li–Yorke chaos, recently Downarowicz [21] proved that positive entropy indeed implies DC2 chaos. An alternative proof can be found in [40] by Huang et al.. We remark that both proofs use ergodic theory heavily and there is no combinatorial proof at this moment.

This survey will be organized as follows. In Section 2, we provide basic definitions in topological dynamics. Li–Yorke chaos, sensitivity and chaos in transitive systems will be discussed in Sections 3–5. In Section 6, we review the results on distributional chaos. In the following two sections, we focus on weakly mixing sets and chaos in the induced spaces.

2 Preliminaries

In this section, we provide some basic notations, definitions and results which will be used later in this survey.

Denote by $\mathbb{N}$ ($\mathbb{Z}_+$, $\mathbb{Z}$, respectively) the set of all positive integers (non-negative integers, integers, respectively). The cardinality of a set A is usually denoted by $|A|$.

Let X be a compact metric space. A subset A of X is called a *perfect set* if it is a closed set with no isolated points; a *Cantor set* if it is homeomorphic to the standard middle third Cantor set; a *Mycielski set* if it is a union of countably many Cantor sets. For convenience, we restate here a version of Mycielski's theorem (see [79, Theorem 1]) which we shall use.

Theorem 2.1 (Mycielski Theorem) *Let X be a perfect compact metric space. If R is a dense G_δ subset of X^n, then there exists a dense Mycielski set $K \subset X$ such that for any distinct n points $x_1, x_2, \dots, x_n \in K$, the tuple $(x_1, x_2, \dots, x_n)$ is in R.*

2.1 Topological Dynamics

By a (*topological*) *dynamical system*, we mean a pair (X, T), where X is a compact metric space and $T\colon X \to X$ is a continuous map. The metric on X is usually denoted by d. One can think of X as a state space for some system, and T as the evolution law of some discrete autonomous dynamics on X: if x is a point in X, denoting the current state of a system, then Tx can be interpreted as the state of the same system after one unit of time has elapsed. For every non-negative integer n, we can define the iterates $T^n\colon X \to X$ as $T^0 = \mathrm{id}$ the identity map on X and $T^{n+1} = T^n \circ T$. One of the main topics of study in dynamical systems is the asymptotic behaviour of T^n as $n \to \infty$.

Note that we always assume that the state space X is not empty. If the state space X contains only one point, then we say that the dynamical system on X is *trivial*, because in this case the unique map on X is the identity map.

For any $n \geq 2$, the *n-th fold product* of (X, T) is denoted by $(X^n, T^{(n)})$, where $X^n = X \times X \times \cdots \times X$ (n-times) and $T^{(n)} = T \times T \times \cdots \times T$ (n-times). We set the diagonal of X^n as $\Delta_n = \{(x, x, \dots, x) \in X^n \colon x \in X\}$, and set $\Delta^{(n)} = \{(x_1, x_2, \dots, x_n) \in X^n \colon \text{there exist } 1 \leq i < j \leq n \text{ such that } x_i = x_j\}$.

The *orbit* of a point x in X is the set $\mathrm{Orb}(x, T) = \{x, Tx, T^2x, \dots\}$; the ω-limit set of x, denoted by $\omega(x, T)$, is the limit set of the orbit of x, that is,

$$\omega(x, T) = \bigcap_{n=1}^{\infty} \overline{\{T^i x \colon i \geq n\}}.$$

A point $x \in X$ is called *recurrent* if there exists an increasing sequence $\{p_i\}$ in $\mathbb{N}$ such that $\lim_{i\to\infty} T^{p_i}x = x$. Clearly, x is recurrent if and only if $x \in \omega(x, T)$.

If Y is a non-empty closed invariant (i.e., $TY \subset Y$) subset of X, then (Y, T) is also a dynamical system, we call it as a *subsystem* of (X, T). A dynamical system (X, T) is called *minimal* if it contains no proper subsystems. A subset A of X is *minimal* if (A, T) forms a minimal subsystem of (X, T). It is easy to see that a non-empty closed invariant set $A \subset X$

is minimal if and only if the orbit of every point of A is dense in A. A point $x \in X$ is called *minimal* or *almost periodic* if it belongs to a minimal set.

A dynamical system (X,T) is called *transitive* if for every two non-empty open subsets U and V of X there is a positive integer n such that $T^nU \cap V \neq \emptyset$; *totally transitive* if (X,T^n) is transitive for all $n \in \mathbb{N}$; *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive; *strongly mixing* if for every two non-empty open subsets U and V of X there is $N > 0$ such that $T^nU \cap V \neq \emptyset$ for all $n \geq N$. Any point with dense orbit is called a *transitive point*. In a transitive system the set of all transitive points is a dense G_δ subset of X and we denote it by $\mathrm{Trans}(X,T)$. For more details related to transitivity see [57].

Let (X,T) be a dynamical system. A pair (x,y) of points in X is called *asymptotic* if $\lim_{n\to\infty} d(T^nx,T^ny) = 0$; *proximal* if $\liminf_{n\to\infty} d(T^nx,T^ny) = 0$; *distal* if $\liminf_{n\to\infty} d(T^nx, T^ny) > 0$. The system (X,T) is called *proximal* if any two points in X form a proximal pair; *distal* if any two distinct points in X form a distal pair.

Let (X,T) and (Y,S) be two dynamical systems. If there is a continuous surjection $\pi\colon X \to Y$ which intertwines the actions (i.e., $\pi \circ T = S \circ \pi$), then we say that π is a *factor map*, (Y,S) is a *factor* of (X,T) or (X,T) is an *extension* of (Y,S). The factor map π is *almost one-to-one* if there exists a residual subset G of X such that $\pi^{-1}(\pi(x)) = \{x\}$ for any $x \in G$.

In 1965, Adler et al. introduced topological entropy in topological dynamics [1]. Let C_X^o be the set of finite open covers of X. Given two open covers $\mathcal{U}$ and $\mathcal{V}$, their *join* is the cover

$$\{U \cap V \colon U \in \mathcal{U}, V \in \mathcal{V}\}.$$

We define $N(\mathscr{U})$ as the minimum cardinality of subcovers of $\mathcal{U}$. The *topological entropy* of T with respect to $\mathcal{U}$ is

$$h_{\mathrm{top}}(T,\mathcal{U}) = \lim_{N\to\infty} \frac{1}{N} \log N\biggl(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{U}\biggr).$$

The *topological entropy* of (X,T) is defined by

$$h_{\mathrm{top}}(T) = \sup_{\mathcal{U}\in C_X^o} h_{\mathrm{top}}(T,\mathcal{U}).$$

We refer the reader to [28, 103] for more information on topological entropy, and [31] for a survey of local entropy theory.

2.2 Furstenberg Family

The idea of using families of subsets of $\mathbb{Z}_+$ when dealing with dynamical properties of maps was first used by Gottschalk and Hedlund [34], and then further developed by Furstenberg [26]. For a systematic study and recent results, see [2, 43, 47, 59, 93, 108]. Here we recall some basic facts related to Furstenberg families.

A *Furstenberg family* (or just *family*) $\mathcal{F}$ is a collection of subsets of $\mathbb{Z}_+$ which is upwards hereditary, that is, if $F_1 \in \mathcal{F}$ and $F_1 \subset F_2$ then $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families, then we define

$$\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 \colon F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}.$$

If $\mathcal{F}$ is a family, the *dual family* of $\mathcal{F}$, denoted by $\kappa\mathcal{F}$, is the family

$$\{F \subset \mathbb{N} \colon F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\}.$$

We denote by $\mathcal{F}_{\text{inf}}$ the family of infinite subsets of $\mathbb{N}$. The dual family of $\mathcal{F}_{\text{inf}}$ is just the collection of co-finite subsets of $\mathbb{Z}_+$, denoted by $\mathcal{F}_{\text{cf}}$. A family $\mathcal{F}$ is called *full* if it is proper and $\mathcal{F} \cdot \kappa\mathcal{F} \subset \mathcal{F}_{\text{inf}}$.

For a sequence $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{N}$, define the finite sums of $\{p_i\}_{i=1}^{\infty}$ as

$$\text{FS}\{p_i\}_{i=1}^{\infty} = \left\{ \sum_{i\in\alpha} p_i \colon \alpha \text{ is a non-empty finite subset of } \mathbb{N} \right\}.$$

A subset F of $\mathbb{Z}_+$ is called an IP-*set* if there exists a sequence $\{p_i\}_{i=1}^{\infty}$ in $\mathbb{N}$ such that $\text{FS}\{p_i\}_{i=1}^{\infty} \subset F$. We denote by $\mathcal{F}_{\text{ip}}$ the family of all IP-sets.

A subset F of $\mathbb{Z}_+$ is called *thick* if it contains arbitrarily long runs of positive integers, i.e., for every $n \in \mathbb{N}$ there exists some $a_n \in \mathbb{Z}_+$ such that $\{a_n, a_n+1, \dots, a_n+n\} \subset F$; *syndetic* if it has bounded gaps, i.e., there is $N \in \mathbb{N}$ such that $[n, n+N] \cap F \neq \emptyset$ for every $n \in \mathbb{Z}_+$. The families of all thick sets and syndetic sets are denoted by $\mathcal{F}_t$ and $\mathcal{F}_s$, respectively. It is easy to see that $\kappa\mathcal{F}_s = \mathcal{F}_t$.

Let (X,T) be a dynamical system. For $x \in X$ and a non-empty subset U of X, we define the *entering time set* of x into U as

$$N(x,U) = \{n \in \mathbb{Z}_+ \colon T^n x \in U\}.$$

If U is a neighborhood of x, then we usually call $N(x,U)$ the *return time set* of x into U. Clearly, a point x is recurrent if and only if for every open neighborhood U of x, the return time set $N(x,U)$ is infinite. In general, we can define recurrence with respect to a Furstenberg family.

Definition 2.2 *Let (X,T) be a dynamical system and $\mathcal{F}$ be a Furstenberg family. A point $x \in X$ is said to be $\mathcal{F}$-*recurrent *if for every open neighborhood U of x, $N(x,U) \in \mathcal{F}$.*

It is well known that the following lemma holds (see, e.g., [2, 26]).

Lemma 2.3 *Let (X,T) be a dynamical system and $x \in X$. Then*

1. *x is a minimal point if and only if it is an $\mathcal{F}_s$-recurrent point;*
2. *x is a recurrent point if and only if it is an $\mathcal{F}_{ip}$-recurrent point.*

For two non-empty subsets U and V of X, we define the *hitting time set* of U and V as

$$N(U,V) = \{n \in \mathbb{Z}_+ \colon T^n U \cap V \neq \emptyset\}.$$

If U is a non-empty open subset of X, then we usually call $N(U,U)$ the *return time set* of U. Clearly, a dynamical system (X,T) is transitive (resp. strongly mixing) if and only if for any two non-empty open subsets U, V of X, the hitting time set $N(U,V)$ is infinite (resp. co-finite).

Definition 2.4 *Let (X,T) be a dynamical system and $\mathcal{F}$ be a family. We say that (X,T) is*

1. $\mathcal{F}$-transitive *if for any two non-empty open subsets U, V of X, $N(U,V) \in \mathcal{F}$;*
2. $\mathcal{F}$-mixing *if the product system $(X \times X, T \times T)$ is $\mathcal{F}$-transitive.*

Theorem 2.5 ([25]) *Let (X,T) be a dynamical system. Then (X,T) is weakly mixing if and only if it is $\mathcal{F}_t$-transitive.*

3 Li–Yorke Chaos

The mathematical terminology "chaos" was first introduced in 1975 by Li and Yorke to describe the complex behavior of trajectories. It turns out that it is the common feature of all known definitions of chaos.

3.1 Li–Yorke Chaos and Its Relation with Topological Entropy

Following the idea in [69], we usually define the Li–Yorke chaos as follows.

Definition 3.1 *Let (X,T) be a dynamical system. A pair $(x,y)\in X\times X$ is called* scrambled *if*

$$\liminf_{n\to\infty} d(T^n x, T^n y)=0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y)>0,$$

that is (x,y) is proximal but not asymptotic. A subset C of X is called scrambled *if any two distinct points $x,y\in C$ form a scrambled pair. The dynamical system (X,T) is called* Li–Yorke chaotic *if there is an uncountable scrambled set in X.*

It is worth to notice that the terminology "scrambled set" was introduced in 1983 by Smítal in [95]. Note that we should assume that a scrambled set contains at least two points. We will keep this convention throughout this paper.

In [69], Li and Yorke showed that

Theorem 3.2 *If a continuous map $f\colon [0,1]\to[0,1]$ has a periodic point of period 3, then it is Li–Yorke chaotic.*

Definition 3.3 *Let (X,T) be a dynamical system. For a given positive number $\delta>0$, a pair $(x,y)\in X\times X$ is called* δ-scrambled, *if*

$$\liminf_{n\to\infty} d(T^n x, T^n y)=0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y)>\delta.$$

A subset C of X is δ-scrambled *if any two distinct points x,y in C form a δ-scrambled pair. The dynamical system (X,T) is called* Li–Yorke δ-chaotic, *if there exists an uncountable δ-scrambled set in X.*

Note that the Auslander–Floyd system [7] is Li–Yorke chaotic, but for any $\delta>0$, there is no uncountable δ-scrambled sets.

In [51], Janková and Smítal showed that, if a continuous map $f:[0,1]\to[0,1]$ has positive topological entropy, then there exists a perfect δ-scrambled set for some $\delta>0$. A natural question is that whether there exists a Li–Yorke chaotic map with zero topological entropy. This was shown, independently, by Xiong [106] and Smítal [96] in 1986.

Theorem 3.4 *There exists a continuous map $f\colon[0,1]\to[0,1]$, which is Li–Yorke chaotic but has zero topological entropy.*

In 1991, using Mycielski Theorem 2.1, Iwanik [50] showed that every weakly mixing system is Li–Yorke chaotic.

Theorem 3.5 ([50]) *If a non-trivial dynamical system (X,T) is weakly mixing, then there exists a dense Mycielski δ-scrambled subset of X for some $\delta>0$.*

As already mentioned above, for interval maps positive topological entropy implies Li–Yorke chaos. It used to be a long-standing open problem whether this also holds for general topological dynamical systems. In 2002, Blanchard et al. gave a positive answer by using ergodic theory method.

Theorem 3.6 ([15]) *If a dynamical system (X,T) has positive topological entropy, then there exists a Mycielski δ-scrambled set for some $\delta > 0$.*

In fact, we know from the proof of Theorem 3.6 that if (X,T) has an ergodic invariant measure which is not measurable distal then same conclusion holds. In 2007, Kerr and Li [53] gave a new proof of Theorem 3.6 by using combinatorial method. First we recall the definition of IE-tuples.

Definition 3.7 *Let (X,T) be a topological dynamical system and $k \geq 2$. For a tuple $\tilde{A} = (A_1,\dots,A_k)$ of subsets of X, we say that a subset J of $\mathbb{Z}_+$ is an* independence set *for $\tilde{A}$ if for any non-empty finite subset I of J, we have*

$$\bigcap_{i\in I} T^{-i}A_{s(i)} \neq \emptyset$$

for any $s \in \{1,\dots,k\}^I$.

A tuple $\tilde{x} = (x_1,\dots,x_k) \in X^k$ is called an IE-tuple *if for every product neighborhood $U_1 \times \cdots \times U_k$ of $\tilde{x}$ the tuple $(U_1,\dots,U_k)$ has an independence set of positive density.*

The following theorem characterizes positive topological entropy and has many applications. It was first proved by Huang and Ye using the notion of interpolating sets [49], and we state it here using the notion of independence by Kerr and Li [53]. Recall for a dynamical system (X,T), a tuple $(x_1,\dots,x_k) \in X^k$ is an *entropy tuple* if $x_i \neq x_j$ for $i \neq j$ and for any disjoint closed neighborhoods V_i of x_i, the open cover $\{V_1^c,\dots,V_k^c\}$ has positive entropy. When $k=2$, we call it an *entropy pair*. A subset A of X is an *entropy set* if any tuple of points in A with pairwise different coordinates is an entropy tuple.

Theorem 3.8 ([49, 53]) *Let (X,T) be a dynamical system. Then a tuple on X is an entropy tuple if and only if it is a non-diagonal* IE*-tuple. In particular, the system (X,T) has zero topological entropy if and only if every* IE*-pair is diagonal (i.e., all of its entries are equal).*

By developing some deep combinatorial tools, Kerr and Li showed

Theorem 3.9 ([53]) *Let (X,T) be a dynamical system. Suppose that $k \geq 2$ and $\tilde{x} = (x_1,\dots,x_k)$ is a non-diagonal* IE*-tuple. For each $1 \leq j \leq k$, let A_j be a neighborhood of x_j. Then there exists a Cantor set $Z_j \subset A_j$ for each $j = 1,\dots,k$ such that the following hold:*

1. *every tuple of finite points in $Z := \bigcup_j Z_j$ is an* IE*-tuple;*
2. *for all $m \in \mathbb{N}$, distinct $y_1,\dots,y_m \in Z$ and $y_1',\dots,y_m' \in Z$ one has*

$$\liminf_{n\to\infty} \max_{1\leq i\leq m} d(T^n y_i, y_i') = 0.$$

In particular, Z is δ-scrambled for some $\delta > 0$.

Recently, Kerr and Li showed that Theorem 3.9 also holds for sofic group actions (see [54, Corollary 8.4]).

3.2 Completely Scrambled Systems and Invariant Scrambled Sets

Definition 3.10 *We say that a dynamical system (X,T) is* completely scrambled *if the whole space X is scrambled.*

It should be noticed that for any $\delta > 0$, the whole space cannot be δ-scrambled [18]. Recall that a dynamical system (X, T) is proximal if any two points in X form a proximal pair. Clearly, every completely scrambled system is proximal. We have the following characterization of proximal systems.

Theorem 3.11 ([6, 42]) *A dynamical system (X, T) is proximal if and only if it has a fixed point which is the unique minimal point in X.*

In 1997, Mai [73] showed that there are some completely scrambled systems on non-compact spaces.

Theorem 3.12 *Let X be a metric space uniformly homeomorphic to the n-dimensional open cube $I^n = (0, 1)^n$, $n \geq 2$. Then there exists a homeomorphism $f: X \to X$ such that the whole space X is a scrambled set of X.*

In 2001, Huang and Ye [45] showed that on some compact spaces there are also some completely scrambled systems.

Theorem 3.13 *There are "many" compacta admitting completely scrambled homeomorphisms, which include some countable compacta, the Cantor set and continua of arbitrary dimension.*

Huang and Ye also mentioned in [45] that an example of completely scrambled transitive homeomorphism is a consequence of construction of uniformly rigid proximal systems by Katznelson and Weiss [52]. Recall that a dynamical system (X, T) is *uniformly rigid* if

$$\liminf_{n\to\infty} \sup_{x\in X} d(T^n(x), x) = 0.$$

Later in [46], they showed that every almost equicontinuous but not minimal system has a completely scrambled factor. These examples are not weakly mixing, so the existence of completely scrambled weakly mixing homeomorphism is left open in [45]. Recently, Foryś et al. in [24] constructed two kinds of completely scrambled systems which are weakly mixing, proximal and uniformly rigid. The first possible approach is derived from results of Akin and Glasner [4] by a combination of abstract arguments. The second method is obtained by modifying the construction of Katznelson and Weiss from [52]. More precisely, we have the following result.

Theorem 3.14 ([24]) *There are completely scrambled systems which are weakly mixing, proximal and uniformly rigid.*

On the other hand, Blanchard et al. [16] proved that any positive topological entropy system cannot be completely scrambled by showing that there are "many" non-diagonal asymptotic pairs in any dynamical system with positive topological entropy.

Theorem 3.15 ([16]) *Let (X, T) be a dynamical system and μ be an ergodic invariant measure on (X, T) with positive entropy. Then for μ-a.e. $x \in X$, there exists a point $y \in X\backslash\{x\}$ such that (x, y) is asymptotic.*

Recently in [44], Huang et al. generalized Theorem 3.15 to positive entropy G-systems for certain countable, discrete, infinite left-orderable amenable groups G. We remark that the following question remains open.

Problem 1 Let T be a homeomorphism on a compact metric space X. If for any two distinct points $x, y \in X$, (x, y) is either Li–Yorke scrambled for T or Li–Yorke scrambled for T^{-1}, does

T have zero topological entropy?

Let S be a scrambled set of a dynamical system (X, T). It is easy to see that for any $n \geq 1$, $T^n S$ is also a scrambled set of (X, T). It is interesting to consider that whether a scrambled set may be invariant under T. Since the space X is compact, if $(x, f(x))$ is scrambled for some $x \in X$, then there is a fixed point in X. It is shown in [22] that

Theorem 3.16 *Let $f : [0, 1] \to [0, 1]$ be a continuous map. Then f has positive topological entropy if and only if f^n has an uncountable invariant scrambled set for some $n > 0$.*

In 2009, Yuan and Lü proved that

Theorem 3.17 ([112]) *Let (X, T) be a non-trivial transitive system. If (X, T) has a fixed point, then there exists a dense Mycielski subset K of X such that K is an invariant scrambled set.*

In 2010, Balibrea et al. studied invariant δ-scrambled sets and showed that

Theorem 3.18 ([9]) *Let (X, T) be a non-trivial strongly mixing system. If (X, T) has a fixed point, then there exist $\delta > 0$ and a dense Mycielski subset S of X such that S is an invariant δ-scrambled set.*

They also conjectured in [9] that there exists a weakly mixing system which has a fixed point but without invariant δ-scrambled sets. The authors in [24] found that the existence of invariant δ-scrambled sets is relative to the property of uniform rigidity. It is shown in [24] that a necessary condition for a dynamical system possessing invariant δ-scrambled sets for some $\delta > 0$ is not uniformly rigid, and this condition (with a fixed point) is also sufficient for transitive systems.

Theorem 3.19 ([24]) *Let (X, T) be a non-trivial transitive system. Then (X, T) contains a dense Mycielski invariant δ-scrambled set for some $\delta > 0$ if and only if it has a fixed point and is not uniformly rigid.*

Combining Theorems 3.14 and 3.19, the above mentioned conjecture from [9] has an affirmative answer. This is because by Theorem 3.14 there exist weakly mixing, proximal and uniformly rigid systems which have a fixed point, but by Theorem 3.19 they do not have any invariant δ-scrambled set.

As far as we know, whenever a dynamical system has been shown to be Li–Yorke chaotic the proof implies the existence of a Cantor or Mycielski scrambled set. This naturally arises the following problem:

Problem 2 If a dynamical system is Li–Yorke chaotic, dose there exist a Cantor scrambled set?

Although we do not know the answer of Problem 2, there are severe restrictions on the Li–Yorke chaotic dynamical systems without a Cantor scrambled set, if they exist (see [18]). On the other hand, we have

Theorem 3.20 ([18]) *If a dynamical system (X, T) is Li–Yorke δ-chaotic for some $\delta > 0$, then it does have some Cantor δ-scrambled set.*

The key point is that the collection of δ-scrambled pairs is a G_δ subset of $X \times X$, then one can apply the Mycielski Theorem to get a Cantor δ-scrambled set. But in general the collection of scrambled pairs is a Bore set of $X \times X$ but may be not G_δ. By [99, Example 3.6] there exists a dynamical system such that the collection of scrambled pairs is dense in $X \times X$ but not residual.

4 Sensitive Dependence on Initial Conditions

We say that a dynamical system (X, T) has *sensitive dependence on initial conditions* (or just *sensitive*) if there exists some $\delta > 0$ such that for each $x \in X$ and each $\varepsilon > 0$ there is $y \in X$ with $d(x, y) < \varepsilon$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \delta$. The initial idea goes back at least to Lorentz [72] and the phrase — sensitive dependence on initial conditions — was used by Ruelle [88] to indicate some exponential rate of divergence of orbits of nearby points. As far as we know the first to formulate the sensitivity was Guckenheimer [35], in his study on maps of the interval (he required the condition to hold for a set of positive Lebesgue measure). The precise expression of sensitivity in the above form was introduced by Auslander and Yorke [8].

4.1 Equicontinuity and Sensitivity

The opposite side of sensitivity is the notion of equicontinuity.

Definition 4.1 *A dynamical system (X, T) is called* equicontinuous *if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for $n = 0, 1, 2, \ldots$, that is the family of maps $\{T^n : n \in \mathbb{Z}_+\}$ is uniformly equicontinuous.*

Equicontinuous systems have simple dynamical behaviors. It is well known that a dynamical system (X, T) with T being surjective is equicontinuous if and only if there exists a compatible metric ρ on X such that T acts on X as an isometry, i.e., $\rho(Tx, Ty) = \rho(x, y)$ for any $x, y \in X$. See [74] for the structure of equicontinuous systems.

We have the following dichotomy result for minimal systems.

Theorem 4.2 ([8]) *A minimal system is either equicontinuous or sensitive.*

Equicontinuity can be localized in an obvious way.

Definition 4.3 *Let (X, T) be a dynamical system. A point $x \in X$ is called an* equicontinuous point *if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{N}$. A transitive system is called* almost equicontinuous *if there exists some equicontinuous point.*

We have the following dichotomy result for transitive systems.

Theorem 4.4 ([3, 29]) *Let (X, T) be a transitive system. Then either*

1. *(X, T) is almost equicontinuous, in this case the collection of equicontinuous points coincides with the collection of transitive points*; *or*
2. *(X, T) is sensitive.*

It is interesting that almost equicontinuity is closely related to the uniform rigid property which was introduced by Glasner and Maon in [33] as a topological analogue of rigidity in ergodic theory.

Theorem 4.5 ([29]) *Let (X, T) be a transitive system. Then it is uniformly rigid if and only if it is a factor of an almost equicontinuous system. In particular, every almost equicontinuous* *system is uniformly rigid.*

It is shown in [33] that every uniformly rigid system has zero topological entropy. Then by Theorem 4.5 every almost equicontinuous system also has zero topological entropy.

4.2 n-sensitivity and Sensitive Sets

Among other things, Xiong [107] introduced a new notion called n-sensitivity, which says roughly that in each non-empty open subset there are n distinct points whose trajectories are apart from (at least for one common moment) a given positive constant pairwise.

Definition 4.6 *Let (X,T) be a dynamical system and $n \geq 2$. The system (X,T) is called n-*sensitive*, if there exists some $\delta > 0$ such that for any $x \in X$ and $\varepsilon > 0$ there are $x_1, x_2, \ldots, x_n \in B(x,\varepsilon)$ and $k \in \mathbb{N}$ satisfying*

$$\min_{1 \leq i < j \leq n} d(T^k x_i, T^k x_j) > \delta.$$

Proposition 4.7 ([107]) *If a dynamical system (X,T) is weakly mixing, then it is n-sensitive for all $n \geq 2$. Moreover, if a dynamical system (X,T) on a locally connected space X is sensitive, then it is n-sensitive for all $n \geq 2$.*

In [94], Shao et al. studied the properties of n-sensitivity for minimal systems, and showed that n-sensitivity and $(n+1)$-sensitivity are essentially different.

Theorem 4.8 ([94]) *For every $n \geq 2$, there exists a minimal system (X,T) which is n-sensitive but not $(n+1)$-sensitive.*

Recently, using ideas and results from local entropy theory, Ye and Zhang [111] and Huang et al. [42] developed a theory of sensitive sets, which measures the "degree" of sensitivity both in the topological and the measure-theoretical setting.

Definition 4.9 *Let (X,T) be a dynamical system. A subset A of X is* sensitive *if for any $n \geq 2$, any n distinct points $x_1, x_2, \ldots, x_n$ in A, any neighborhood U_i of x_i, $i = 1, 2, \ldots, n$, and any non-empty open subset U of X there exist $k \in \mathbb{N}$ and $y_i \in U$ such that $T^k y_i \in U_i$ for $i = 1, 2, \ldots, n$.*

It is shown in [111] that a transitive system is n-sensitive if and only if there exists a sensitive set with cardinality n. Moreover, a dynamical system is weakly mixing if and only if the whole space X is sensitive.

Theorem 4.10 ([111]) *If a dynamical system is transitive, then every entropy set is also a sensitive set. This implies that if a transitive system has positive topological entropy, then there exists an uncountable sensitive set.*

The number of minimal subsets is related to the cardinality of sensitive sets. For example, it was shown in [111] that if a transitive system has a dense set of minimal points but is not minimal, then there exists an infinite sensitive set. Moreover, if there are uncountable pairwise disjoint minimal subsets, then there exists an uncountable sensitive set.

In 2011, Huang et al. got a fine structure of sensitive sets in minimal systems.

Theorem 4.11 ([42]) *Let (X,T) be a minimal dynamical system and $\pi : (X,T) \to (Y,S)$ be the factor map to the maximal equicontinuous factor of (X,T). Then*

1. *each sensitive set of (X,T) is contained in some $\pi^{-1}(y)$ for some $y \in Y$;*

2. *for each* $y \in Y$, $\pi^{-1}(y)$ *is a sensitive set of* (X,T).

Consequently, (X,T) *is* n*-sensitive, not* $(n+1)$*-sensitive, if and only if* $\max_{y\in Y} \#(\pi^{-1}(y)) = n$.

4.3 Li–Yorke Sensitivity

A concept that combines sensitivity and Li–Yorke δ-scrambled pairs was proposed by Akin and Kolyada in [6], which is called Li–Yorke sensitivity.

Definition 4.12 *A dynamical system* (X,T) *is called* Li–Yorke sensitive *if there exists some* $\delta > 0$ *such that for any* $x \in X$ *and* $\varepsilon > 0$, *there is* $y \in X$ *satisfying* $d(x,y) < \varepsilon$ *such that*

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \textit{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > \delta.$$

Let (X,T) be a dynamical system and $x \in X$. The *proximal cell* of x is the set $\{y \in X : (x,y) \textit{ is proximal}\}$. First, it was proved in [55] that for a weakly mixing (X,T) the set of points x at which the proximal cell is residual in X is itself residual in X. Later it was proved by Furstenberg [26] that the proximal cell of every point is residual, provided that the system is minimal and weakly mixing. Finally, Akin and Kolyada proved in [6] that in any weakly mixing system the proximal cell of every point is residual. Note that the authors in [43] got more about the structure of the proximal cells of $\mathcal{F}$-mixing systems, where $\mathcal{F}$ is an Furstenberg family.

In [6], Akin and Kolyada also proved that

Theorem 4.13 ([6]) *If a non-trivial dynamical system* (X,T) *is weakly mixing, then it is Li–Yorke sensitive.*

See a survey paper [56] for more results about Li–Yorke sensitivity and its relation to other concepts of chaos. But the following question remains open.

Problem 3 Are all Li–Yorke sensitive systems Li–Yorke chaotic?

4.4 Mean Equicontinuity and Mean Sensitivity

Definition 4.14 *A dynamical system* (X,T) *is called* mean equicontinuous *if for every* $\varepsilon > 0$, *there exists some* $\delta > 0$ *such that whenever* $x, y \in X$ *with* $d(x,y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

This definition is equivalent to the notion of mean-L-stability which was first introduced by Fomin [23]. It is an open question if every ergodic invariant measure on a mean-L-stable system has discrete spectrum [91]. The authors in [65] firstly gave an affirmative answer to this question. See [27] for another approach to this question.

Theorem 4.15 ([65]) *If a dynamical system* (X,T) *is mean equicontinuous, then every ergodic invariant measure on* (X,T) *has discrete spectrum and hence the topological entropy of* (X,T) *is zero.*

Similarly, we can define the local version and the opposite side of mean equicontinuity.

Definition 4.16 *Let* (X,T) *be a dynamical system. A point* $x \in X$ *is called* mean equicontinuous *if for every* $\varepsilon > 0$*, there exists some* $\delta > 0$ *such that for every* $y \in X$ *with* $d(x, y) < \delta$*,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

A transitive system is called almost mean equicontinuous *if there is at least one mean equicontinuous point.*

Definition 4.17 *A dynamical system* (X, T) *is called* mean sensitive *there exists some* $\delta > 0$ *such that for every* $x \in X$ *and every neighborhood* U *of* x*, there exists* $y \in U$ *and* $n \in \mathbb{N}$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

We have the following dichotomy for transitive systems and minimal systems.

Theorem 4.18 ([65]) *If a dynamical system* (X, T) *is transitive, then* (X, T) *is either almost mean equicontinuous or mean sensitive. In particular, if* (X, T) *is a minimal system, then* (X, T) *is either mean equicontinuous or mean sensitive.*

Recall that an almost equicontinuous system is uniformly rigid and thus has zero topological entropy. The following Theorem 4.19 shows that an almost mean equicontinuous system behaves quite differently.

Theorem 4.19 ([65]) *In the full shift* (Σ_2, σ)*, every minimal subshift* (Y, σ) *is contained in an almost mean equicontinuous subshift* (X, σ)*.*

Since it is well known that there are many minimal subshifts of (Σ_2, σ) with positive topological entropy, an immediate corollary of Theorem 4.19 is the following result.

Corollary 4.20 *There exist many almost mean equicontinuous systems which have positive topological entropy.*

Globally speaking a mean equicontinuous system is "simple", since it is a Banach proximal extension of an equicontinuous system and each of its ergodic measures has discrete spectrum. Unfortunately, the local version does not behave so well, as Theorem 4.19 shows. We will introduce the notion of Banach mean equicontinuity, whose local version has the better behavior that we are looking for.

Let (X, T) be a dynamical system. We say that (X, T) is *Banach mean equicontinuous* if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\limsup_{M-N\to\infty} \frac{1}{M-N} \sum_{i=N}^{M-1} d(T^i x, T^i y) < \varepsilon.$$

A point $x \in X$ is called *Banach mean equicontinuous* if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $y \in B(x, \delta)$,

$$\limsup_{M-N\to\infty} \frac{1}{M-N} \sum_{i=N}^{M-1} d(T^i x, T^i y) < \varepsilon.$$

We say that a transitive system (X, T) is *almost Banach mean equicontinuous* if there exists a transitive point which is Banach mean equicontinuous.

A dynamical system (X,T) is *Banach mean sensitive* if there exists some $\delta>0$ such that for every $x\in X$ and every $\varepsilon>0$ there is $y\in B(x,\varepsilon)$ satisfying

$$\limsup_{M-N\to\infty}\frac{1}{M-N}\sum_{i=N}^{M-1}d(T^ix,T^iy)>\delta.$$

We also have the following dichotomy for transitive systems and minimal systems.

Theorem 4.21 ([65]) *If a dynamical system (X,T) is transitive, then (X,T) is either almost Banach mean equicontinuous or Banach mean sensitive. If (X,T) is a minimal system, then (X,T) is either Banach mean equicontinuous or Banach mean sensitive.*

Theorem 4.22 ([65]) *Let (X,T) be a transitive system. If the topological entropy of (X,T) is positive, then (X,T) is Banach mean sensitive.*

Combining Theorems 4.21 and 4.22, we have the following corollary.

Corollary 4.23 *If (X,T) is almost Banach mean equicontinuous, then the topological entropy of (X,T) is zero.*

By Corollary 4.20, there are many almost mean equicontinuous systems which have positive entropy. Then by Corollary 4.23, they are not almost Banach mean equicontinuous. But the following question is still open.

Problem 4 Is there a minimal system which is mean equicontinuous but not Banach mean equicontinuous?

4.5 Other Generalization of Sensitivity

Note that sensitivity can be generalized in other ways, see [38, 76, 104] for example.

Let (X,T) be a dynamical system. For $\delta>0$ and a non-empty open subset U, let

$$\begin{aligned}N(\delta,U)&=\{n\in\mathbb{N}\colon \exists x,y\in U \text{ with } d(T^nx,T^ny)>\delta\}\\&=\{n\in\mathbb{N}\colon \operatorname{diam}(T^n(U))>\delta\}.\end{aligned}$$

Let $\mathcal{F}$ be a Furstenberg family. According to Moothathu [76], we say that a dynamical system (X,T) is *$\mathcal{F}$-sensitive* if there exists some $\delta>0$ such that for any non-empty open subset U of X, $N(\delta,U)\in\mathcal{F}$. $\mathcal{F}$-sensitivity for some special families were discussed in [27, 67, 71, 102].

A dynamical system (X,T) is called *multi-sensitive* if there exists some $\delta>0$ such that for any finite open non-empty subsets $U_1,\dots,U_n$ of X,

$$\bigcap_{i=1}^{n}N(\delta,U_i)\neq\emptyset.$$

In [38] among other things, Huang et al. proved that for a minimal system thick sensitivity is equivalent to multi-sensitivity. Moreover, they showed the following dichotomy for minimal systems.

Theorem 4.24 *Let (X,T) be a minimal system. Then (X,T) is multi-sensitive if and only if it is not an almost one-to-one extension of its maximal equicontinuous factor.*

Results similar to Theorem 4.24 for other families will be appeared in [110] by Ye and Yu.

5 Chaos in Transitive Systems

In this section, we discuss various kinds of chaos in transitive systems.

5.1 Auslander–Yorke Chaos

In [8] Auslander and Yorke defined a kind of chaos as "topological transitivity plus pointwise instability". This leads to the following definition of Auslander–Yorke chaos.

Definition 5.1 *A dynamical system* (X,T) *is called* Auslander–Yorke chaotic *if it both transitive and sensitive.*

Due to Ruelle and Takens' work on turbulence [89], Auslander–Yorke chaos was also called *Ruelle–Takens chaos* (see [107] for example). It should be noticed that there is no implication relation between Li–Yorke chaos and Auslander–Yorke chaos. For example, any non-equicontinuous distal minimal system is sensitive, so it is Auslander–Yorke chaotic, but it has no Li–Yorke scrambled pairs. On the other hand, there are non-periodic transitive systems with a fixed point that are not sensitive [4]; by Theorem 5.6 they are Li–Yorke chaotic.

The work of Wiggins [105] leads to the following definition.

Definition 5.2 *A dynamical system* (X,T) *is called* Wiggins chaotic *if there exists a subsystem* (Y,T) *of* (X,T) *such that* (Y,T) *is both transitive and sensitive.*

In [90], Ruette investigated transitive and sensitive subsystems for interval maps. She showed that

Theorem 5.3 1. *For a transitive map* $f\colon [0,1]\to[0,1]$, *if it is Wiggins chaotic, then it is also Li–Yorke chaotic.*

2. *There exists a continuous map* $f\colon [0,1]\to[0,1]$ *of zero topological entropy which is Wiggins chaotic.*

3. *There exists a Li–Yorke chaotic continuous map* $f\colon [0,1]\to[0,1]$ *which is not Wiggins chaotic.*

5.2 Devaney Chaos

In his book [19], Devaney proposed a new kind of chaos, which is usually called Devaney chaos.

Definition 5.4 *A dynamical system* (X,T) *is called* Devaney chaotic *if it satisfies the following three properties:*

1. (X,T) *is transitive;*
2. (X,T) *has sensitive dependence on initial conditions;*
3. *the set of periodic points of* (X,T) *is dense in* X.

Sensitive dependence is widely understood as the central idea in Devaney chaos, but it is implied by transitivity and density of periodic points, see [12] or [29]. In 1993, Li [68] showed that

Theorem 5.5 *Let* $f : [0,1]\to[0,1]$ *be a continuous map. Then* f *has positive topological entropy if and only if there exists a Devaney chaotic subsystem.*

In 1996, Akin et al. [3] discussed in details when a transitive map is sensitive. An important question is that: does Devaney chaos imply Li–Yorke chaos? In 2002, Huang and Ye gave a positive answer to this long-standing open problem.

Theorem 5.6 ([46]) *Let (X,T) be a non-periodic transitive system. If there exists a periodic point, then it is Li–Yorke chaos. Particularly, Devaney chaos implies Li–Yorke chaos.*

A key result in the proof of Theorem 5.6 is called *Huang–Ye equivalences* in [6]. To state this result, we need some notations.

For every $\varepsilon > 0$, let $\overline{V_\varepsilon} = \{(x,y) \in X \times X \colon d(x,y) \leq \varepsilon\}$ and

$$\mathrm{Asym}_\varepsilon(T) = \bigcup_{n=0}^{\infty} \left(\bigcap_{k=n}^{\infty} T^{-k}\overline{V_\varepsilon} \right).$$

For every $x \in X$, let

$$\mathrm{Asym}_\varepsilon(T)(x) = \big\{ y \in X \colon (x,y) \in \mathrm{Asym}_\varepsilon(T) \big\}.$$

Theorem 5.7 (Huang–Ye Equivalences) *For a dynamical system (X,T) the following conditions are equivalent.*

1. *(X,T) is sensitive.*
2. *There exists $\varepsilon > 0$ such that $\mathrm{Asym}_\varepsilon(T)$ is a first category subset of $X \times X$.*
3. *There exists $\varepsilon > 0$ such that for every $x \in X$, $\mathrm{Asym}_\varepsilon(T)(x)$ is a first category subset of X.*
4. *There exists $\varepsilon > 0$ such that for every $x \in X$, $x \in \overline{X \setminus \mathrm{Asym}_\varepsilon(T)(x)}$.*
5. *There exists $\varepsilon > 0$ such that the set*

$$\Big\{ (x,y) \in X \times X \colon \limsup_{n\to\infty} d(T^n x, T^n y) > \varepsilon \Big\}$$

is dense in $X \times X$.

Note that the first part of Theorem 5.6 does not involve the sensitive property directly. In 2004, Mai proved that Devaney chaos implies the existence of δ-scrambled sets for some $\delta > 0$. Using a method of direct construction, he showed that

Theorem 5.8 ([75]) *Let (X,T) be a non-periodic transitive system. If there exists a periodic point of period p, then there exist Cantor sets $C_1 \subset C_2 \subset \cdots$ in X consisting of transitive points such that*

1. *each C_n is a synchronously proximal set, that is $\liminf_{k\to\infty} \mathrm{diam}(T^k C_n) = 0$;*
2. *$S := \bigcup_n C_n$ is scrambled, and $\bigcup_{i=0}^{p-1} T^i S$ is dense in X;*
3. *furthermore, if (X,T) is sensitive, then S is δ-scrambled for some $\delta > 0$.*

5.3 Multivariant Li–Yorke Chaos

The scrambled set in Li–Yorke chaos only compares the trajectories of two points. It is natural to consider the trajectories of finite points. In 2005, Xiong introduced the following multivariant chaos in the sense of Li–Yorke.

Definition 5.9 ([107]) *Let (X,T) be a dynamical system and $n \geq 2$. A tuple $(x_1, x_2, \ldots, x_n) \in X^n$ is called* n-scrambled *if*

$$\liminf_{k\to\infty} \max_{1\leq i<j\leq n} d\big(f^k(x_i), f^k(x_j)\big) = 0$$

and

$$\limsup_{k\to\infty} \min_{1\leq i<j\leq n} d\big(f^k(x_i), f^k(x_j)\big) > \delta > 0.$$

A subset C of X is called n-scrambled *if any pairwise distinct n points in C form an n-scrambled tuple. The dynamical system (X,T) is called* Li–Yorke n-chaotic *if there exists an uncountable n-scrambled set.*

Similarly, if the separated constant δ is uniform for all pairwise distinct n-tuples in C, we can define n-δ-scrambled sets *and* Li–Yorke n-δ-chaos.

Theorem 5.10 ([107]) *Let (X,T) be a non-trivial transitive system. If (X,T) has a fixed point, then there exists a dense Mycielski subset C of X such that C is n-scrambled for all $n \geq 2$.*

In 2011, Li proved that there is no 3-scrambled tuples for an interval map with zero topological entropy.

Theorem 5.11 ([58]) *Let $f : [0,1] \to [0,1]$ be a continuous map. If f has zero topological entropy, then there is no 3-scrambled tuples.*

Note that there exists a continuous map $f : [0,1] \to [0,1]$, which is Li–Yorke chaotic but has zero topological entropy (see Theorem 3.4). Then we have the following result, which shows that Li–Yorke 2-chaos and 3-chaos are essentially different.

Corollary 5.12 *There exists a Li–Yorke 2-chaotic system which has no 3-scrambled tuples.*

5.4 Uniform Chaos

Recall that a dynamical system (X,T) is *scattering* if for any minimal system (Y,S) the product system $(X \times Y, T \times S)$ is transitive. It is not hard to show that every weakly mixing system is scattering. In [46], Huang and Ye also proved that scattering implies Li–Yorke chaos.

Theorem 5.13 *If a non-trivial dynamical system (X,T) is scattering, then there is a dense Mycielski scrambled set.*

Since every system has a minimal subsystem, for a scattering system (X,T) there exists a subsystem (Y,S) of (X,T) such that $(X \times Y, T \times T)$ is transitive. In 2010, Akin et al. showed that a dynamical system satisfying this property is more complicated than Li–Yorke chaos, and proposed the concept of uniform chaos [5].

Definition 5.14 *Let (X,T) be a dynamical system. A subset K of X is said to be*

(1) uniformly recurrent *if for every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ with $d(T^n x, x) < \varepsilon$ for all $x \in K$;*

(2) recurrent *if every finite subset of K is uniformly recurrent;*

(3) uniformly proximal *if for every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ with* $\mathrm{diam}(T^n K) < \varepsilon$;

(4) proximal *if every finite subset of K is uniformly proximal.*

Definition 5.15 *Let (X,T) be a dynamical system. A subset $K \subset X$ is called a* uniformly chaotic set *if there are Cantor sets $C_1 \subset C_2 \subset \cdots$ such that*

1. *for each $N \in \mathbb{N}$, C_N is uniformly recurrent;*
2. *for each $N \in \mathbb{N}$, C_N is uniformly proximal;*
3. *$K := \bigcup_{i=1}^{\infty} C_n$ is a recurrent subset of X and also a proximal subset of X.*

The system (X,T) is called (densely) uniformly chaotic *if it has a (dense) uniformly chaotic subset of X.*

By the definition, any uniformly chaotic set is scrambled, then uniform chaos implies Li–Yorke chaos. The following is the main result of [5].

Theorem 5.16 *Let (X,T) be a non-trivial dynamical system. If there exists a subsystem (Y,T) of (X,T) such that $(X\times Y, T\times T)$ is transitive, then (X,T) is densely uniformly chaotic.*

As a corollary, we can show that many transitive systems are uniformly chaotic. Note that a dynamical system is *weakly scattering* if its product with any minimal equicontinuous system is transitive.

Corollary 5.17 *If (X,T) is a dynamical system without isolated points and satisfies one of the following properties, then it is densely uniformly chaotic*:

1. (X,T) *is transitive and has a fixed point*;
2. (X,T) *is totally transitive with a periodic point*;
3. (X,T) *is scattering*;
4. (X,T) *is weakly scattering with an equicontinuous minimal subsystem*;
5. (X,T) *is weakly mixing.*

Furthermore, if (X,T) is transitive and has a periodic point of period p, then there is a closed T^p-invariant subset X_0 of X, such that (X_0,T^p) is densely uniformly chaotic and $X=\bigcup_{j=0}^{p-1}T^jX_0$. In particular, (X,T) is uniformly chaotic.

Remark 5.18 By Corollary 5.17, Devaney chaos implies uniform chaos.

6 Various Types of Distributional Chaos and Its Generalization

6.1 Distributional chaos

In [92], Schweizer and Smítal used ideas from probability theory to develop a new definition of chaos, so called *distributional chaos*. Let (X,T) be a dynamical system. For a pair of points x,y in X and a positive integer n, we define a function Φ_{xy}^n on the real line by

$$\Phi_{xy}^n(t)=\frac{1}{n}\#\big\{0\le i\le n-1\colon d(T^ix,T^iy)<t\big\},$$

where $\#(\cdot)$ denotes the number of elements of a set. Clearly, the function Φ_{xy}^n is non-decreasing, has minimal value 0 (since $\Phi_{xy}^n(t)=0$ for all $t\le 0$), has maximum value 1 (since $\Phi_{xy}^n(t)=1$ for all t greater than the diameter of X), and is left continuous. Then Φ_{xy}^n is a distribution function whose value at t may be interpreted as the probability that the distance between the initial segment of length n of the trajectories of x and y is less than t.

We are interested in the asymptotic behavior of the function Φ_{xy}^n as n gets large. To this end, we consider the functions Φ_{xy}^* and Φ_{xy} defined by

$$\Phi_{xy}^*(t)=\limsup_{n\to\infty}\Phi_{xy}^n(t)\quad\text{and}\quad\Phi_{xy}(t)=\liminf_{n\to\infty}\Phi_{xy}^n(t).$$

Then Φ_{xy}^* and Φ_{xy} are distribution functions with $\Phi_{xy}(t)\le\Phi_{xy}^*(t)$ for all t. It follows that Φ_{xy}^* is an asymptotic measure of how close x and y can come together, while Φ_{xy} is an asymptotic measure of their maximum separation. We shall refer to Φ_{xy}^* as the *upper distribution*, and to Φ_{xy} as the *lower distribution* of x and y.

Definition 6.1 *Let (X,T) be a dynamical system. A pair $(x,y)\in X\times X$ is called* distributionally scrambled *if*

1. *for every* $t > 0$, $\Phi^*_{xy}(t) = 1$, *and*
2. *there exists some* $\delta > 0$ *such that* $\Phi_{xy}(\delta) = 0$.

A subset C *of* X *is called* distributionally scrambled *if any two distinct points in* C *form a distributionally scrambled pair. The dynamical system* (X, T) *is called* distributionally chaotic *if there exists an uncountable distributionally scrambled set.*

Similarly, if the separated constant δ *is uniform for all non-diagonal pairs in* C, *we can define* distributionally δ-scrambled sets *and* distributional δ-chaos.

According to the definitions, it is clear that any distributionally scrambled pair is scrambled, and then distributional chaos is stronger than Li–Yorke chaos. In [92], Schweizer and Smítal showed that

Theorem 6.2 ([92]) *Let* $f\colon [0,1] \to [0,1]$ *be a continuous map. Then* f *is distributionally chaotic if and only if it has positive topological entropy.*

In 1998, Liao and Fan [70] constructed a minimal system with zero topological entropy which is distributionally chaotic. In 2006, Oprocha [80] showed that weak mixing and Devaney chaos do not imply distributional chaos.

In Section 5.3, Li–Yorke chaos is extended to a multivariant version. In fact, it is clear that any kind of chaos defined by scrambled pairs can be extended to multivariant version.

In [100], Tan and Fu showed that distributionally n-chaos and $(n+1)$-chaos are essentially different.

Theorem 6.3 ([100]) *For every* $n \geq 2$, *there exists a transitive system which is distributionally* n*-chaotic but without any distributionally* $(n+1)$*-scrambled tuples.*

In 2013, Li and Oprocha showed that

Theorem 6.4 ([62]) *For every* $n \geq 2$, *there exists a dynamical system which is distributionally* n*-chaotic but not Li–Yorke* $(n+1)$*-chaotic.*

Note that the example constructed in the proof of Theorem 6.4 contains $(n+1)$-scrambled tuples. Then, a natural question is as following.

Problem 5 Is there a dynamical system (X, T) with an uncountable distributionally 2-scrambled set but without any 3-scrambled tuples?

Recently, Doleželová made some progress on Problem 5, but the original problem is still open. She showed that

Theorem 6.5 ([20]) *There exists a dynamical system* X *with an infinite extremal distributionally scrambled set but without any scrambled triple.*

Theorem 6.6 ([20]) *There exists an invariant Mycielski* (*not closed*) *set* X *in the full shift with an uncountable extremal distributionally* 2*-scrambled set but without any* 3*-scrambled tuple.*

6.2 The Three Versions of Distributional Chaos

Presently, we have at least three different definitions of distributionally scrambled pair (see [98] and [10]). The original distributionally scrambled pair is said to be *distributionally scrambled of type* 1.

Definition 6.7 *A pair* $(x, y) \in X \times X$ *is called* distributionally scrambled of type 2 *if*

1. *for any* $t>0$, $\Phi^*_{xy}(t)=1$, *and*
2. *there exists some* $\delta>0$ *such that* $\Phi_{xy}(\delta)<1$.

A pair $(x,y)\in X\times X$ *is called* distributionally scrambled of type 3 *if there exists an interval* $[a,b]\subset(0,\infty)$ *such that* $\Phi_{xy}(t)<\Phi^*_{xy}(t)$ *for all* $t\in[a,b]$.

A subset C *of* X *is called* distributionally scrambled of type i ($i=1,2,3$) *if any two distinct points in* C *form a distributionally scrambled pair of type* i. *The dynamical system* (X,T) *is called* distributionally chaotic of type i (DCi *for short*) *if there exists an uncountable distributionally scrambled set of type* i.

It should be noticed that in [98] and [10] DCi chaos only requires the existence of one distributionally scrambled pair of type i. It is not hard to see that any distributionally scrambled pair of type 1 or 2 is topological conjugacy invariant [98]. But a distributionally scrambled pair of type 3 may not be topological conjugacy invariant [10]. It is also not hard to construct a dynamical system has a distributionally scrambled pair of type 2, but no distributionally scrambled pairs of type 1. A really interesting example constructed in [101] shows that there exists a minimal subshift which is distributionally chaotic of type 2, while it does not contain any distributionally scrambled pair of type 1.

6.3 The Relation between Distributional Chaos and Positive Topological Entropy

It is known that in general there is no implication relation between DC1 and positive topological entropy (see [70] and [86]). In [97] Smítal conjectured that positive topological entropy does imply DC2. Oprocha showed that this conjecture is true for uniformly positive entropy minimal systems [81]. Finally, Downarowicz solved this problem by proving this conjecture in general case [21].

Theorem 6.8 ([21]) *If a dynamical system* (X,T) *has positive topological entropy, then there exists a Cantor distributional* δ*-scrambled set of type* 2 *for some* $\delta>0$.

It was observed in [21] that a pair (x,y) is DC2-scrambled if and only if it is mean scrambled in the Li–Yorke sense, that is,

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}d\big(T^kx,T^ky\big)=0$$

and

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}d\big(T^kx,T^ky\big)>0.$$

Recently, Huang et al. showed that positive topological entropy implies a multivariant version of mean Li–Yorke chaos.

Theorem 6.9 ([40]) *If a dynamical system* (X,T) *has positive topological entropy, then there exists a Mycielski multivariant mean Li–Yorke* (δ_n)*-scrambled subset* K *of* X, *that is, for every* $n\geq 2$ *and every* n *pairwise distinct points* $x_1,\dots,x_n\in K$, *we have*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\max_{1\leq i<j\leq n}d\big(T^kx_i,T^kx_j\big)=0$$

and

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \min_{1\le i<j\le n} d\big(T^k x_i, T^k x_j\big) > \delta_n > 0.$$

We remark that the key tool in the proof of Theorem 6.9 is the excellent partition constructed in [16, Lemma 4], which is different from the one used in [21]. So among other things for $n = 2$ the authors also obtained a new proof of Theorem 6.8.

Note that in [16] Blanchard et al. showed that the closure of the set of asymptotic pairs contains the set of entropy pairs. Kerr and Li [53] proved that the intersection of the set of scrambled pairs with the set of entropy pairs is dense in the set of entropy pairs. We extended the above mentioned results to the following result.

Theorem 6.10 ([40]) *If a dynamical system has positive topological entropy, then for any $n \ge 2$,*

1. *the intersection of the set of asymptotic n-tuples with the set of topological entropy n-tuples is dense in the set of topological entropy n-tuples;*
2. *the intersection of the set of mean n-scrambled tuples with the set of topological entropy n-tuples is dense in the set of topological entropy n-tuples.*

6.4 Chaos via Furstenberg Families

In Section 2, we have shown that there is a powerful connection between topological dynamics and Furstenberg families. In 2007, Xiong et al. introduced the notion of chaos via Furstenberg families. It turned out that the Li–Yorke chaos and some version of distributional chaos can be described as chaos in Furstenberg families sense.

Definition 6.11 ([101, 108]) *Let (X,T) be a dynamical system. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Furstenberg families. A pair $(x,y) \in X\times X$ is called* $(\mathcal{F}_1,\mathcal{F}_2)$-scrambled *if it satisfies*

1. *for any $t>0$, $\{n\in\mathbb{Z}_+ : d(T^n x, T^n y) < t\} \in \mathcal{F}_1$, and*
2. *there exists some $\delta>0$ such that $\{n\in\mathbb{Z}_+ : d(T^n x, T^n y) > \delta\} \in \mathcal{F}_2$.*

A subset C of X is called $(\mathcal{F}_1,\mathcal{F}_2)$-scrambled *if any two distinct points in C form a $(\mathcal{F}_1,\mathcal{F}_2)$-scrambled pair. The dynamical system (X,T) is called* $(\mathcal{F}_1,\mathcal{F}_2)$-chaotic *if there is an uncountable $(\mathcal{F}_1,\mathcal{F}_2)$-scrambled set in X.*

Similarly, if the separated constant δ is uniform for all non-diagonal pairs in C, we can define $(\mathcal{F}_1,\mathcal{F}_2)$-$\delta$-scrambled sets *and* $(\mathcal{F}_1,\mathcal{F}_2)$-$\delta$-chaos.

Let F be a subset of $\mathbb{Z}_+$. The *upper density* of F is defined as

$$\overline{d}(F) = \limsup_{n\to\infty} \frac{1}{n}\#(F\cap\{0,1,\dots,n-1\}).$$

For any $a\in[0,1]$, let

$$\overline{\mathcal{M}}(a) = \{F\subset\mathbb{Z}_+ : F \text{ is infinite and } \overline{d}(F)\ge a\}.$$

Clearly, every $\overline{\mathcal{M}}(a)$ is a Furstenberg family. The following proposition can be easily verified by the definitions, which shows that Li–Yorke chaos and distributional chaos can be characterized via Furstenberg families.

Proposition 6.12 *Let (X,T) be a dynamical system and $(x,y)\in X\times X$. Then*

(1) *(x,y) is scrambled if and only if it is $(\overline{\mathcal{M}}(0),\overline{\mathcal{M}}(0))$-scrambled.*

(2) (x, y) *is distributionally scrambled of type* 1 *if and only if it is* $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$*-scrambled.*

(3) (x, y) *is distributionally scrambled of type* 2 *if and only if it is* $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(t))$*-scrambled with some* $t > 0$*.*

A Furstenberg family $\mathcal{F}$ is said to be *compatible* with the dynamical system (X, T), if for every non-empty open subset U of X the set $\{x \in X \colon \{n \in \mathbb{Z}_+ \colon T^n x \in U\} \in \mathcal{F}\}$ is a G_δ subset of X.

Lemma 6.13 ([108]) *If* $\mathcal{F} = \overline{\mathcal{M}}(t)$ *for some* $t \in [0, 1]$*, then* $\mathcal{F}$ *is compatible with any dynamical system* (X, T)*.*

We have the following criteria for chaos via compatible Furstenberg families.

Theorem 6.14 ([101, 108]) *Let* (X, T) *be a dynamical system. Suppose that there exists a fixed point* $p \in X$ *such that the set* $\bigcup_{i=1}^{\infty} T^{-i}(p)$ *is dense in* X*, and a non-empty closed invariant subset* A *of* X *disjoint with the point* p *such that* $\bigcup_{i=1}^{\infty} T^{-i}A$ *is dense in* X*. Then for every two Furstenberg families* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *compatible with the system* $(X \times X, T \times T)$*, there exists a dense Mycielski* $(\mathcal{F}_1, \mathcal{F}_2)$*-*$\delta$*-scrambled set for some* $\delta > 0$*. In particular, the dynamical system* (X, T) *is* $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$*-*$\delta$*-chaotic for some* $\delta > 0$*.*

There are two important Furstenberg families: the collections of all syndetic sets and all sets with Banach density one. They are so specifical that we use terminologies: syndetically scrambled sets and Banach scrambled sets, when studying chaos via those two Furstenberg families.

Recall that a subset F of $\mathbb{Z}_+$ is *syndetic* if there is $N \in \mathbb{N}$ such that $[n, n+N] \cap F \neq \emptyset$ for every $n \in \mathbb{Z}_+$. Let (X, T) be a dynamical system. A pair of points $(x, y) \in X^2$ is called *syndetically proximal* if for every $\varepsilon > 0$ the set $\{n \in \mathbb{Z}_+ \colon d(T^n x, T^n y) < \varepsilon\}$ is syndetic. If (X, T) is proximal, that is every pair in X^2 is proximal, then every pair in X^2 is also syndetically proximal [77]. Moothathu studied the syndetically proximal relation, and identified certain sufficient conditions for the syndetically proximal cell of each point to be small [77]. A subset S of X is called *syndetically scrambled* if for any two distinct points $x, y \in S$, (x, y) is syndetically proximal but not asymptotic. In [78] Moothathu and Oprocha systematically studied the syndetically proximal relation and the possible existence of syndetically scrambled sets for many kinds of dynamical systems, including various classes of transitive subshifts, interval maps, and topologically Anosov maps.

A subset F of $\mathbb{Z}_+$ is said to have *Banach density one* if for every $\lambda < 1$ there exists $N \geq 1$ such that $\#(F \cap I) \geq \lambda \#(I)$ for every subinterval I of $\mathbb{Z}_+$ with $\#(I) \geq N$, where $\#(I)$ denotes the number of elements of I. A pair of points $(x, y) \in X^2$ is called *Banach proximal* if for every $\varepsilon > 0$ the set $\{n \in \mathbb{Z}_+ \colon d(T^n x, T^n y) < \varepsilon\}$ has Banach density one. Clearly, every set with Banach density one is syndetic, then a Banach proximal pair is syndetically proximal. In [64] Li and Tu studied the structure of the Banach proximal relation. Particularly, they showed that for a non-trivial minimal system the Banach proximal cell of every point is of first category.

A subset S of X is called *Banach scrambled* if for any two distinct points $x, y \in S$, (x, y) is Banach proximal but not asymptotic. Note that it was shown in [58] that for an interval map with zero topological entropy, every proximal pair is Banach proximal. Moreover we have

Theorem 6.15 ([64]) *Let* $f \colon [0, 1] \to [0, 1]$ *be a continuous map. If* f *is Li–Yorke chaotic,* *then it has a Cantor Banach scrambled set.*

Theorem 6.16 ([64]) *Let $f\colon [0,1] \to [0,1]$ be a continuous map. Then f has positive topological entropy if and only if there is Cantor set $S \subset [0,1]$ such that S is a Banach scrambled set and $f^n(S) \subset S$ for some $n \geq 1$.*

There exists a dynamical system with the whole space being a scrambled set (see Theorems 3.13 and 3.14), and then every pair is syndetically proximal. It is shown in [64] that the whole space also can be a Banach scrambled set.

Theorem 6.17 ([64]) *There exists a transitive system with the whole space being a Banach scrambled set.*

7 Local Aspects of Mixing Properties

As we said before for a dynamical system with positive topological entropy, it is possible that there is no weakly mixing subsystem. The notion of weakly mixing sets captures some complicated subset of the system.

7.1 Weakly Mixing Sets

In 1991, Xiong and Yang [109] showed that for in a weakly mixing system there are considerably many points in the domain whose orbits display highly erratic time dependence. More specifically, they showed that

Theorem 7.1 *Let (X,T) be a non-trivial dynamical system. Then*

1. *(X,T) is weakly mixing if and only if there exists a c-dense F_σ subset C of X satisfying for any subset D of C and any continuous map $f\colon D \to X$, there exists an increasing sequence $\{q_i\}$ of positive integers such that $\lim_{i\to\infty} T^{q_i}x = f(x)$ for all $x \in D$;*

2. *(X,T) is strongly mixing if and only if for any increasing sequence $\{p_i\}$ of positive integers there exists a c-dense F_σ subset C of X satisfying for any subset D of C and any continuous map $f\colon D \to X$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that $\lim_{i\to\infty} T^{p_{n_i}}x = f(x)$ for all $x \in D$.*

In particular, if (X,T) is weakly mixing, then there exists some $\delta > 0$ such that (X,T) is Li–Yorke δ-chaotic.

Note that a subset C is called *c-dense* in X if for every non-empty open subset U of X, the intersection $C \cap U$ has the cardinality of the continuum c.

Inspired by Xiong–Yang's result (Theorem 7.1), in 2008, Blanchard and Huang [17] introduced the concept of weakly mixing sets, which can be regraded as a local version of weak mixing.

Definition 7.2 *Let (X,T) be a dynamical system. A closed subset A of X is said to be* weakly mixing *if there exists a dense Mycielski subset C of A such that for any subset D of C and any continuous map $f\colon D \to A$, there exists an increasing sequence $\{q_i\}$ of positive integers satisfying $\lim_{i\to\infty} T^{q_i}x = f(x)$ for all $x \in D$.*

Blanchard and Huang showed that in any positive entropy system there are many weakly mixing sets. More precisely, let $\mathrm{WM}(X,T)$ be the set of weakly mixing sets of (X,T) and $H(X,T)$ be the closure of the set entropy sets in the hyperspace.

Theorem 7.3 ([17]) *If a dynamical system (X,T) has positive topological entropy, then the set $H(X,T)\cap \mathrm{WM}(X,T)$ is a dense G_δ subset of $H(X,T)$.*

Moreover in [37], Huang showed that in any positive entropy system there is a measure-theoretically rather big set such that the closure of the stable set of any point from the set contains a weakly mixing set. Recall that the *stable set* of a point $x\in X$ for T is

$$W^s(X,T)=\Big\{y\in X\colon \liminf_{n\to\infty} d(T^n x,T^n y)=0\Big\}.$$

Theorem 7.4 ([37]) *Let (X,T) be a dynamical system and μ be an ergodic invariant measure on (X,T) with positive entropy. Then for μ-a.e. $x\in X$ the closure of the stable set $\overline{W^s(X,T)}$ contains a weakly mixing set.*

Recently in [44], Huang et al. showed the existence of certain chaotic sets in the stable set of positive entropy G-systems for certain countable, discrete, infinite left-orderable amenable groups G. We restate [44, Theorem 1.2] in our setting as follows.

Theorem 7.5 ([44]) *Let T be a homeomorphism on a compact metric space X. If μ is an ergodic invariant measure on (X,T) with positive entropy, then there exists $\delta>0$ such that for μ-a.e. $x\in X$ the stable set $W^s(X,T)$ contains a Cantor δ-scrambled set for T^{-1}.*

Remark 7.6 The authors in [17] also discussed the relation between weakly mixing sets and other chaotic properties.

1. Positive entropy is strictly stronger than the existence of weakly mixing sets, which in turn is strictly stronger than Li–Yorke chaos.
2. There exists a Devaney chaotic system without any weakly mixing sets.

The following result is the well known Furstenberg intersection lemma, which shows that weak mixing implies weak mixing of all finite orders.

Lemma 7.7 ([25]) *If a dynamical system (X,T) is weakly mixing, then for any $k\in\mathbb{N}$ and any non-empty open subsets $U_1,\dots,U_k$, $V_1,\dots,V_k$ of X,*

$$\bigcap_{i=1}^{k} N(U_i,V_i)\neq\emptyset.$$

Using the idea in Lemma 7.7 we can give the following alternative definition of weakly mixing sets.

Proposition 7.8 ([17]) *Let (X,T) be a dynamical system and let A be a closed subset of X but not a singleton. Then A is weakly mixing if and only if for any $k\in\mathbb{N}$ and any choice of non-empty open subsets $U_1,\dots,U_k$, $V_1,\dots,V_k$ of X intersecting A (that is $A\cap U_i\neq\emptyset$, $A\cap V_i\neq\emptyset$ for $i=1,\dots,k$), one has*

$$\bigcap_{i=1}^{k} N(U_i\cap A,V_i)\neq\emptyset.$$

Definition 7.9 *Let (X,T) be a dynamical system, A be a closed subset of X but not a singleton and $k\geq 2$. The set A is said to be* weakly mixing of order k *if for any choice of non-empty open subsets $U_1,\dots,U_k$, $V_1,\dots,V_k$ of X intersecting A, one has*

$$\bigcap_{i=1}^{k} N(U_i\cap A,V_i)\neq\emptyset.$$

Then A is weakly mixing if and only if it is weakly mixing of order k for all $k \geq 2$.

In 2011, Oprocha and Zhang [82] studied weakly mixing sets of finite order, and constructed an example showing that the concepts of weakly mixing sets of order 2 and of order 3 are different. They generalized this result to general weakly mixing sets of finite order in [84].

Theorem 7.10 ([84]) *For every $n \geq 2$, there exists a minimal subshift on n symbols such that it contains a perfect weakly mixing set of order n but no non-trivial weakly mixing set of order $n+1$.*

Recall that it was shown in [6] that if (X,T) is weakly mixing, then for every $x \in X$, the proximal cell of x is residual in X. In [83] Oprocha and Zhang proved that for every closed weakly mixing set A and every $x \in A$, the proximal cell of x in A is residual in A. In [63] Li, Oprocha and Zhang showed that the same is true if we consider proximal tuples instead of pairs. First recall that an n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ is *proximal* if

$$\liminf_{k\to\infty} \max_{1\leq i<j\leq n} d(T^k(x_i), T^k(x_j)) = 0.$$

For $x \in X$, define the n-proximal cell of x as

$$P_n(x) = \{(x_1, \ldots, x_{n-1}) \in X^{n-1} \colon (x, x_1, \ldots, x_{n-1}) \text{ is proximal}\}.$$

Theorem 7.11 ([63]) *Let (X,T) be a dynamical system and $A \subset X$ be a weakly mixing set. Then for every $x \in A$ and $n \geq 2$, the set $P_n(x) \cap A^{n-1}$ is residual in A^{n-1}.*

In fact, they proved even more as presented in the following theorem, where $LY_n^\delta(x)$ is the n-scrambled cell of x with modular $\delta > 0$.

Theorem 7.12 ([63]) *Let (X,T) be a dynamical system and $A \subset X$ be a weakly mixing set. Then for every $n \geq 2$, there exists some $\delta > 0$ such that for every $x \in A$, the set $LY_n^\delta(x) \cap A^{n-1}$ is residual in A^{n-1}.*

The following result shows that, when we look only at separation of trajectories of tuples, weak mixing of order 2 is enough to obtain rich structure of such points.

Theorem 7.13 ([63]) *Let (X,T) be a dynamical system and $A \subset X$ a weakly mixing set of order 2. Then A is a sensitive set in $\big(\overline{\mathrm{Orb}(A,T)}, T\big)$, where $\mathrm{Orb}(A,T) = \{T^n x \colon n \geq 0, x \in A\}$. In particular, the system $\big(\overline{\mathrm{Orb}(A,T)}, T\big)$ is n-sensitive for every $n \geq 2$.*

Recall that we have given the definition of independent sets in Definition 3.7. Huang et al. in [39] studied independent sets via Furstenberg families. In particular, they showed the following connection between weak mixing and independent sets of open sets.

Theorem 7.14 *Let (X,T) be a dynamical system. Then the following conditions are equivalent:*

1. *(X,T) is weakly mixing;*
2. *for any two non-empty open subsets U_1, U_2 of X, (U_1, U_2) has an infinite independent set;*
3. *for any $n \in \mathbb{N}$ and non-empty open subsets $U_1, U_2, \ldots, U_n$ of X, $(U_1, U_2, \ldots, U_n)$ has an* IP*-independent set.*

In the spirit of [39] we introduce a local version of independence sets as follows.

Definition 7.15 *Let (X,T) be a dynamical system and $A \subset X$. Let $U_1, U_2, \dots, U_n$ be open subsets of X intersecting A. We say that a non-empty subset F of $\mathbb{Z}_+$ is an* independence set for $(U_1, U_2, \dots, U_n)$ with respect to A, *if for every non-empty finite subset $J \subset F$, and $s \in \{1, 2, \dots, n\}^J$,*

$$\bigcap_{j \in J} T^{-j}(U_{s(j)})$$

is a non-empty open subset of X intersecting A.

Now we can employ introduced notion, to state a theorem analogous to Theorem 7.14.

Theorem 7.16 ([63]) *Let (X,T) be a dynamical system and $A \subset X$ a closed set. Then the following conditions are equivalent*:

1. *A is a weakly mixing set*;

2. *for every $n \geq 2$ and every open subsets $U_1, U_2, \dots, U_n$ of X intersecting A, there exists $t \in \mathbb{N}$ such that $\{0, t\}$ is an independence set for $(U_1, U_2, \dots, U_n)$ with respect to A*;

3. *for every $n \geq 2$ and every open subsets $U_1, U_2, \dots, U_n$ of X intersecting A, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ in $\mathbb{N}$ such that $\{0\} \cup \mathrm{FS}(\{t_j\}_{j=1}^{\infty})$ is an independence set for $(U_1, U_2, \dots, U_n)$ with respect to A.*

7.2 Weakly Mixing Sets via Furstenberg Families

In 2004, Huang et al. generalized Theorem 7.1 to $\mathcal{F}$-mixing systems. Let $\mathcal{F}$ be a Furstenberg family. Recall that a dynamical system (X,T) is called *$\mathcal{F}$-mixing* if it is weakly mixing and for any two non-empty open subsets U and V of X, $N(U,V) \in \mathcal{F}$.

Theorem 7.17 ([43]) *Let (X,T) be a non-trivial dynamical system and $\mathcal{F}$ be a Furstenberg family. Then (X,T) is $\mathcal{F}$-mixing if and only if for any $S \in \kappa\mathcal{F}$, there exists a dense Mycielski subset C of X satisfying for any subset D of C and any continuous map $f \colon D \to X$, there exists an increasing sequence $\{q_i\}$ in S such that $\lim_{i\to\infty} T^{q_i}x = f(x)$ for all $x \in D$.*

It is natural that weakly mixing sets can be also generalized via Furstenberg families.

Definition 7.18 *Let (X,T) be a dynamical system and $\mathcal{F}$ be a Furstenberg family. Suppose that A is a closed subset of X with at least two points. The set A is said to be $\mathcal{F}$-*mixing *if for any $k \in \mathbb{N}$, any open subsets $U_1, U_2, \dots, U_k$, $V_1, V_2, \dots, V_k$ of X intersecting A,*

$$\bigcap_{i=1}^{k} N(U_i \cap A, V_i) \in \mathcal{F}.$$

Inspired by the proof of Theorem 7.17, we have the following characterization of $\mathcal{F}$-mixing sets.

Theorem 7.19 ([61]) *Let (X,T) be a dynamical system and $\mathcal{F}$ be a Furstenberg family. Suppose that A is a closed subset of X with at least two points. Then A is an $\mathcal{F}$-mixing set if and only if for every $S \in \kappa\mathcal{F}$ (the dual family of $\mathcal{F}$) there are Cantor subsets $C_1 \subset C_2 \subset \cdots$ of A such that*

(i) *$K = \bigcup_{n=1}^{\infty} C_n$ is dense in A*;

(ii) *for any $n \in \mathbb{N}$ and any continuous function $g \colon C_n \to A$, there exists a subsequence $\{q_i\}$ of S such that $\lim_{i\to\infty} T^{q_i}(x) = g(x)$ uniformly on $x \in C_n$*;

(iii) *for any subset E of K and any continuous map $g\colon E \to A$, there exists a subsequence $\{q_i\}$ of S such that* $\lim_{i\to\infty} T^{q_i}(x) = g(x)$ *for every $x \in E$.*

It is shown in [61] that two classes of important dynamical systems have weakly mixing sets via proper Furstenberg families.

Let F be a subset of $\mathbb{Z}_+$. The *upper Banach density of F* is defined by

$$\mathrm{BD}^*(F) = \limsup_{|I|\to\infty} \frac{|F \cap I|}{|I|},$$

where I is taken over all non-empty finite intervals of $\mathbb{Z}_+$. The family of sets with positive upper Banach density is denoted by $\mathcal{F}_{\mathrm{pubd}} = \{F \subset \mathbb{Z}_+ \colon \mathrm{BD}^*(F) > 0\}$. We say that F is *piecewise syndetic* if it is the intersection of a thick set and a syndetic set. The family of piecewise syndetic sets is denoted by $\mathcal{F}_{\mathrm{ps}}$.

Theorem 7.20 ([61]) *Let (X,T) be a dynamical system.*

1. *If (X,T) has positive topological entropy, then it has some $\mathcal{F}_{\mathrm{pubd}}$-mixing sets.*
2. *If (X,T) is a non-*PI *minimal system, then it has some $\mathcal{F}_{\mathrm{ps}}$-mixing sets.*

8 Chaos in the Induced Spaces

A dynamical system (X,T) induces two natural systems, one is $(K(X),T_K)$ on the hyperspace $K(X)$ consisting of all closed non-empty subsets of X with the Hausdorff metric, and the other one is $(M(X),T_M)$ on the probability space $M(X)$ consisting of all Borel probability measures with the weak*-topology. Bauer and Sigmund [13] first gave a systematic investigation on the connection of dynamical properties among (X,T), $(K(X),T_K)$ and $(M(X),T_M)$. It was proved that (X,T) is weakly mixing (resp. mildly mixing, strongly mixing) if and only if $(K(X),T_K)$ (resp. $(M(X),T_M)$) has the same property. This leads to a natural question:

Problem 6 If one of the dynamical systems (X,T), $(K(X),T_K)$ and $(M(X),T_M)$ is chaotic in some sense, how about the other two systems?

This question attracted a lot of attention, see, e.g., [11, 36, 87] and references therein, and many partial answers were obtained. We first show that when the induced system is weakly mixing.

Theorem 8.1 ([11, 13]) *Let (X,T) be a dynamical system. Then (X,T) is weakly mixing if and only if $(K(X),T_K)$ is weakly mixing if and only if $(K(X),T_K)$ is transitive.*

Theorem 8.2 ([13, 66]) *Let (X,T) be a dynamical system. Then (X,T) is weakly mixing if and only if $(M(X),T_M)$ is weakly mixing if and only if $(M(X),T_M)$ is transitive.*

In [30], Glasner and Weiss studied the topological entropy of $(K(X),T_K)$ and $(M(X),T_M)$. They proved that

Theorem 8.3 *Let (X,T) be a dynamical system.*

1. *The topological entropy of (X,T) is zero if and only if the one of $(M(X),T_M)$ is also zero, and the topological entropy of (X,T) is positive if and only if the one of $(M(X),T_M)$ is infinite.*

2. *If the topological entropy of (X,T) is positive, then the topological entropy of $(K(X),T_K)$ is infinite, while there exists a minimal system (X,T) of zero topological entropy and $(K(X),T_K)$*

with positive topological entropy.

To show that when the dynamical system on the hyperspace is Devaney chaotic, we need to introduce some concepts firstly. We say that a dynamical system (X, T) has *dense small periodic sets* [48] if for any non-empty open subset U of X there exists a non-empty closed subset Y of U and $k \in \mathbb{N}$ such that $T^k Y \subset Y$. Clearly, if a dynamical system has a dense set of periodic points, then it also has dense small periodic sets. The dynamical system (X, T) is called an HY-*system* if it is totally transitive and has dense small periodic sets. Note that there exists an HY-system without periodic points (see [48, Example 3.7]). Recently, Li showed in [60] that $(K(X), T_K)$ is Devaney chaotic is equivalent to the origin system (X, T) is an HY-system.

Theorem 8.4 ([60]) *Let (X, T) be a dynamical system with X being infinite. Then the following conditions are equivalent*:

1. $(K(X), T_K)$ *is Devaney chaotic*;
2. $(K(X), T_K)$ *is an* HY-*system*;
3. (X, T) *is an* HY-*system.*

In order to characterize Devaney chaos on the space of probability measures, we need a notion of an almost HY-system. We say that (X, T) has *almost dense periodic sets* if for each non-empty open subset $U \subset X$ and $\epsilon > 0$, there are $k \in \mathbb{N}$ and $\mu \in M(X)$ with $T_M^k \mu = \mu$ such that $\mu(U^c) < \epsilon$, where $U^c = \{x \in X : x \notin U\}$. We say that (X, T) is an *almost* HY-*system* if it is totally transitive and has almost dense periodic sets.

Theorem 8.5 ([66]) *Let (X, T) be a dynamical system with X being infinite. Then the following conditions are equivalent*:

1. $(M(X), T_M)$ *is Devaney chaotic*;
2. $(M(X), T_M)$ *is an almost* HY-*system*;
3. (X, T) *is an almost* HY-*system.*

It is clear that every HY-system is also an almost HY-system. There is a non-trivial minimal weakly mixing almost-HY-system (see [66, Theorem 4.11]), which is not an HY-system, since every minimal HY-system is trivial.

Recall that a dynamical system (X, T) is *proximal* if any pair $(x, y) \in X^2$ is proximal. The following proposition shows that if $(K(X), T_K)$ is proximal then (X, T) is “almost” trivial.

Proposition 8.6 ([64]) *Let (X, T) be a dynamical system. Then the following conditions are equivalent*:

1. $(K(X), T_K)$ *is proximal*;
2. $\bigcap_{n=1}^{\infty} T^n X$ *is a singleton*;
3. X *is a uniformly proximal set.*

If $(M(X), T_M)$ is proximal, then (X, T) is called *strongly proximal* [32]. Note that if (X, T) is strongly proximal, then it is proximal, since (X, T) can be regarded as a subsystem of the proximal system $(M(X), T_M)$. We have the following characterization of strongly proximal systems.

Theorem 8.7 ([64]) *Let (X, T) be a dynamical system. Then the following conditions are equivalent*:

1. (X, T) *is strongly proximal*;
2. (X, T) *is proximal and unique ergodic*;
3. *every pair* $(x, y) \in X^2$ *is Banach proximal.*

Acknowledgements The authors would like to thank Wen Huang, Jie Li and Guohua Zhang for the careful reading, and thank the anonymous reviewer for his/her valuable comments and suggestions.

References

[1] Adler, R. L., Konheim, A. G., McAndrew, M. H.: Topological entropy. *Trans. Amer. Math. Soc.*, **114**, 309–319 (1965)

[2] Akin, E.: Recurrence in topological dynamics: Furstenberg families and Ellis actions. The University Series in Mathematics, Plenum Press, New York, 1997

[3] Akin, E., Auslander, J., Berg, K.: When is a transitive map chaotic? In: Convergence in ergodic theory and probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., Vol. 5, de Gruyter, Berlin, 1996, 25–40

[4] Akin, E., Glasner, E.: Residual properties and almost equicontinuity. *J. Anal. Math.*, **84**, 243–286 (2001)

[5] Akin, E., Glasner, E., Huang, W., et al.: Sufficient conditions under which a transitive system is chaotic. *Ergodic Theory Dynam. Systems*, **30**(5), 1277–1310 (2010)

[6] Akin, E., Kolyada, S.: Li–Yorke sensitivity. *Nonlinearity*, **16**(4), 1421–1433 (2003)

[7] Auslander, J.: Minimal flows and their extensions. In: North-Holland Mathematics Studies, Vol. 153 North-Holland Publishing Co., Amsterdam, 1988

[8] Auslander, J., Yorke, J. A.: Interval maps, factors of maps, and chaos. *Tôhoku Math. J.* (2), **32**(2), 177–188 (1980)

[9] Balibrea, F., Guirao, J. L. G., Oprocha, P.: On invariant ϵ-scrambled sets. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **20**(9), 2925–2935 (2010)

[10] Balibrea, F., Smítal, J., Štefánková, M.: The three versions of distributional chaos. *Chaos Solitons Fractals*, **23**(5), 1581–1583 (2005)

[11] Banks, J.: Chaos for induced hyperspace maps. *Chaos Solitons Fractals*, **25**(3), 681–685 (2005)

[12] Banks, J., Brooks, J., Cairns, G., et al.: On Devaney's definition of chaos. *Amer. Math. Monthly*, **99**(4), 332–334 (1992)

[13] Bauer, W., Sigmund, K.: Topological dynamics of transformations induced on the space of probability measures. *Monatsh. Math.*, **79**, 81–92 (1975)

[14] Blanchard, F.: Topological chaos: what may this mean? *J. Difference Equ. Appl.*, **15**(1), 23–46 (2009)

[15] Blanchard, F., Glasner, E., Kolyada, S., et al.: On Li–Yorke pairs. *J. Reine Angew. Math.*, **547**, 51–68 (2002)

[16] Blanchard, F., Host, B., Ruette, S.: Asymptotic pairs in positive-entropy systems. *Ergodic Theory Dynam. Systems*, **22**(3), 671–686 (2002)

[17] Blanchard, F., Huang, W.: Entropy sets, weakly mixing sets and entropy capacity. *Discrete Contin. Dyn. Syst.*, **20**(2), 275–311 (2008)

[18] Blanchard, F., Huang, W., Snoha, L.: Topological size of scrambled sets. *Colloq. Math.*, **110**(2), 293–361 (2008)

[19] Devaney, R. L.: An Introduction to Chaotic Dynamical Systems. Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989

[20] Doleželová, J.: Scrambled and distributionally scrambled n-tuples. *J. Difference Equ. Appl.*, **20**(8), 1169–1177 (2014)

[21] Downarowicz, T.: Positive topological entropy implies chaos DC2. *Proc. Amer. Math. Soc.*, **142**(1), 137–149 (2014)

[22] Du, B.-S.: On the invariance of Li–Yorke chaos of interval maps. *J. Difference Equ. Appl.*, **11**(9), 823–828 (2005)

[23] Fomin, S.: On dynamical systems with a purely point spectrum. *Doklady Akad. Nauk SSSR* (*N.S.*), **77**, 29–32 (1951)

[24] Foryś, M., Huang, W., Li, J., et al.: Invariant scrambled sets, uniform rigidity and weak mixing. *Israel J. Math.*, to appear (2014)

[25] Furstenberg, H.: Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory*, **1**, 1–49 (1967)

[26] Furstenberg, H.: Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, Princeton, N.J., 1981

[27] García-Ramos, F.: Weak forms of topological and measure theoretical equicontinuity: relationships with discrete spectrum and sequence entropy. Preprint, arXiv:1402.7327[math.DS]

[28] Glasner, E.: Ergodic theory via joinings. In: Mathematical Surveys and Monographs, Vol. 101, American Mathematical Society, Providence, RI, 2003

[29] Glasner, E., Weiss, B.: Sensitive dependence on initial conditions. *Nonlinearity*, **6**(6), 1067–1075 (1993)

[30] Glasner, E., Weiss, B.: Quasi-factors of zero-entropy systems. *J. Amer. Math. Soc.*, **8**(3), 665–686 (1995)

[31] Glasner, E., Ye, X.: Local entropy theory. *Ergodic Theory Dynam. Systems*, **29**(2), 321–356 (2009)

[32] Glasner, S.: Proximal flows. In: Lecture Notes in Mathematics, Vol. 517, Springer-Verlag, Berlin-New York, 1976

[33] Glasner, S., Maon, D.: Rigidity in topological dynamics. *Ergodic Theory Dynam. Systems*, **9**(2), 309–320 (1989)

[34] Gottschalk, W. H., Hedlund, G. A.: Topological dynamics. In: American Mathematical Society Colloquium Publications, Vol. 36, American Mathematical Society, Providence, R. I., 1955

[35] Guckenheimer, J.: Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, **70**(2), 133–160 (1979)

[36] Guirao, J. L. G., Kwietniak, D., Lampart, M., et al.: Chaos on hyperspaces. *Nonlinear Anal.*, **71**(1–2), 1–8 (2009)

[37] Huang, W.: Stable sets and ϵ-stable sets in positive-entropy systems. *Comm. Math. Phys.*, **279**(2), 535–557 (2008)

[38] Huang, W., Kolyada, S., Zhang, G.: Multi-sensitivity, Lyapunov numbers and almost automorphic maps. Preprint (2014)

[39] Huang, W., Li, H., Ye, X.: Family independence for topological and measurable dynamics. *Trans. Amer. Math. Soc.*, **364**(10), 5209–5242 (2012)

[40] Huang, W., Li, J., Ye, X.: Stable sets and mean Li–Yorke chaos in positive entropy systems. *J. Funct. Anal.*, **266**(6), 3377–3394 (2014)

[41] Huang, W., Li, J., Ye, X., Zhou, X.: Topological entropy and diagonal-weakly mixing sets. Preprint (2014)

[42] Huang, W., Lu, P., Ye, X.: Measure-theoretical sensitivity and equicontinuity. *Israel J. Math.*, **183**, 233–283 (2011)

[43] Huang, W., Shao, S., Ye, X.: Mixing and proximal cells along sequences. *Nonlinearity*, **17**(4), 1245–1260 (2004)

[44] Huang, W., Xu, L., Yi, Y.: Asymptotic pairs, stable sets and chaos in positive entropy systems. *J. Funct. Anal.*, to appear

[45] Huang, W., Ye, X.: Homeomorphisms with the whole compacta being scrambled sets. *Ergodic Theory Dynam. Systems*, **21**(1), 77–91 (2001)

[46] Huang, W., Ye, X.: Devaney's chaos or 2-scattering implies Li-Yorke's chaos. *Topology Appl.*, **117**(3), 259–272 (2002)

[47] Huang, W., Ye, X.: Topological complexity, return times and weak disjointness. *Ergodic Theory Dynam. Systems*, **24**(3), 825–846 (2004)

[48] Huang, W., Ye, X.: Dynamical systems disjoint from any minimal system. *Trans. Amer. Math. Soc.*, **357**(2), 669–694 (2005)

[49] Huang, W., Ye, X.: A local variational relation and applications. *Israel J. Math.*, **151**, 237–279 (2006)

[50] Iwanik, A.: Independence and scrambled sets for chaotic mappings. In: The mathematical heritage of C. F. Gauss, World Sci. Publ., River Edge, NJ, 1991, 372–378

[51] Janková, K., Smítal, J.: A characterization of chaos. *Bull. Austral. Math. Soc.*, **34**(2), 283–292 (1986)

[52] Katznelson, Y., Weiss, B.: When all points are recurrent/generic. In: Ergodic theory and dynamical systems, I (College Park, Md., 1979–80), Progr. Math., Vol. 10, Birkhäuser Boston, Mass., 1981, 195–210
[53] Kerr, D., Li, H.: Independence in topological and C^*-dynamics. *Math. Ann.*, **338**(4), 869–926 (2007)
[54] Kerr, D., Li, H.: Combinatorial independence and sofic entropy. *Commun. Math. Stat.*, **1**(2), 213–257 (2013)
[55] Keynes, H. B., Robertson, J. B.: Eigenvalue theorems in topological transformation groups. *Trans. Amer. Math. Soc.*, **139**, 359–369 (1969)
[56] Kolyada, S.: Li–Yorke sensitivity and other concepts of chaos. *Ukraïn. Mat. Zh.*, **56**(8), 1043–1061 (2004)
[57] Kolyada, S., Snoha, L.: Some aspects of topological transitivity — a survey. In: Iteration theory (ECIT 94) (Opava), Grazer Math. Ber., Vol. 334, Karl-Franzens-Univ. Graz, Graz, 1997, 3–35
[58] Li, J.: Chaos and entropy for interval maps. *J. Dynam. Differential Equations*, **23**(2), 333–352 (2011)
[59] Li, J.: Transitive points via Furstenberg family. *Topology Appl.*, **158**(16), 2221–2231 (2011)
[60] Li, J.: Equivalent conditions of devaney chaos on the hyperspace. *J. Univ. Sci. Technol. China*, **44**(2), 93–95 (2014)
[61] Li, J.: Localization of mixing property via Furstenberg families. *Discrete Contin. Dyn. Syst.*, **35**(2), 725–740 (2015)
[62] Li, J., Oprocha, P.: On n-scrambled tuples and distributional chaos in a sequence. *J. Difference Equ. Appl.*, **19**(6), 927–941 (2013)
[63] Li, J., Oprocha, P., Zhang, G.: On recurrence over subsets and weak mixing. preprint (2013)
[64] Li, J., Tu, S.: On proximality with Banach density one. *J. Math. Anal. Appl.*, **416**(1), 36–51 (2014)
[65] Li, J., Tu, S., Ye, X.: Mean equicontinuity and mean sensitivity. *Ergodic Theory Dynam. Systems,* to appear
[66] Li, J., Yan, K., Ye, X.: Recurrence properties and disjointness on the induced spaces. *Discrete Contin. Dyn. Syst.*, **35**(3), 1059–1073 (2015)
[67] Li, R., Shi, Y.: Stronger forms of sensitivity for measure-preserving maps and semiflows on probability spaces. *Abstr. Appl. Anal.*, Art. ID 769523, 10 pages (2014)
[68] Li, S.: ω-chaos and topological entropy. *Trans. Amer. Math. Soc.*, **339**(1), 243–249 (1993)
[69] Li, T., Yorke, J. A.: Period three implies chaos. *Amer. Math. Monthly*, **82**(10), 985–992 (1975)
[70] Liao, G., Fan, Q.: Minimal subshifts which display Schweizer–Smítal chaos and have zero topological entropy. *Sci. China Ser. A*, **41**(1), 33–38 (1998)
[71] Liu, H., Liao, L., Wang, L.: Thickly syndetical sensitivity of topological dynamical system. *Discrete Dyn. Nat. Soc.*, Art. ID 583431, 4 pages (2014)
[72] Lorenz, E. N.: Deterministic nonperiodic flow. *J. Atmospheric Sci.*, **20**(2), 130–148 (1963)
[73] Mai, J.: Continuous maps with the whole space being a scrambled set. *Chinese Sci. Bull.*, **42**(19), 1603–1606 (1997)
[74] Mai, J.: The structure of equicontinuous maps. *Trans. Amer. Math. Soc.*, **355**(10), 4125–4136 (2003)
[75] Mai, J.: Devaney's chaos implies existence of s-scrambled sets. *Proc. Amer. Math. Soc.*, **132**(9), 2761–2767 (2004)
[76] Moothathu, T. K. S.: Stronger forms of sensitivity for dynamical systems. *Nonlinearity*, **20**(9), 2115–2126 (2007)
[77] Moothathu, T. K. S.: Syndetically proximal pairs. *J. Math. Anal. Appl.*, **379**(2), 656–663 (2011)
[78] Moothathu, T. K. S., Oprocha, P.: Syndetic proximality and scrambled sets. *Topol. Methods Nonlinear Anal.*, **41**(2), 421–461 (2013)
[79] Mycielski, J.: Independent sets in topological algebras. *Fund. Math.*, **55**, 139–147 (1964)
[80] Oprocha, P.: Relations between distributional and Devaney chaos. *Chaos*, **16**(3), 033112, 5 (2006)
[81] Oprocha, P.: Minimal systems and distributionally scrambled sets. *Bull. Soc. Math. France*, **140**(3), 401–439 (2012)
[82] Oprocha, P., Zhang, G.: On local aspects of topological weak mixing in dimension one and beyond. *Studia Math.*, **202**(3), 261–288 (2011)
[83] Oprocha, P., Zhang, G.: On weak product recurrence and synchronization of return times. *Adv. Math.*, **244**, 395–412 (2013)
[84] Oprocha, P., Zhang, G.: On local aspects of topological weak mixing, sequence entropy and chaos. *Ergodic Theory Dynam. Systems*, **34**(5), 1615–1639 (2014)

[85] Oprocha, P., Zhang, G.: Topological aspects of dynamics of pairs, tuples and sets. In: Recent progress in general topology, III, Atlantis Press, Paris, 2014, 665–709

[86] Pikuła, R.: On some notions of chaos in dimension zero. *Colloq. Math.*, **107**(2), 167–177 (2007)

[87] Román-Flores, H.: A note on transitivity in set-valued discrete systems. *Chaos Solitons Fractals*, **17**(1), 99–104 (2003)

[88] Ruelle, D.: Dynamical systems with turbulent behavior. In: Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977), Lecture Notes in Phys., Vol. 80, Springer, Berlin, 1978, 341–360

[89] Ruelle, D., Takens, F.: On the nature of turbulence. *Comm. Math. Phys.*, **20**, 167–192 (1971)

[90] Ruette, S.: Transitive sensitive subsystems for interval maps. *Studia Math.*, **169**(1), 81–104 (2005)

[91] Scarpellini, B.: Stability properties of flows with pure point spectrum. *J. London Math. Soc.* (2), **26**(3), 451–464 (1982)

[92] Schweizer, B., Smítal, J.: Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.*, **344**(2), 737–754 (1994)

[93] Shao, S.: Proximity and distality via Furstenberg families. *Topology Appl.*, **153**(12), 2055–2072 (2006)

[94] Shao, S., Ye, X., Zhang, R.: Sensitivity and regionally proximal relation in minimal systems. *Sci. China Ser. A*, **51**(6), 987–994 (2008)

[95] Smítal, J.: A chaotic function with some extremal properties. *Proc. Amer. Math. Soc.*, **87**(1), 54–56 (1983)

[96] Smítal, J.: Chaotic functions with zero topological entropy. *Trans. Amer. Math. Soc.*, **297**(1), 269–282 (1986)

[97] Smítal, J.: Topological entropy and distributional chaos. *Real Anal. Exchange*, (30th Summer Symposium Conference), 61–65 (2006)

[98] Smítal, J., Štefánková, M.: Distributional chaos for triangular maps. *Chaos Solitons Fractals*, **21**(5), 1125–1128 (2004)

[99] Snoha, L.: Generic chaos. *Comment. Math. Univ. Carolin.*, **31**(4), 793–810 (1990)

[100] Tan, F., Fu, H.: On distributional n-chaos. *Acta Math. Sci. Ser. B Engl. Ed.*, **34**(5), 1473–1480 (2014)

[101] Tan, F., Xiong, J.: Chaos via Furstenberg family couple. *Topology Appl.*, **156**(3), 525–532 (2009)

[102] Tan, F., Zhang, R.: On $\mathscr{F}$-sensitive pairs. *Acta Math. Sci. Ser. B Engl. Ed.*, **31**(4), 1425–1435 (2011)

[103] Walters, P.: An introduction to ergodic theory. In: Graduate Texts in Mathematics, Vol. 79, Springer-Verlag, New York, 1982

[104] Wang, H., Xiong, J., Tan, F.: Furstenberg families and sensitivity. *Discrete Dyn. Nat. Soc.*, Art. ID 649348, 12 pages (2010)

[105] Wiggins, S.: Introduction to applied nonlinear dynamical systems and chaos. In: Texts in Applied Mathematics, Vol 2, Springer-Verlag, New York, 1990

[106] Xiong, J.: A chaotic map with topological entropy. *Acta Math. Sci. English Ed.*, **6**(4), 439–443 (1986)

[107] Xiong, J.: Chaos in a topologically transitive system. *Sci. China Ser. A*, **48**(7), 929–939 (2005)

[108] Xiong, J., Lü, J., Tan, F.: Furstenberg family and chaos. *Sci. China Ser. A*, **50**(9), 1325–1333 (2007)

[109] Xiong, J., Yang, Z.: Chaos caused by a topologically mixing map. In: Dynamical systems and related topics (Nagoya, 1990), Adv. Ser. Dynam. Systems, Vol. 9, 550–572, World Sci. Publ., River Edge, NJ, 1991

[110] Ye, X., Yu, T.: Sensitivity, proximal extension and higher order almost automorphy. Preprint (2014)

[111] Ye, X., Zhang, R.: On sensitive sets in topological dynamics. *Nonlinearity*, **21**(7), 1601–1620 (2008)

[112] Yuan, D., Lü, J.: Invariant scrambled sets in transitive systems. *Adv. Math. (China)*, **38**(3), 302–308 (2009)