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Characterization of the Generalized Calabi Composition of Affine Hyperspheres

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Abstract In this paper, continuing with Hu–Li–Vrancken and the recent work of Antić–Dillen-Schoels–Vrancken, we obtain a decomposition theorem which settled the problem of how to determine whether a given locally strongly convex affine hypersurface can be decomposed as a generalized Calabi composition of two affine hyperspheres, based on the properties of its difference tensor K and its affine shape operator S.

Keywords Generalized Calabi composition, affine hyperspheres, warped product

MR(2010) Subject Classification 53A15, 53B20, 53B25

1 Introduction

In affine differential geometry, Calabi [3] initiated the study of compositions of affine hyperspheres. In particular, he introduced a construction, nowadays called the Calabi composition, which shows how to associate with one (or two) hyperbolic affine hypersphere(s) a new hyperbolic affine hypersphere. Such Calabi composition was later generalized by Dillen and

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Vrancken [6] systematically to obtain a large class of examples of equiaffine *homogeneous* affine hypersurfaces, some of which have appeared in the list of the partial classification of equiaffine homogeneous affine hypersurfaces [4, 5, 10]. Most recently and importantly, Hu et al. [8] considered the inverse construction and obtained characterizations of the Calabi composition of hyperbolic hyperspheres, applying these characterizations they can successfully complete the classification of locally strongly convex affine hypersurfaces with parallel cubic form [9], and little later also that of the Lorentzian case [7].

Recently, Antić et al. [1] made further contribution by constructing several generalized Calabi compositions of two affine hyperspheres, and importantly, they characterized these compositions of an affine hypersphere and a point using properties of the difference tensor K and the affine shape operator S.

Theorem 1.1 ([1]) *Let M^{n+1} $(n \geq 2)$ be a locally strongly convex affine hypersurface of the $(n+2)$-dimensional affine space $\mathbb{R}^{n+2}$ such that its tangent bundle is an orthogonal sum, with respect to the affine metric h, of two distributions, a one-dimensional distribution $\mathcal{D}_1$ spanned by a unit vector field T and an n-dimensional distribution $\mathcal{D}_2$, such that*

$$K(T,T) = \lambda_1 T, \quad K(T,X) = \lambda_2 X,$$
$$ST = \mu_1 T, \quad SX = \mu_2 X, \quad \forall\, X \in \mathcal{D}_2.$$

Then either M^{n+1} is an affine hypersphere such that $K_T = 0$ or is affine congruent to one of the following immersions:

(1) $F(t, x_1, \ldots, x_n) = (\gamma_1(t), \gamma_2(t) g_2(x_1, \ldots, x_n))$ *for* γ_1, γ_2 *such that* $\gamma_1'\gamma_2 - \gamma_1\gamma_2' \neq 0$ *and*

$$\epsilon\gamma_1'\gamma_2(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') < 0;$$

(2) $F(t, x_1, \ldots, x_n) = \gamma_1(t) C(x_1, \ldots, x_n) + \gamma_2(t) e_{n+1}$ *for* γ_1, γ_2 *such that*

$$\operatorname{sgn}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sgn}(\gamma_1\gamma_1') \neq 0;$$

(3) $F(t, x_1, \ldots, x_n) = C(x_1, \ldots, x_n) + \gamma_2(t) e_{n+1} + \gamma_1(t) e_{n+2}$ *for* γ_1, γ_2 *such that*

$$\operatorname{sgn}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sgn}\gamma_1' \neq 0,$$

where $g_2 : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a proper affine hypersphere centered at the origin with affine mean curvature ϵ, and $C : \mathbb{R}^n \to \mathbb{R}^{n+2}$ is an improper affine hypersphere, given by

$$C(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n), 1),$$

with the affine normal e_{n+1}.

Remark 1.2 The proof in [1] shows that the case that $\lambda_1 = 2\lambda_2$ exactly occurs when M^{n+1} is an affine hypersphere with $K_T = 0$. From the classification of such 3-dimensional affine hypersphere in [11, 12], we see that they are not related to generalized Calabi compositions. Therefore in order to exclude such examples, we will assume that $\lambda_1 \neq 2\lambda_2$.

Note that as Riemannian manifold the Calabi composition is, up to a constant factor, a Riemannian product, whereas the above mentioned generalized Calabi composition is usually a warped product. The following natural problem in [2] for a composition theorem, related to the Calabi composition and its generalizations, gives another motivation for studying the characterization of the generalized Calabi composition.

Problem 1.3 Let M be a non-degenerate affine hypersurface in $\mathbb{R}^{n+m+2}$. Under what conditions do there exist affine hyperspheres M_2^n in $\mathbb{R}^{n+1}$ and M_3^m in $\mathbb{R}^{m+1}$, such that $M = I \times_{\rho_2} M_2^n \times_{\rho_3} M_3^m$, where $I \subset \mathbb{R}$ and the functions ρ_2 and ρ_3 depend only on I (i.e., M admits a warped product structure)? How can the original immersion be recovered starting from the immersion of the affine hyperspheres?

In this paper, by extending Theorem 1.1 we will consider a further problem on the reverse construction of generalized Calabi composition with more factors. Our main result can be stated as follows:

Theorem 1.4 *Let M^{n+m+1} $(n \geq 2, m \geq 2)$ be a locally strongly convex affine hypersurface of the affine space $\mathbb{R}^{n+m+2}$ such that its tangent bundle is an orthogonal sum, with respect to the affine metric h, of three distributions, a 1-dimensional distribution $\mathcal{D}_1$ spanned by a unit vector field T, an n-dimensional distribution $\mathcal{D}_2$ and an m-dimensional distribution $\mathcal{D}_3$ such that*

$$\begin{cases} K_T T = \lambda_1 T, \quad K_T X = \lambda_2 X, \quad K_T Y = \lambda_3 Y, \ K_X Y = 0, \\ ST = \mu_1 T, \quad SX = \mu_2 X, \quad SY = \mu_3 Y, \\ \lambda_1 \neq 2\lambda_2, \ \lambda_1 \neq 2\lambda_3, \ \lambda_2\lambda_3 \neq 0, \ \forall\ X \in \mathcal{D}_2, \ Y \in \mathcal{D}_3. \end{cases} \tag{1.1}$$

Then M^{n+m+1} is affine congruent to the hypersurface immersions (1)–(3) *of Theorem* 1.1 *if $\lambda_2 = \lambda_3$. Otherwise, M^{n+m+1} is affine congruent to one of the following immersions*:

(1) *the generalized Calabi composition of two proper affine hyperspheres ϕ_1 and ϕ_2, defined by $F(t, p_1, p_2) = (\gamma_1(t)\phi_1(p_1), \gamma_2(t)\phi_2(p_2))$, where $\phi_1 : \mathbb{R}^n \to \mathbb{R}^{n+1}$ and $\phi_2 : \mathbb{R}^m \to \mathbb{R}^{m+1}$ are both proper affine hypersphere centered at the origin with affine mean curvature ϵ_1 and ϵ_2, for $\gamma_1, \ \gamma_2$ such that $\gamma_1'\gamma_2 - \gamma_1\gamma_2' \neq 0$, and*

$$\epsilon_2\gamma_1'\gamma_2(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') < 0, \quad \epsilon_1\gamma_1\gamma_2'(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') > 0;$$

(2) *the generalized Calabi composition of two improper affine hyperspheres normalized by two functions $F_1(x_1, \ldots, x_n)$ and $F_2(y_1, \ldots, y_m)$, constructed by*

$$\begin{aligned} F(t, x_1, \ldots, x_n, y_1, \ldots, y_m) = (&x_1, \ldots, x_n, F_1(x_1, \ldots, x_n) + \gamma_1(t), \\ &y_1, \ldots, y_m, F_2(y_1, \ldots, y_m) + \gamma_2(t)) \end{aligned}$$

for $\gamma_1, \ \gamma_2$ such that $\operatorname{sgn}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sgn}\gamma_1' = -\operatorname{sgn}\gamma_2' \neq 0$;

(3) *the generalized Calabi composition of a proper and an improper affine hyperspheres, denoted the former by ϕ and its affine mean curvature by ϵ and the latter by a normalized function $F_1(x_1, \ldots, x_n)$, constructed by*

$$F(t, x_1, \ldots, x_n, p) = (x_1, \ldots, x_n, F_1(x_1, \ldots, x_n) + \gamma_1(t), \gamma_2(t)\phi(p))$$

for $\gamma_1, \ \gamma_2$ such that $\operatorname{sgn}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sgn}(-\epsilon\gamma_1'\gamma_2) = -\operatorname{sgn}\gamma_2' \neq 0$.

Remark 1.5 The assumptions that $\lambda_1 \neq 2\lambda_2, \ \lambda_1 \neq 2\lambda_3$ in Theorem 1.4 are necessary. In fact, non-composition examples indeed appear in [9], i.e., the standard immersions of $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m)$, $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m)$, $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$ and $\mathbf{E}_{6(-26)}/\mathbf{F}_4$, satisfy the above properties for K and S with $\lambda_1 = 2\lambda_2$ or $\lambda_1 = 2\lambda_3$.

Remark 1.6 Straightforward computation shows that the immersions constructed in Theorem 1.4 satisfy conditions (1.1). Moreover, they all satisfy $\widehat{\nabla}_T T = 0$. For instance, let

M_1 and M_2 be proper affine hyperspheres in A^{n+1} and A^{m+1} respectively given by the vector valued functions f_1 and f_2; and a plane curve γ parametrised such that $|\gamma\gamma'| = 1$, i.e., $\gamma'' = \kappa\gamma$. Then, we can write their composition as $g(t,x,y) = (\gamma_1 f_1, \gamma_2 f_2)$. Straightforwardly, $g_{tt} = \kappa g$, $g_{tx_i} \| g_{x_i}$ and $g_{ty_j} \| g_{y_j}$. Moreover, the affine normal is given by $\xi = \eta g + \alpha g_t$, where $\eta^{n+m+3} = |\gamma_1^{-n}\gamma_2^{-m}\kappa\gamma_1'^m\gamma_2'^n|$ and $\alpha = -\eta'/\kappa$. Then, $g_{tt} = \kappa g = \kappa/\eta(\xi - \alpha g_t) = \kappa/\eta(\xi + \eta'/\kappa g_t)$ and, further $\eta'/\eta g_t$ is the tangential part of g_{tt}. If we denote by T the unit vector field parallel to $g_t = \partial t$, it follows that the right-hand side of the equality $\widehat{\nabla}_{\partial t}\partial t = \eta'/\eta\partial t - K(\partial t, \partial t)$ is parallel to T, and therefore, so is the left-hand side. However, this is possible only if $\widehat{\nabla}_T T = 0$.

Also we finally remark that the condition $\lambda_2\lambda_3 \neq 0$ is necessary. This is illustrated by the following two examples.

Example 1.7 Let M be a hypersurface of $\mathbb{R}^6$ given by

$$\begin{aligned} f(t,x_0,x_1,y_0,y_1) = c_1 \mathrm{e}^{2t}\sqrt{x_0}/(1+\mathrm{e}^{8t}2^{\frac{6}{5}})^{\frac{1}{4}}(\sqrt{2}(g(t) + 2^{\frac{2}{5}}(y_0^2 + y_1^2))e_1 \\ + 2^{\frac{1}{5}}(y_0 e_2 + y_1 e_3) + \sqrt{2}e_4) + x_1 e_5 + (x_0 + x_1^2/2)e_6, \end{aligned}$$

where $\{e_1, \dots, e_6\}$ is the standard basis of $\mathbb{R}^6$, $x_0 > 0$ and $g(t) = 2^{\frac{8}{5}} \int \mathrm{e}^{-2t}/(1+\mathrm{e}^{8t}2^{\frac{6}{5}})^{\frac{1}{2}} dt$ and $c_1 = 2^{\frac{14}{25}}$. A straightforward computation shows that M is an improper affine hypersphere with affine normal being e_6. Moreover, if we denote $\lambda_3 = 2^{\frac{119}{50}}(\mathrm{e}^{-\frac{4}{5}} + 2^{\frac{6}{5}}\mathrm{e}^{\frac{4}{5}})x_0^{-\frac{1}{2}}$ and $k = 2^{\frac{59}{50}}\mathrm{e}^t x_0^{\frac{1}{2}}(1+2^{\frac{6}{5}}\mathrm{e}^{8t})^{-\frac{1}{4}}$, the vector fields

$$T = \lambda_3\frac{\partial}{\partial t}, \quad X_0 = 2^{\frac{11}{50}}x_0^{\frac{1}{2}}\frac{\partial}{\partial x_0}, \quad X_1 = 2^{-\frac{7}{25}}\frac{\partial}{\partial x_1}, \quad Y_0 = k^{-1}\frac{\partial}{\partial y_0}, \quad Y_1 = k^{-1}\frac{\partial}{\partial y_1}$$

form an orthonormal basis with respect to the second fundamental form and the nonzero components of the difference tensor K in this basis are

$$K_T T = -2\lambda_3 T, \quad K_T Y_i = \lambda_3 Y_i, \quad i = 0, 1.$$

However, $\widehat{\nabla}_T T = -2^{-\frac{39}{50}}x_0^{-\frac{1}{2}}X_0 \neq 0$, and therefore M is not a composition of two affine hyperspheres.

Example 1.8 Let, now, M be a hypersurface of $\mathbb{R}^7$ given by

$$\begin{aligned} &f(t,x_0,x_1,x_2,y_1,y_2) \\ &\quad = \frac{\rho}{16}(16e_2 - c_1\sqrt{t}(4e_3 + 2(y_1+y_2)e_4 + (y_1^2 + 2y_1y_2 - y_2^2)e_5)) \\ &\quad\quad + \frac{\rho}{2}(2t^2 + 3x_0^2 + x_1^3 + 3x_2(x_1^2 - x_1x_2 + x_2^2) + 2\sqrt{t}(y_1^3 + 3y_2(y_1^2 - y_1y_2 + y_2^2)))e_1 \\ &\quad\quad + \frac{\rho}{4}((x_1^2 + 2x_1x_2 - x_2^2)e_6 + 2(x_1+x_2)e_7), \end{aligned}$$

where $\{e_1, \dots, e_7\}$ is the standard basis of the space $\mathbb{R}^7$, $\rho = \frac{4}{4c_2 - c_1\sqrt{t} + 4x_0}$ with $c_1 = 2^4 3^{-\frac{1}{3}}$ and c_2 constant such that $\rho > 0$ and $x_2, y_2, t > 0$. If we denote $\lambda_3 = \frac{1}{4t\rho}$, $\nu = \sqrt{2x_2}\rho$ and $\sigma = 2\sqrt[4]{t}\sqrt{y_2}\rho$, the vector fields

$$T = \frac{1}{\rho}\frac{\partial}{\partial t}, \quad X_0 = \frac{1}{\rho}\frac{\partial}{\partial x_0}, \quad X_1 = \frac{1}{\nu}\frac{\partial}{\partial x_1}, \quad X_2 = \frac{1}{\nu}\frac{\partial}{\partial x_2}, \quad Y_1 = \frac{1}{\sigma}\frac{\partial}{\partial y_1}, \quad Y_2 = \frac{1}{\sigma}\frac{\partial}{\partial y_2}$$

form an orthonormal basis with respect to the second fundamental form. Taking $\tau = \frac{c_1}{8\rho}t^{-\frac{3}{2}}$, a straightforward computation shows that

$$ST = -\left(1 + \frac{\tau(\tau - 16\lambda_3^2)}{16\lambda_3^2}\right)T, \quad SX_i = -\left(1 + \frac{\tau^2}{16\lambda_3^2}\right)X_i, \quad i = 0, 1, 2$$

and

$$SY_i = -\left(1 + \frac{\tau(8\lambda_3^2 + \tau)}{16\lambda_3^2}\right) Y_i, \quad i = 1, 2.$$

Also, the nonzero components of the difference tensor K in this basis are $K_T T = -2\lambda_3 T$, $K_T Y_i = \lambda_3 Y_i, i = 1, 2$. However, $\widehat{\nabla}_T T = X_0 \neq 0$, and therefore M is not a composition of two affine hyperspheres.

This paper is organized as follows. In Section 2, we introduce the theory of local affine hypersurfaces. In Section 3, we study the basic properties of the difference tensor and the affine shape operator. The proof of Theorem 1.4 are given in Section 4 for case $\widehat{\nabla}_T T = 0$ and in Section 5 for case $\widehat{\nabla}_T T \neq 0$, respectively.

2 Preliminaries

We briefly recall the theory of local equiaffine hypersurfaces in [13, 15]. Let $\mathbb{R}^{n+1}$ be the standard $(n+1)$-dimensional real affine space, i.e., $\mathbb{R}^{n+1}$ endowed with the standard flat connection D and its parallel volume form w, given by the determinant. Let $F : M \hookrightarrow \mathbb{R}^{n+1}$ be an oriented hypersurface, and ξ be any transversal vector field on M, i.e., $T_p\mathbb{R}^{n+1} = T_pM \oplus \mathrm{span}\{\xi_p\}$, $\forall\, p \in M$. For any tangent vector fields $X, Y, X_1, \dots, X_n$, we write

$$D_X F_*(Y) = F_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.1}$$

$$\theta(X_1, \dots, X_n) = w(F_*(X_1), \dots, F_*(X_n), \xi), \tag{2.2}$$

thus defining a torsion-free affine connection ∇, a symmetric bilinear form h, and a volume element θ on M. M is said to be non-degenerate if h is non-degenerate (this condition is independent of the choice of the transversal vector field). If M is non-degenerate, up to sign there exists a unique choice of transversal vector field such that $\nabla\theta = 0$ and $\theta = w_h$, where w_h is the metric volume element induced by h. This special transversal vector field ξ, called the *affine normal*, induces the *affine connection* ∇ and a pseudo-Riemannian metric h on M. We call h the *affine metric, or Berwald–Blaschke metric* and $C := \nabla h$ the *cubic form*.

The condition $\nabla\theta = 0$ shows that $D_X\xi$ is tangent to M for all X. Hence we can define a $(1,1)$-type tensor S on M, called *affine shape operator*, by

$$D_X\xi = -F_*(SX), \tag{2.3}$$

and the *affine mean curvature* by $H = \frac{1}{n}\mathrm{tr}\, S$. Here S has the property of self-adjoint relative to h. The hypersurface M is called an *affine hypersphere* if $S = H\,\mathrm{id}$, then one easily proves that $H = \mathrm{const}$ if $n \geq 2$. M is called a *proper affine hypersphere* if $H \neq 0$ and an *improper affine hypersphere* if $H = 0$. For a proper affine hypersphere the affine normal satisfies $\xi = -H(F - c)$, where c is a fixed point in $\mathbb{R}^{n+1}$, called the *center* of $F(M)$, for simplicity, we choose c as the origin of $\mathbb{R}^{n+1}$. For an improper affine hypersphere, the affine normal ξ is constant.

The classical Pick–Berwald theorem states that the affine connection coincides with the Levi–Civita connection $\widehat{\nabla}$ of affine metric h if and only if the hypersurface is a hyperquadric. For that reason, the difference tensor $K(X, Y) := \nabla_X Y - \widehat{\nabla}_X Y$, related to the cubic form by

$$C(X, Y, Z) = -2h(K(X, Y), Z),$$

plays a fundamental role in affine differential geometry. Recall that the curvature tensor $\widehat{R}$ of the affine metric, affine shape operator S and the difference tensor K are related by the Gauss and Codazzi equations:

$$\widehat{R}(X,Y)Z = \frac{1}{2}\big[h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\big] - [K_X, K_Y]Z, \quad (2.4)$$

$$(\widehat{\nabla}_X K)(Y,Z) - (\widehat{\nabla}_Y K)(X,Z) = \frac{1}{2}\big[h(Y,Z)SX - h(X,Z)SY - h(SY,Z)X + h(SX,Z)Y\big], \quad (2.5)$$

$$(\widehat{\nabla}_X S)Y - (\widehat{\nabla}_Y S)X = K(SX,Y) - K(SY,X). \quad (2.6)$$

From the Gauss equation (2.4), we obtain

$$\chi = H + J, \quad (2.7)$$

where $J = \frac{1}{n(n-1)}h(K,K)$ is the *Pick invariant* and χ is the *normalized scalar curvature* of h. Moreover, h and K satisfy the apolarity condition

$$\mathrm{tr}_h(X \mapsto K_Z X) = 0, \quad (2.8)$$

or equivalently $\mathrm{tr}\, K_Z = 0$ for all Z.

Finally, for later use we recall some notions of distributions. Let (M,h) be a Riemannian manifold and $\widehat{\nabla}$ its Levi–Civita connection. Then a subbundle $E \subset TM$ is called autoparallel if $\widehat{\nabla}_X Y \in E$ holds for all $X, Y \in E$, whereas a subbundle E is called totally umbilical if there exists a vector field $H \in E^{\perp}$ such that $h(\widehat{\nabla}_X Y, Z) = h(X,Y)h(H,Z)$ for all $X, Y \in E$ and $Z \in E^{\perp}$, here we call H the mean curvature vector of E. If, moreover, $h(\widehat{\nabla}_X H, Z) = 0$ holds, we say that E is spherical. We recall the following decomposition theorem of Riemannian manifold.

Theorem 2.1 ([14, Theorem 4]) *Let M be a Riemannian manifold, and let $TM = \bigoplus_{i=0}^k E_i$ be an orthogonal decomposition into nontrivial vector subbundles such that E_i is spherical and $E_i^{\perp}$ is autoparallel for $i = 1, \dots, k$. Then*

(a) *For every point $\tilde{p} \in M$, there is an isometry ψ of a warped product $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ onto a neighbourhood of $\tilde{p}$ in M such that*

$$\rho_1(\tilde{p}_0) = \cdots = \rho_k(\tilde{p}_0) = 1, \quad (2.9)$$

where $\tilde{p}_0$ is the component of $\psi^{-1}(\tilde{p})$ in M_0, and such that

$$\begin{gathered}\psi\big(\{p_0\} \times \cdots \times \{p_{i-1}\} \times M_i \times \{p_{i+1}\} \times \cdots \times \{p_k\}\big) \text{ is an integral} \\ \text{manifold of } E_i \text{ for } i = 0, \dots, k \text{ and all } p_0 \in M_0, \dots, p_k \in M_k.\end{gathered} \quad (2.10)$$

(b) *If M is simply connected and complete, then for every point $\tilde{p} \in M$ there exists an isometry ψ of a warped product $M_0 \times_{\rho_1} M_1 \times \cdots \times_{\rho_k} M_k$ onto all of M with the properties* (2.9) *and* (2.10).

3 Basic Relations

From now on, we assume that $F : M \to \mathbb{R}^{n+m+2}$ $(n \geq 2, m \geq 2)$ is a locally strongly convex $(n+m+1)$-dimensional affine hypersurface such that its tangent bundle is an orthogonal sum of three distributions with respect to the affine metric h, i.e., $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$, where $\mathcal{D}_1$ is spanned by a unit vector field T, $\dim\mathcal{D}_2 = n$ and $\dim\mathcal{D}_3 = m$ such that for arbitrary

$X \in \mathcal{D}_2,\ Y \in \mathcal{D}_3$,

$$\begin{cases} K_T T = \lambda_1 T, \quad K_T X = \lambda_2 X, \quad K_T Y = \lambda_3 Y, \quad K_X Y = 0, \\ ST = \mu_1 T, \quad SX = \mu_2 X, \quad SY = \mu_3 Y. \end{cases}$$

The apolarity condition $\operatorname{tr} K_T = 0$ shows that

$$\lambda_1 + n\lambda_2 + m\lambda_3 = 0. \tag{3.1}$$

There exists a pair of unit vector fields $X_0 \in \mathcal{D}_2, Y_0 \in \mathcal{D}_3$ such that

$$\widehat{\nabla}_T T = aX_0 + bY_0.$$

We assume that $\mathcal{D}_2$ (resp. $\mathcal{D}_3$) is spanned by orthonormal vector fields $\{X_0, X_1, \dots, X_{n-1}\}$ (resp. $\{Y_0, Y_1, \dots, Y_{m-1}\}$). The apolarity condition shows that

$$\sum_i K(X_i, X_i) = -\sum_j K(Y_j, Y_j) - K(T, T).$$

Together with $K(X_i, Y_j) = 0$ and

$$h\Big(\sum_i K(X_i, X_i), T\Big) = n\lambda_2, \quad h\Big(\sum_j K(Y_j, Y_j), T\Big) = m\lambda_3,$$

we see that

$$\sum_i K(X_i, X_i) = n\lambda_2 T, \quad \sum_j K(Y_j, Y_j) = m\lambda_3 T. \tag{3.2}$$

In the following, we also assume $X \in \mathcal{D}_2$ and $Y \in \mathcal{D}_3$ are unit vector fields.

Using (2.6) on vector fields T and X, we have

$$\begin{aligned} &T(\mu_2)X + \mu_2 \widehat{\nabla}_T X + \mu_2 K_T X - S(\widehat{\nabla}_T X) \\ &\quad = X(\mu_1)T + \mu_1 \widehat{\nabla}_X T + \mu_1 K_T X - S(\widehat{\nabla}_X T). \end{aligned} \tag{3.3}$$

Then, since $h(K_T X, T) = 0$, multiplying (3.3) with T w.r.t. h yields

$$X(\mu_1) = a(\mu_1 - \mu_2)h(X_0, X). \tag{3.4}$$

Similarly, multiplying (3.3) with Y and arbitrary $X' \in \mathcal{D}_2$ respectively, we obtain

$$\begin{aligned} &(\mu_2 - \mu_3)h(\widehat{\nabla}_T X, Y) = (\mu_1 - \mu_3)h(\widehat{\nabla}_X T, Y), \\ &\big(T(\mu_2) + \lambda_2(\mu_2 - \mu_1)\big)h(X, X') = (\mu_1 - \mu_2)h(\widehat{\nabla}_X T, X'). \end{aligned} \tag{3.5}$$

Using (2.6) for Y and X, we have

$$Y(\mu_2)X + \mu_2(\widehat{\nabla}_Y X) - S(\widehat{\nabla}_Y X) = X(\mu_3)Y + \mu_3(\widehat{\nabla}_X Y) - S(\widehat{\nabla}_X Y).$$

Then, multiplying it with arbitrary $X' \in \mathcal{D}_2$ yields

$$Y(\mu_2)h(X, X') = (\mu_3 - \mu_2)h(\widehat{\nabla}_X Y, X').$$

If we now put $X = X' = X_i$ and we sum the previous equation, as well as the second equation in (3.5), we deduce that

$$\begin{aligned} Y(\mu_2) &= \frac{\mu_3 - \mu_2}{n} \sum_i h(\widehat{\nabla}_{X_i} Y, X_i), \\ T(\mu_2) + \lambda_2(\mu_2 - \mu_1) &= \frac{\mu_1 - \mu_2}{n} \sum_i h(\widehat{\nabla}_{X_i} T, X_i). \end{aligned} \tag{3.6}$$

Exchanging the role of X and Y, for arbitrary $Y' \in \mathcal{D}_3$ we can repeat the previous computations in order to get that

$$\begin{cases} Y(\mu_1) = b(\mu_1 - \mu_3)h(Y_0, Y), \\ (\mu_3 - \mu_2)h(\widehat{\nabla}_T Y, X) = (\mu_1 - \mu_2)h(\widehat{\nabla}_Y T, X), \\ X(\mu_3) = \dfrac{\mu_2 - \mu_3}{m} \displaystyle\sum_j h(\widehat{\nabla}_{Y_j} X, Y_j), \\ T(\mu_3) + \lambda_3(\mu_3 - \mu_1) = \dfrac{\mu_1 - \mu_3}{m} \displaystyle\sum_j h(\widehat{\nabla}_{Y_j} T, Y_j). \end{cases} \tag{3.7}$$

Remark 3.1 For $n \geq 2$, X_i and X_j $(i \neq j)$ in (2.6), we have

$$X_i(\mu_2)X_j + \mu_2(\widehat{\nabla}_{X_i} X_j) - S(\widehat{\nabla}_{X_i} X_j) = X_j(\mu_2)X_i + \mu_2(\widehat{\nabla}_{X_j} X_i) - S(\widehat{\nabla}_{X_j} X_i).$$

Multiplying this with X_j yields $X_i(\mu_2) = 0$, and thus $X(\mu_2) = 0$. Similarly, for $m \geq 2$ we have $Y(\mu_3) = 0$.

Taking $Y = Z = T$ in (2.5), we have

$$(\widehat{\nabla}_X K)(T, T) - (\widehat{\nabla}_T K)(X, T) = \frac{1}{2}(\mu_2 - \mu_1)X,$$

and calculating the left-hand side gives

$$\begin{aligned} \frac{1}{2}(\mu_2 - \mu_1)X = X(\lambda_1)T + \lambda_1 \widehat{\nabla}_X T - T(\lambda_2)X - \lambda_2 \widehat{\nabla}_T X \\ - 2K(\widehat{\nabla}_X T, T) + K(\widehat{\nabla}_T X, T) + aK(X, X_0). \end{aligned} \tag{3.8}$$

Multiplying (3.8) with T, Y and arbitrary $X' \in \mathcal{D}_2$ respectively, we see that

$$\begin{cases} X(\lambda_1) = a(\lambda_1 - 2\lambda_2)h(X_0, X), \\ (\lambda_1 - 2\lambda_3)h(\widehat{\nabla}_X T, Y) + (\lambda_3 - \lambda_2)h(\widehat{\nabla}_T X, Y) = 0, \\ (\lambda_1 - 2\lambda_2)h(\widehat{\nabla}_X T, X') + ah(K(X, X_0), X') = \left(T(\lambda_2) + \dfrac{1}{2}(\mu_2 - \mu_1)\right)h(X, X'). \end{cases} \tag{3.9}$$

Similarly, by (2.5) we have

$$(\widehat{\nabla}_X K)(T, Y) - (\widehat{\nabla}_T K)(X, Y) = 0,$$

and calculating the left-hand side gives

$$X(\lambda_3)Y + \lambda_3 \widehat{\nabla}_X Y + K(\widehat{\nabla}_T X, Y) + K(X, \widehat{\nabla}_T Y) = K(\widehat{\nabla}_X T, Y) + K(T, \widehat{\nabla}_X Y), \tag{3.10}$$

and multiplying (3.10) with arbitrary $X' \in \mathcal{D}_2$ and $Y' \in \mathcal{D}_3$ respectively, we get

$$\begin{aligned} h(K(X, X'), \widehat{\nabla}_T Y) = (\lambda_2 - \lambda_3)h(\widehat{\nabla}_X Y, X'), \\ X(\lambda_3)h(Y, Y') = h(K(Y, Y'), \widehat{\nabla}_X T - \widehat{\nabla}_T X). \end{aligned} \tag{3.11}$$

Summing this and the last formula in (3.9) for $X = X' = X_i,\ Y = Y' = Y_j$ respectively, we see that

$$\begin{aligned} & b\lambda_2 h(Y_0, Y) = \frac{\lambda_3 - \lambda_2}{n} \sum_i h(\widehat{\nabla}_{X_i} Y, X_i), \\ & X(\lambda_3) = a\lambda_3 h(X_0, X), \\ & T(\lambda_2) + \frac{1}{2}(\mu_2 - \mu_1) = \frac{\lambda_1 - 2\lambda_2}{n} \sum_i h(\widehat{\nabla}_{X_i} T, X_i). \end{aligned} \tag{3.12}$$

As before, exchanging the role of X and Y, we can repeat the computations of (3.9), (3.11) and (3.12) in order to get that

$$\begin{cases} Y(\lambda_1) = b(\lambda_1 - 2\lambda_3)h(Y_0, Y), \\ (\lambda_1 - 2\lambda_2)h(\widehat{\nabla}_Y T, X) + (\lambda_2 - \lambda_3)h(\widehat{\nabla}_T Y, X) = 0, \\ (\lambda_1 - 2\lambda_3)h(\widehat{\nabla}_Y T, Y') + bh(K(Y, Y_0), Y') = \Big(T(\lambda_3) + \dfrac{1}{2}(\mu_3 - \mu_1)\Big)h(Y, Y'), \\ (\lambda_3 - \lambda_2)h(\widehat{\nabla}_Y X, Y') = h(K(Y, Y'), \widehat{\nabla}_T X), \\ Y(\lambda_2)h(X, X') = h(K(X, X'), \widehat{\nabla}_Y T - \widehat{\nabla}_T Y), \\ a\lambda_3 h(X_0, X) = \dfrac{\lambda_2 - \lambda_3}{m} \displaystyle\sum_j h(\widehat{\nabla}_{Y_j} X, Y_j), \\ Y(\lambda_2) = b\lambda_2 h(Y_0, Y), \\ T(\lambda_3) + \dfrac{1}{2}(\mu_3 - \mu_1) = \dfrac{\lambda_1 - 2\lambda_3}{m} \displaystyle\sum_j h(\widehat{\nabla}_{Y_j} T, Y_j). \end{cases} \tag{3.13}$$

By (2.5) we calculate both the left-hand sides of

$$h((\widehat{\nabla}_X K)(Y, T), X) - h((\widehat{\nabla}_Y K)(X, T), X) = 0,$$
$$h((\widehat{\nabla}_{X'} K)(X, T), X') - h((\widehat{\nabla}_X K)(X', T), X') = 0,$$

where X, $X' \in \mathcal{D}_2$ are unitary and orthogonal, we obtain respectively that

$$(\lambda_3 - \lambda_2)h(\widehat{\nabla}_X Y, X) + h(K(X, X), \widehat{\nabla}_Y T) = Y(\lambda_2),$$
$$h(K(X', X'), \widehat{\nabla}_X T) - h(K(X, X'), \widehat{\nabla}_{X'} T) = X(\lambda_2).$$

Summing the first formula for $X = X_i$ and the second for $X' = X_i$ $(i \geq 1)$ and $X = X_0$, respectively, we see that

$$\begin{aligned} &Y(\lambda_2) = \frac{\lambda_3 - \lambda_2}{n} \sum_i h(\widehat{\nabla}_{X_i} Y, X_i), \\ &\sum_i h(K(X_0, X_i), \widehat{\nabla}_{X_i} T) + (n-1)X_0(\lambda_2) = 0. \end{aligned} \tag{3.14}$$

Similarly, we have

$$\begin{aligned} &X(\lambda_3) = \frac{\lambda_2 - \lambda_3}{m} \sum_j h(\widehat{\nabla}_{Y_j} X, Y_j), \\ &\sum_j h(K(Y_0, Y_j), \widehat{\nabla}_{Y_j} T) + (m-1)Y_0(\lambda_3) = 0. \end{aligned} \tag{3.15}$$

Now we can prove the following lemma:

Lemma 3.2 *For arbitrary $X \in \mathcal{D}_2$, it holds that*

(1) $X(\lambda_1) = a(\lambda_1 - 2\lambda_2)h(X_0, X)$, $X(\lambda_3) = a\lambda_3 h(X_0, X)$, $X(\lambda_2) = \frac{n+2}{n} a\lambda_2 h(X_0, X)$;

(2) $X(\mu_1) = a(\mu_1 - \mu_2)h(X_0, X)$, $(\lambda_3 - \lambda_2)X(\mu_3) = a(\mu_3 - \mu_2)\lambda_3 h(X_0, X)$, $X(\mu_2) = 0$;

(3) $(\mu_1 - \mu_2)T(\lambda_2) + (2\lambda_2 - \lambda_1)T(\mu_2) = \frac{1}{2}(\mu_1 - \mu_2)^2 + (2\lambda_2 - \lambda_1)\lambda_2(\mu_1 - \mu_2)$, $a\lambda_2(\lambda_1 - 2\lambda_2)(\mu_1 - \mu_2) = 0$, $a(\mu_1 - \mu_2)\sum_{i,j} h(K(X_0, X_i), X_j)^2 = 0$;

(4) $a[-\frac{(n+2)(n-1)}{n}\lambda_2(\lambda_1 - 2\lambda_2) + \sum_{i,j} h(K(X_0, X_i), X_j)^2] = 0$.

Proof The first two conclusions in (1) follow immediately from the first equation of (3.9) and the second of (3.12). From (3.1), we deduce that

$$X(\lambda_1) + nX(\lambda_2) + mX(\lambda_3) = 0.$$

So using (3.1) oncemore shows $X(\lambda_2) = \frac{n+2}{n} a\lambda_2 h(X_0, X)$.

The conclusions in (2) separately follow from (3.4), the third equation of (3.7) combined with the sixth equation of (3.13), and Remark 3.1.

The first conclusion in (3) follows from the second equation of (3.6) and the third of (3.12). In order to obtain the remaining conclusions, we introduce

$$V_X := (\lambda_1 - 2\lambda_2)\widehat{\nabla}_X T + aK(X_0, X) - T(\lambda_2)X - \frac{1}{2}(\mu_2 - \mu_1)X.$$

By the third equation of (3.9), we have $h(V_X, X') = 0$ for arbitrary $X' \in \mathcal{D}_2$. Moreover, we see that

$$h(V_X, T) = a\lambda_2 h(X_0, X), \quad h(V_X, Y) = (\lambda_1 - 2\lambda_2)h(\widehat{\nabla}_X T, Y).$$

Hence we can write $V_X = a\lambda_2 h(X_0, X)T + (\lambda_1 - 2\lambda_2)\sum_j h(\widehat{\nabla}_X T, Y_j)Y_j$. Multiplying this with $K(X_0, X)$ yields $h(V_X, K(X_0, X)) = a\lambda_2^2 h(X_0, X)^2$. Summing this equation for $X = X_i$, we obtain that

$$(\lambda_1 - 2\lambda_2)\sum_i h(\widehat{\nabla}_{X_i} T, K_{X_0} X_i) + a\sum_i h(K_{X_0} X_i, K_{X_0} X_i) = a\lambda_2^2.$$

Substituting $K_{X_0} X_i = \lambda_2 h(X_0, X_i)T + \sum_j h(K_{X_0} X_i, X_j)X_j$ into the previous equation, we get

$$(\lambda_1 - 2\lambda_2)\sum_i h(\widehat{\nabla}_{X_i} T, K(X_0, X_i)) + a\sum_{i,j} h(K(X_0, X_i), X_j)^2 = 0. \tag{3.16}$$

It follows from the second equation of (3.14) and the conclusion (1) that

$$a\Big[-\frac{(n+2)(n-1)}{n}\lambda_2(\lambda_1 - 2\lambda_2) + \sum_{i,j} h(K(X_0, X_i), X_j)^2 \Big] = 0. \tag{3.17}$$

On the other hand, we write $K_{X_0} X_i := \sum_j \alpha_i^j X_j + \lambda_2 h(X_0, X_i)T$. We also remark that $\widehat{\nabla}_{X_i} T$ is orthogonal to T. Therefore substituting the $\mathcal{D}_2$ component of $K(X_0, X_i)$ for X' in the second equation of (3.5) corresponds with replacing X' in that equation with $K(X_0, X_i)$, we find that

$$\begin{aligned}
(\mu_1 - \mu_2)\sum_i h(\widehat{\nabla}_{X_i} T, K(X_0, X_i)) &= (\mu_1 - \mu_2)\sum_{i,j} \alpha_i^j h(\widehat{\nabla}_{X_i} T, X_j) \\
&= \big(T(\mu_2) + \lambda_2(\mu_2 - \mu_1)\big)\sum_{i,j} \alpha_i^j h(X_i, X_j) \\
&= \big(T(\mu_2) + \lambda_2(\mu_2 - \mu_1)\big)\sum_i h(X_0, K(X_i, X_i)) \\
&= 0.
\end{aligned}$$

It follows from (3.16) that

$$a(\mu_1 - \mu_2)\sum_{i,j} h(K(X_0, X_i), X_j)^2 = 0.$$

Combining this with (3.17) we get $a\lambda_2(\lambda_1 - 2\lambda_2)(\mu_1 - \mu_2) = 0$. The last two conclusions in (3) now follow.

Finally, the conclusion (4) follows from (3.17).

Similar to Lemma 3.2, we have the following

Lemma 3.3 *For arbitrary $Y \in \mathcal{D}_3$, there hold*

(1) $Y(\lambda_1) = b(\lambda_1 - 2\lambda_3)h(Y_0, Y)$, $Y(\lambda_2) = b\lambda_2 h(Y_0, Y)$, $Y(\lambda_3) = \frac{m+2}{m} b\lambda_3 h(Y_0, Y)$;

(2) $Y(\mu_1) = b(\mu_1 - \mu_3)h(Y_0, Y)$, $(\lambda_2 - \lambda_3)Y(\mu_2) = b(\mu_2 - \mu_3)\lambda_2 h(Y_0, Y)$, $Y(\mu_3) = 0$;

(3) $(\mu_1 - \mu_3)T(\lambda_3) + (2\lambda_3 - \lambda_1)T(\mu_3) = \frac{1}{2}(\mu_1 - \mu_3)^2 + (2\lambda_3 - \lambda_1)\lambda_3(\mu_1 - \mu_3)$, $b\lambda_3(\lambda_1 - 2\lambda_3)(\mu_1 - \mu_3) = 0$, $b(\mu_1 - \mu_3)\sum_{j,\ell} h(K(Y_0, Y_j), Y_\ell)^2 = 0$;

(4) $b[-\frac{(m+2)(m-1)}{m}\lambda_3(\lambda_1 - 2\lambda_3) + \sum_{i,j} h(K(Y_0, Y_i), Y_j)^2] = 0$.

By the first equation of (3.5), the second equations of (3.9), (3.7) and (3.13), respectively, we easily obtain the following

Lemma 3.4 *For arbitrary vector fields $X \in \mathcal{D}_2$ and $Y \in \mathcal{D}_3$, we have*

$$(\mu_2 - \mu_3)h(\widehat{\nabla}_T X, Y) = (\mu_1 - \mu_3)h(\widehat{\nabla}_X T, Y) = (\mu_1 - \mu_2)h(\widehat{\nabla}_Y T, X),$$
$$(\lambda_2 - \lambda_3)h(\widehat{\nabla}_T X, Y) = (\lambda_1 - 2\lambda_3)h(\widehat{\nabla}_X T, Y) = (\lambda_1 - 2\lambda_2)h(\widehat{\nabla}_Y T, X).$$

From now on, we assume that $\lambda_1 \neq 2\lambda_2$ and $\lambda_1 \neq 2\lambda_3$, and we will further always denote

$$\alpha := \frac{T(\lambda_2) + \frac{1}{2}(\mu_2 - \mu_1)}{2\lambda_2 - \lambda_1}, \quad \beta := \frac{T(\lambda_3) + \frac{1}{2}(\mu_3 - \mu_1)}{2\lambda_3 - \lambda_1}.$$

Then, both the last equations in (3.12) and (3.13) can be written respectively as

$$\sum_i h(\widehat{\nabla}_{X_i} T, X_i) = -n\alpha, \quad \sum_j h(\widehat{\nabla}_{Y_j} T, Y_j) = -m\beta. \tag{3.18}$$

Hence, we obtain

Lemma 3.5 *We have*

$$\alpha\lambda_3 - \beta\lambda_2 = \frac{1}{2}(\mu_3 - \mu_2).$$

Proof By (2.5), we have

$$h((\widehat{\nabla}_{Y_j} K)(X_i, X_i) - (\widehat{\nabla}_{X_i} K)(Y_j, X_i), Y_j) = \frac{1}{2}(\mu_3 - \mu_2),$$

which could reduce to

$$h(\widehat{\nabla}_{Y_j} K(X_i, X_i) + K(Y_j, \widehat{\nabla}_{X_i} X_i), Y_j) = \frac{1}{2}(\mu_3 - \mu_2).$$

Summarizing above over orthonormal basis of $\mathcal{D}_2$ and $\mathcal{D}_3$, we obtain

$$\begin{aligned}\frac{nm}{2}(\mu_3 - \mu_2) &= n\lambda_2 \sum_j h(\widehat{\nabla}_{Y_j} T, Y_j) - m\lambda_3 \sum_i h(\widehat{\nabla}_{X_i} T, X_i) \\ &= -mn\beta\lambda_2 + mn\alpha\lambda_3,\end{aligned}$$

where we used (3.2) and (3.18). Our conclusion follows. □

Note that, if $\lambda_2 = \lambda_3 = 0$, the apolarity condition gives $\lambda_1 = 0$, a contradiction to $\lambda_1 \neq 2\lambda_2$. On the other hand, if $\lambda_2 = \lambda_3 \neq 0$, by Lemma 3.5 we obtain

$$2(\alpha - \beta)\lambda_2 = \frac{(\mu_2 - \mu_3)\lambda_2}{2\lambda_2 - \lambda_1} = \mu_3 - \mu_2.$$

Moreover, if $\mu_2 \neq \mu_3$, then we get $\lambda_1 = 3\lambda_2$, by apolarity condition we get $\lambda_2 = \lambda_3 = 0$, a contradiction. Hence, in this case we have $\mu_2 = \mu_3$, which reduces to the case of Theorem 1.1.

Therefore, from now on, regardless on the case, we may assume that $\lambda_2 \neq \lambda_3$.

We now look at the three systems of equations in Lemma 3.4 more carefully.

If there exist some $X \in \mathcal{D}_2, Y \in \mathcal{D}_3$ such that, for instance, $h(\widehat{\nabla}_X T, Y) \neq 0$ at some point, and therefore in some neighborhood, then Lemma 3.4 implies

$$\begin{aligned}&(\mu_1 - \mu_3)(\lambda_2 - \lambda_3) - (\mu_3 - \mu_2)(2\lambda_3 - \lambda_1) = 0,\\&(\mu_1 - \mu_3)(2\lambda_2 - \lambda_1) - (\mu_1 - \mu_2)(2\lambda_3 - \lambda_1) = 0,\end{aligned} \tag{3.19}$$

and it follows that $(\mu_1 - \mu_3)(\lambda_2 - \lambda_3) = 0$, so we have $\mu_1 = \mu_3$. It then follows from the first equation of (3.19) that $\mu_2 = \mu_3$, and the hypersurface is in fact an affine hypersphere.

Similar conclusion follows from $h(\widehat{\nabla}_T X, Y) \neq 0$ or $h(\widehat{\nabla}_Y T, X) \neq 0$.

We conclude that two situations can occur, i.e., either we are dealing with an affine hypersphere, or for all $X \in \mathcal{D}_2, Y \in \mathcal{D}_3$ we have

$$h(\widehat{\nabla}_T X, Y) = h(\widehat{\nabla}_X T, Y) = h(\widehat{\nabla}_Y T, X) = 0. \tag{3.20}$$

4 Affine Hypersurfaces with $\widehat{\nabla}_T T = 0$

In this section we assume that $\widehat{\nabla}_T T = 0$, i.e., $a = b = 0$. Recall that $\lambda_2 \neq \lambda_3$ and $\lambda_2\lambda_3 \neq 0$. By Lemmas 3.2 and 3.3, we have $X(\lambda_i) = Y(\lambda_i) = 0, X(\mu_i) = Y(\mu_i) = 0, i = 1, 2, 3$.

Lemma 4.1 *For $X, X' \in \mathcal{D}_2$ and for $Y, Y' \in \mathcal{D}_3$, we have*

$$h(\widehat{\nabla}_X X', Y) = 0, \quad h(\widehat{\nabla}_Y Y', X) = 0. \tag{4.1}$$

Proof Assume first that M is not an affine hypersphere. Therefore, for all $X \in \mathcal{D}_2, Y \in \mathcal{D}_3$, we have $h(\widehat{\nabla}_T X, Y) = 0$, then the first equation of (3.11) together with the fourth equation of (3.13) implies that (4.1) holds.

Therefore, we may assume that M is an affine hypersphere, i.e., $\mu_1 = \mu_2 = \mu_3$.

For any fixed $X, X' \in \mathcal{D}_2$, we can put $K(X, X') = \lambda_2 h(X, X')T + \widehat{X}$ where $\widehat{X} \in \mathcal{D}_2$. Then the fifth equation of (3.13) yields $h(\widehat{X}, \widehat{\nabla}_Y T - \widehat{\nabla}_T Y) = 0$. Along with Lemma 3.4, we obtain a system for $h(\widehat{X}, \widehat{\nabla}_Y T)$ and $h(\widehat{X}, \widehat{\nabla}_T Y)$ with determinant $\lambda_1 - \lambda_2 - \lambda_3$.

If $\lambda_1 - \lambda_2 - \lambda_3 \neq 0$, we get $h(K(X, X'), \widehat{\nabla}_T Y) = 0$, and putting this into the first equation of (3.11), we get (4.1).

So we are left with the case that $\lambda_1 - \lambda_2 - \lambda_3 = 0$. From the apolarity condition, it follows now that $\lambda_2 = -\frac{m+1}{n+1}\lambda_3$, $\lambda_1 = \frac{n-m}{n+1}\lambda_3$. The second equation of Lemma 3.4 implies that

$$h(\widehat{\nabla}_X T, Y) = h(\widehat{\nabla}_T X, Y) = h(\widehat{\nabla}_Y X, T).$$

Furthermore, Lemma 3.5 gives $T(\lambda_3) = 0$ and so λ_3 is a nonzero constant. Moreover the third equations of both (3.9) and (3.13) are simplified to be $h(\widehat{\nabla}_X T, X') = h(\widehat{\nabla}_Y T, Y') = 0$. The Gauss equation gives $\widehat{R}(X, T)T = (\mu_1 - \lambda_1\lambda_2 + \lambda_2^2)X$, and on the other hand

$$\begin{aligned}&\widehat{\nabla}_X \widehat{\nabla}_T T = 0, \quad \widehat{\nabla}_T \widehat{\nabla}_X T = \sum_j T(h(\widehat{\nabla}_X T, Y_j))Y_j + \sum_j h(\widehat{\nabla}_X T, Y_j)\widehat{\nabla}_T Y_j,\\&\widehat{\nabla}_X T - \widehat{\nabla}_T X = -\sum_i h(\widehat{\nabla}_T X, X_i)X_i, \quad \widehat{\nabla}_{[X,T]} T = -\sum_i h(\widehat{\nabla}_T X, X_i)\widehat{\nabla}_{X_i} T.\end{aligned}$$

Then,

$$(\mu_1 - \lambda_1\lambda_2 + \lambda_2^2)h(X, X') = \widehat{R}(X, T, T, X')$$

$$
\begin{aligned}
&= -\sum_j h(\widehat{\nabla}_X T, Y_j) h(\widehat{\nabla}_T Y_j, X') + \sum_i h(\widehat{\nabla}_T X, X_i) h(\widehat{\nabla}_{X_i} T, X') \\
&= \sum_j h(\widehat{\nabla}_X T, Y_j) h(\widehat{\nabla}_{X'} T, Y_j) = h(\widehat{\nabla}_X T, \widehat{\nabla}_{X'} T).
\end{aligned}
$$

Therefore, $|\widehat{\nabla}_X T|^2 = p^2 = \mu_1 + \frac{m+1}{n+1}\lambda_3^2 = \text{const}$ for all unit X. Since for $p = 0$ we have $\widehat{\nabla}_X T = 0$, thus $h(\widehat{\nabla}_X T, Y) = h(\widehat{\nabla}_T X, Y) = 0$. Similar to the case of non affine hypersphere, the lemma holds. Next we assume $p \neq 0$. Note that $\widehat{\nabla}_X T \in \mathcal{D}_3$. For an orthonormal basis X_i of $\mathcal{D}_2$ we get that the vector fields $Y_i = \widehat{\nabla}_{X_i} T/p \in \mathcal{D}_3$ are mutually orthogonal and have unit length. So $n \leq m$. Similarly changing the role of both distributions we have $\widehat{\nabla}_{Y_i} T = -pX_i$, so it follows $m = n$ and therefore, we deduce that $\lambda_1 = 0$, $\lambda_2 = -\lambda_3$ and $p^2 = \mu_1 + \lambda_3^2$.

We now take corresponding basis for both distributions, i.e., for an orthonormal basis X_i of $\mathcal{D}_2$, we take $Y_i = \widehat{\nabla}_{X_i} T/p \in \mathcal{D}_3$. It then follows that $\widehat{\nabla}_{Y_i} T = -pX_i$. We then get that

$$
\begin{aligned}
0 &= \widehat{R}(X_k, T, T, Y_l) \\
&= -\sum_j h(\widehat{\nabla}_{X_k} T, Y_j) h(\widehat{\nabla}_T Y_j, Y_l) + \sum_i h(\widehat{\nabla}_T X_k, X_i) p h(Y_l, Y_i) \\
&= p[h(\widehat{\nabla}_T X_k, X_l) - h(\widehat{\nabla}_T Y_k, Y_l)],
\end{aligned}
$$

and from the fourth equation of (3.13), we have

$$
h(\widehat{\nabla}_{Y_j} X_i, Y_k) = \frac{p}{2\lambda_3} h(K(Y_j, Y_k), Y_i), \quad h(\widehat{\nabla}_{X_j} Y_i, X_k) = \frac{p}{2\lambda_3} h(K(X_j, X_k), X_i).
$$

From above we see that

$$
\begin{aligned}
0 &= \widehat{R}(X_i, Y_j, T, X_l) \\
&= h(\widehat{\nabla}_{X_i}(-pX_j), X_l) - h(\widehat{\nabla}_{Y_j}(pY_i), X_l) - h(\widehat{\nabla}_{[X_i, Y_j]} T, X_l) \\
&= -p(h(\widehat{\nabla}_{X_i} X_j, X_l) + h(\widehat{\nabla}_{Y_j} Y_i, X_l)) + p h(\widehat{\nabla}_{X_i} Y_j - \widehat{\nabla}_{Y_j} X_i, Y_l) \\
&= p(-h(\widehat{\nabla}_{X_i} X_j, X_l) - h(\widehat{\nabla}_{Y_j} Y_i, X_l) + h(\widehat{\nabla}_{X_i} Y_j, Y_l) - h(\widehat{\nabla}_{Y_j} X_i, Y_l)) \\
&= p(h(\widehat{\nabla}_{X_i} Y_j, Y_l) - h(\widehat{\nabla}_{X_i} X_j, X_l)),
\end{aligned}
$$

and by exchanging X and Y we get $h(\widehat{\nabla}_V X_j, X_l) = h(\widehat{\nabla}_V Y_j, Y_l)$ for any V.

We can resume the previous results by

$$
\begin{cases}
\widehat{\nabla}_T T = 0, \quad \widehat{\nabla}_{X_i} T = pY_i, \quad \widehat{\nabla}_{Y_i} T = -pX_i, \\
\widehat{\nabla}_T X_i = \sum_j h(\widehat{\nabla}_T X_i, X_j) X_j + pY_i, \\
\widehat{\nabla}_{X_j} X_i = \sum_k h(\widehat{\nabla}_{X_j} X_i, X_k) X_k - \frac{p}{2\lambda_3} \sum_k h(K(X_i, X_j), X_k) Y_k, \\
\widehat{\nabla}_{Y_j} X_i = p\delta_{ij} T + \sum_k h(\widehat{\nabla}_{Y_j} X_i, X_k) X_k + \frac{p}{2\lambda_3} \sum_k h(K(Y_k, Y_j), Y_i) Y_k, \\
\widehat{\nabla}_T Y_i = -pX_i + \sum_j h(\widehat{\nabla}_T Y_i, Y_j) Y_j, \\
\widehat{\nabla}_{X_j} Y_i = -p\delta_{ij} T + \frac{p}{2\lambda_3} \sum_k h(K(X_j, X_i), X_k) X_k + \sum_k h(\widehat{\nabla}_{X_j} Y_i, Y_k) Y_k, \\
\widehat{\nabla}_{Y_j} Y_i = -\frac{p}{2\lambda_3} \sum_k h(K(Y_j, Y_i), Y_k) X_k + \sum_k h(\widehat{\nabla}_{Y_j} Y_i, Y_k) Y_k.
\end{cases}
\tag{4.2}
$$

Straightforwardly, by Gauss equations we obtain

$$\begin{cases} \widehat{R}(X,T)T = p^2 X, \quad \widehat{R}(Y,T)T = p^2 Y, \\ \widehat{R}(X,Y_i)Y_j = p^2 X\delta_{ij}, \quad \widehat{R}(X_i,Y)X_j = -p^2 Y\delta_{ij}, \\ \widehat{R}(X_i,T)X_j = \widehat{R}(Y_i,T)Y_j = -\delta_{ij}p^2 T, \\ \widehat{R}(X_i,X_j)T = \widehat{R}(X,T)Y = \widehat{R}(Y,T)X = 0, \\ \widehat{R}(X_i,X_j)Y = \widehat{R}(Y_i,Y_j)X = \widehat{R}(X_i,Y_j)T = 0. \end{cases} \tag{4.3}$$

We now define a homomorphism $\tau : \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 \to \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ given by $V \mapsto \widehat{\nabla}_V T/p$. Then from (4.2) we obtain

$$\widehat{\nabla}_{X_j}\tau(X_i) = -p\delta_{ij}T + \tau(\widehat{\nabla}_{X_j}X_i), \quad \widehat{\nabla}_{X_j}\tau(Y_i) = \tau(\widehat{\nabla}_{X_j}Y_i).$$

Then

$$\begin{aligned} &\widehat{\nabla}_{X_i}(\widehat{\nabla}_{X_j}\tau(Y_k)) \\ &\quad = \widehat{\nabla}_{X_i}\tau(\widehat{\nabla}_{X_j}Y_k) \\ &\quad = \widehat{\nabla}_{X_i}\Big(\sum_l h(\widehat{\nabla}_{X_j}Y_k, X_l)\tau(X_l) + \sum_l h(\widehat{\nabla}_{X_j}Y_k, Y_l)\tau(Y_l)\Big) \\ &\quad = \sum_l X_i(h(\widehat{\nabla}_{X_j}Y_k, X_l))\tau(X_l) + \sum_l h(\widehat{\nabla}_{X_j}Y_k, X_l)(-p\delta_{il}T + \tau(\widehat{\nabla}_{X_i}X_l)) \\ &\qquad + \sum_l X_i(h(\widehat{\nabla}_{X_j}Y_k, Y_l))\tau(Y_l) + \sum_l h(\widehat{\nabla}_{X_j}Y_k, Y_l)\tau(\widehat{\nabla}_{X_i}Y_l) \\ &\quad = \tau\Big[\widehat{\nabla}_{X_i}\Big(\sum_l h(\widehat{\nabla}_{X_j}Y_k, X_l)X_l + \sum_l h(\widehat{\nabla}_{X_j}Y_k, Y_l)Y_l\Big)\Big] - ph(\widehat{\nabla}_{X_j}Y_k, X_i)T \\ &\quad = \tau(\widehat{\nabla}_{X_i}(\widehat{\nabla}_{X_j}Y_k + p\delta_{jk}T)) - ph(\widehat{\nabla}_{X_j}Y_k, X_i)T \\ &\quad = \tau(\widehat{\nabla}_{X_i}(\widehat{\nabla}_{X_j}Y_k)) - p^2\delta_{jk}X_i - ph(\widehat{\nabla}_{X_j}Y_k, X_i)T. \end{aligned}$$

It follows that

$$\begin{aligned} &\widehat{R}(X_i,X_j)X_k \\ &\quad = -\widehat{R}(X_i,X_j)\tau(Y_k) \\ &\quad = ph(\widehat{\nabla}_{X_j}Y_k, X_i)T - \tau(\widehat{\nabla}_{X_i}(\widehat{\nabla}_{X_j}Y_k)) + p^2\delta_{jk}X_i \\ &\qquad - ph(\widehat{\nabla}_{X_i}Y_k, X_j)T + \tau(\widehat{\nabla}_{X_j}(\widehat{\nabla}_{X_i}Y_k)) - p^2\delta_{ik}X_j + \widehat{\nabla}_{[X_i,X_j]}\tau(Y_k) \\ &\quad = -\tau\big(\widehat{R}(X_i,X_j)Y_k\big) + p^2(\delta_{jk}X_i - \delta_{ik}X_j) \\ &\quad = p^2\sum_l(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})X_l. \end{aligned}$$

Similarly, we have $\widehat{R}(Y_i,Y_j)Y_k = p^2\sum_l(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})Y_l$. These together with (4.3) show that this affine hypersphere is of constant sectional curvature p^2. We refer the reader to [16] where such hyperspheres have been classified and it was shown that they are either flat ($p^2 = 0$), or the hyperquadrics ($K = 0$, and thus $\lambda_3 = 0$). However, we have excluded both possibilities which ends the proof of Lemma 4.1.

Also, by the last equation in (3.9) and the third equation in (3.13), we obtain respectively that

$$h(\widehat{\nabla}_X T, X') = -\alpha h(X, X'), \quad h(\widehat{\nabla}_Y T, Y') = -\beta h(Y, Y'). \tag{4.4}$$

Lemma 4.2 *There holds*

$$\begin{aligned} &-\alpha\beta = \frac{1}{2}(\mu_2 + \mu_3) - \lambda_2\lambda_3, \\ &h(X, \widehat{\nabla}_T Y) = h(X, \widehat{\nabla}_Y T) = h(Y, \widehat{\nabla}_X T) = 0, \quad \forall\, X \in \mathcal{D}_2, Y \in \mathcal{D}_3. \end{aligned}$$

Proof For arbitrary unit vectors X and Y, we have the Gauss equation

$$\begin{aligned} \widehat{R}(X, Y, Y, X) &= \frac{1}{2}(\mu_2 + \mu_3) - h(K_X K_Y Y, X) + h(K_Y K_X Y, X) \\ &= \frac{1}{2}(\mu_2 + \mu_3) - h(K(X, X), K(Y, Y)) \\ &= \frac{1}{2}(\mu_2 + \mu_3) - \lambda_2\lambda_3. \end{aligned}$$

On the other hand, by using (4.1), (4.4) and Lemma 3.4, we obtain

$$\begin{aligned} h(\widehat{\nabla}_X \widehat{\nabla}_Y Y, X) &= h\Big(\widehat{\nabla}_X\Big(\beta T + \sum_j h(\widehat{\nabla}_Y Y, Y_j) Y_j\Big), X\Big) = \beta h(\widehat{\nabla}_X T, X) = -\alpha\beta, \\ h(\widehat{\nabla}_Y \widehat{\nabla}_X Y, X) &= h(\widehat{\nabla}_X Y, T) h(\widehat{\nabla}_Y T, X) + h\Big(\widehat{\nabla}_Y\Big(\sum_j h(\widehat{\nabla}_X Y, Y_j) Y_j\Big), X\Big) \\ &= -\frac{(\lambda_3 - \lambda_2)^2}{(2\lambda_2 - \lambda_1)(2\lambda_3 - \lambda_1)} h(X, \widehat{\nabla}_T Y)^2, \\ h(\widehat{\nabla}_{\widehat{\nabla}_X Y - \widehat{\nabla}_Y X} Y, X) &= h(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X, T) h(\widehat{\nabla}_T Y, X) + h(\widehat{\nabla}_{\sum a_i X_i + \sum b_j Y_j} Y, X) \\ &= (\lambda_3 - \lambda_2) h(X, \widehat{\nabla}_T Y)^2 \Big(\frac{1}{2\lambda_3 - \lambda_1} + \frac{1}{\lambda_1 - 2\lambda_2}\Big), \end{aligned}$$

where $a_i = h(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X, X_i)$, $b_j = h(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X, Y_j)$. Thus we get

$$\widehat{R}(X, Y, Y, X) = -\alpha\beta + 3\frac{(\lambda_3 - \lambda_2)^2}{(2\lambda_2 - \lambda_1)(2\lambda_3 - \lambda_1)} h(X, \widehat{\nabla}_T Y)^2.$$

However, since $n \geq 2$, we can choose $X \in \mathcal{D}_2$ being orthogonal to the projection of $\widehat{\nabla}_T Y$ on $\mathcal{D}_2$ so that $h(\widehat{\nabla}_T Y, X) = 0$, and therefore we obtain the first assertion. From Lemma 3.4, the fact $\lambda_2 \neq \lambda_3$ and the first assertion we have the second assertion.

Now, it follows from (4.4) and Lemma 4.2 that

$$\widehat{\nabla}_X T = -\alpha X, \quad \widehat{\nabla}_Y T = -\beta Y. \tag{4.5}$$

Together with previous lemmas we see that $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ are integrable, and both $\mathcal{D}_1 \oplus \mathcal{D}_3$ and $\mathcal{D}_1 \oplus \mathcal{D}_2$ are autoparallel. Moreover, taking $H_2 = \alpha T$ (resp. $H_3 = \beta T$), one can show that $\mathcal{D}_2$ (resp. $\mathcal{D}_3$) is spherical with the mean curvature vector H_2 (resp. H_3). Therefore, by Theorem 2.1 we conclude that M is locally a warped product $\mathbb{R} \times_{\rho_2} M_2 \times_{\rho_3} M_3$, where $\mathbb{R}$, M_2 and M_3 are, respectively, the leaf of the distributions $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. The warping functions ρ_2 and ρ_3 are determined by

$$H_2 = -T(\ln \rho_2) T, \quad H_3 = -T(\ln \rho_3) T.$$

Now we assume that $\frac{\partial}{\partial t} = T$ and the warping functions satisfy the initial condition $\rho_2(0) = \rho_3(0) = 1$. Therefore, we get $\rho_2(t) = \mathrm{e}^{-\int_0^t \alpha(\tau)d\tau}$, $\rho_3(t) = \mathrm{e}^{-\int_0^t \beta(\tau)d\tau}$.

From now on, if not stated otherwise, we take the local coordinates $\{t, x_1, \ldots, x_n, y_1, \ldots, y_m\}$ on M such that $\frac{\partial}{\partial t} = T$, $\mathrm{span}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\} = \mathcal{D}_2$ and $\mathrm{span}\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\} = \mathcal{D}_3$.

Lemma 4.3 *The following relations hold*:

$$\begin{aligned}
&X(\alpha) = Y(\alpha) = X(\beta) = Y(\beta) = 0, \quad \forall\, X \in \mathcal{D}_2, Y \in \mathcal{D}_3,\\
&T(\mu_2) = (\alpha - \lambda_2)(\mu_2 - \mu_1), \quad T(\mu_3) = (\beta - \lambda_3)(\mu_3 - \mu_1),\\
&T(\alpha) = \alpha^2 + \frac{1}{2}(\mu_2 + \mu_1) - \lambda_1\lambda_2 + \lambda_2^2, \quad T(\beta) = \beta^2 + \frac{1}{2}(\mu_3 + \mu_1) - \lambda_1\lambda_3 + \lambda_3^2.
\end{aligned}$$

Proof Lemma 4.2 and the relation (4.1) imply that

$$\widehat{\nabla}_X Y = \sum_j b_j Y_j, \quad \widehat{\nabla}_Y X = \sum a_i X_i. \tag{4.6}$$

From (4.1) and (4.4), it follows that $\widehat{\nabla}_X X' - \widehat{\nabla}_{X'} X = \sum_i \alpha_i X_i$, which simplifies the Gauss equation as

$$\begin{aligned}
0 = \widehat{R}(X, X')T &= \widehat{\nabla}_X(-\alpha X') - \widehat{\nabla}_{X'}(-\alpha X) - \sum_i \alpha_i \widehat{\nabla}_{X_i} T\\
&= -X(\alpha)X' + X'(\alpha)X,
\end{aligned}$$

so we have $X(\alpha) = 0$. Similarly, we get $Y(\beta) = 0$.

Analogously, using (4.6), from

$$\begin{aligned}
0 = \widehat{R}(X, Y)T &= -\widehat{\nabla}_X(\beta Y) + \widehat{\nabla}_Y(\alpha X) - \sum_j b_j \widehat{\nabla}_{Y_j} T + \sum_i a_i \widehat{\nabla}_{X_i} T\\
&= -X(\beta)(Y) + Y(\alpha)X,
\end{aligned}$$

we get $X(\beta) = Y(\alpha) = 0$.

Expressions for $T(\mu_2)$ and $T(\mu_3)$ directly follow from (3.5) and the last equation of (3.7).

From the Gauss equation, we see that $h(\widehat{R}(X, T)T, X) = \frac{1}{2}(\mu_1 + \mu_2) - \lambda_1\lambda_2 + \lambda_2^2$ for unit vector field $X \in \mathcal{D}_2$. On the other hand, by Lemma 3.4 and (4.5) we see that $\widehat{\nabla}_T X \in \mathcal{D}_2$. Thus

$$\begin{aligned}
h(\widehat{R}(X, T)T, X) &= h(-\widehat{\nabla}_T\widehat{\nabla}_X T - \widehat{\nabla}_{\widehat{\nabla}_X T - \widehat{\nabla}_T X} T, X),\\
&= h(\widehat{\nabla}_T(\alpha X) + \alpha\widehat{\nabla}_X T - \alpha\widehat{\nabla}_T X, X)\\
&= T(\alpha) - \alpha^2.
\end{aligned}$$

Then the assertion for $T(\alpha)$ follows. Similarly we have the expression of $T(\beta)$.

Now, in view of the symmetry between $\mathcal{D}_2$ and $\mathcal{D}_3$ it is sufficient to discuss the following three subcases:

Case I-(1) $\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0$, $\mu_3^2 + (\beta - \lambda_3)^2 \neq 0$.

Case I-(1)-(i) $\mu_3(\lambda_2 - \alpha) - \mu_2(\lambda_3 - \beta) \neq 0$.

Case I-(1)-(ii) $\mu_3(\lambda_2 - \alpha) - \mu_2(\lambda_3 - \beta) = 0$.

Case I-(2) $\mu_2 = 0$, $\alpha = \lambda_2$, $\mu_3 = 0$, $\beta = \lambda_3$.

Case I-(3) $\mu_2 = 0, \alpha = \lambda_2$, $\mu_3^2 + (\beta - \lambda_3)^2 \neq 0$.

Case I-(3)-(i) $\mu_3 = 0, \beta \neq \lambda_3$.

Case I-(3)-(ii) $\mu_3 \neq 0$.

Case I-(1) Let $\beta_1(t)$ and $\beta_2(t)$ be functions such that

$$\beta_1' = -\beta_2, \quad \beta_2' = 1 + \beta_1\mu_1 - \beta_2\lambda_1.$$

We denote $\delta_1 = 1 + \mu_2\beta_1 + \beta_2(\alpha - \lambda_2)$ and $\delta_2 = 1 + \mu_3\beta_1 + \beta_2(\beta - \lambda_3)$. As $\mu_2^2 + (\alpha - \lambda_2)^2 \neq 0$, $\mu_3^2 + (\beta - \lambda_3)^2 \neq 0$ holds, straightforward computation shows that by choosing the initial conditions for β_1 and β_2 appropriately we may assume that $\delta_1 = \delta_2 = 0$. Then, straight computation shows

$$\begin{aligned} &D_X(\beta_1\xi + \beta_2 T) = X, \quad \forall\, X \in \mathcal{D}_2,\\ &D_Y(\beta_1\xi + \beta_2 T) = Y, \quad \forall\, Y \in \mathcal{D}_3,\\ &D_T(\beta_1\xi + \beta_2 T) = T. \end{aligned}$$

It follows that, up to translation we can write the immersion

$$F : M^{n+m+1} \to \mathbb{R}^{n+m+2}$$

as

$$F = \beta_1\xi + \beta_2 T.$$

We now define a vector field by

$$g_1 = M((\lambda_2 - \alpha)\xi + \mu_2 T), \tag{4.7}$$

where $M(t)$ is a nonzero solution of the equation $M' + M(\alpha - \lambda_2 + \lambda_1) = 0$.

Then direct computations give that

$$\begin{aligned} D_T g_1 &= (M' + M(\alpha - \lambda_2 + \lambda_1))((\lambda_2 - \alpha)\xi + \mu_2 T) = 0,\\ D_X g_1 &= M(-\mu_2(\lambda_2 - \alpha)X + \mu_2(\widehat{\nabla}_X T + K(X,T))) = 0,\\ D_Y g_1 &= M(-\mu_3(\lambda_2 - \alpha) + \mu_2(\lambda_3 - \beta))Y,\\ D_{Y'} D_Y g_1 &= M(-\mu_3(\lambda_2 - \alpha) + \mu_2(\lambda_3 - \beta))\\ &\quad\cdot [\widehat{\nabla}^{\perp}_{Y'} Y + K^{\perp}(Y,Y') + h(Y,Y')(\xi + (\beta + \lambda_3)T)], \end{aligned} \tag{4.8}$$

where $\widehat{\nabla}^{\perp}_{Y'} Y$ and $K^{\perp}(Y,Y')$ are projections of $\widehat{\nabla}_{Y'} Y$ and $K(Y,Y')$ on $\mathcal{D}_3$ (their projections on $\mathcal{D}_2$ vanish). From Lemmas 3.5 and 4.2, we have

$$\mu_2 = (\lambda_2 - \alpha)(\lambda_3 + \beta), \quad \mu_3 = (\lambda_2 + \alpha)(\lambda_3 - \beta). \tag{4.9}$$

Then we see that

$$\mu_3(\lambda_2 - \alpha) - \mu_2(\lambda_3 - \beta) = (\lambda_2 - \alpha)(\lambda_3 - \beta)(\lambda_2 - \lambda_3 + \alpha - \beta). \tag{4.10}$$

Case I-(1)-(i) In this case, we can see from (4.8) and (4.9) that

$$D_{Y'} D_Y g_1 \in \mathcal{D}_3 + \mathrm{span}(g_1).$$

Since $\sum K^{\perp}(Y_j, Y_j)$ is a projection of $\sum K(Y_j, Y_j) = -\sum K(X_i, X_i) - K(T,T)$ on $\mathcal{D}_3$, $K^{\perp}$ satisfies apolarity condition and g_1 defines a proper affine hypersphere.

Similarly as before, the vector field

$$g_2 = N((\lambda_3 - \beta)\xi + \mu_3 T), \tag{4.11}$$

where $N(t)$ is a nonzero function satisfies $N' = -N(\beta - \lambda_3 + \lambda_1)$, defines a proper affine hypersphere. From (4.7) and (4.11), we can express ξ and T to obtain

$$F = \gamma_1(t)g_1 + \gamma_2(t)g_2,$$

where

$$\gamma_1(t) = \frac{\beta_1' + \beta_1(\lambda_2 + \alpha)}{M(\lambda_2 - \alpha)(\lambda_2 - \lambda_3 + \alpha - \beta)}, \quad \gamma_2(t) = \frac{\beta_1' + \beta_1(\lambda_3 + \beta)}{N(\lambda_3 - \beta)(\lambda_3 - \lambda_2 + \beta - \alpha)}.$$

We have obtained (1) of Theorem 1.4.

Case I-(1)-(ii) In this case by (4.10), we have

$$(\lambda_2 - \alpha)(\lambda_3 - \beta)(\lambda_2 - \lambda_3 + \alpha - \beta) = 0.$$

Since $\lambda_2 = \alpha$ implies $\mu_2 = 0$, it follows from $\mu_2^2 + (\lambda_2 - \alpha)^2 \neq 0$ that $\lambda_2 \neq \alpha$. Similarly $\lambda_3 \neq \beta$. So we have

$$\lambda_2 - \lambda_3 + \alpha - \beta = 0.$$

Then also $D_Y g_1 = 0$, so $g_1 = M(\lambda_2 - \alpha)(\xi + (\beta + \lambda_3)T)$ is a constant vector field. We can put $\xi = \frac{1}{\beta_1}(F - \beta_2 T)$ and further

$$F = \frac{\beta_1}{M(\lambda_2 - \alpha)} g_1 + (\beta_2 - \beta_1(\beta + \lambda_3)) D_T F,$$

and by integrating we obtain

$$F = \gamma_1(t)g_3 + \gamma_2(t)g_1,$$

where g_3 does not depend on t and $\gamma_2 = \frac{\beta_1}{M(\lambda_2 - \alpha)} + (\beta_2 - \beta_1(\beta + \lambda_3))\gamma_2'$. Furthermore,

$$X = D_X F = \gamma_1(t) D_X g_3, \quad Y = D_Y F = \gamma_1(t) D_Y g_3.$$

Then $\frac{\partial}{\partial x_i} g_3 \in \mathcal{D}_2$, and since $\widehat{\nabla}_Y X \in \mathcal{D}_2$ it follows that $\frac{\partial}{\partial y_j}\frac{\partial}{\partial x_i} g_3 \in \mathcal{D}_2$. Similarly, $\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} g_3 \in \mathcal{D}_3$ and therefore $\frac{\partial}{\partial y_j}\frac{\partial}{\partial x_i} g_3 = 0$. Hence,

$$g_3 = C_2(x_1, \ldots, x_n) + C_3(y_1, \ldots, y_m).$$

Furthermore,

$$\begin{aligned} D_X C_2 = D_X g_3 &= \frac{1}{\gamma_1(t)} X, \\ D_{X'} D_X C_2 &= \frac{1}{\gamma_1(t)} (\widehat{\nabla}^{\perp}_{X'} X + K^{\perp}(X, X') + h(X, X')((\alpha + \lambda_2)T + \xi)) \\ &= \frac{1}{\gamma_1(t)} \left(\widehat{\nabla}^{\perp}_{X'} X + K^{\perp}(X, X') + h(X, X') \frac{g_1}{M(\lambda_2 - \alpha)} \right), \end{aligned}$$

so C_2 is an improper affine hypersphere whose affine normal is parallel to g_1.

Similarly, C_3 is an improper affine hypersphere whose affine normal is parallel to g_1. Summing above, we see

$$F(t, x_1, \ldots, x_n) = \gamma_1(t)\big(C_2(x_1, \ldots, x_n) + C_3(y_1, \ldots, y_m)\big) + \gamma_2(t) g_1,$$

where $C_2 + C_3$ is also an improper affine hypersphere whose affine normal is parallel to g_1. This is the hypersurface immersion (2) of Theorem 1.1.

Case I-(2) Now, we have $\mu_2 = \mu_3 = 0$, $\alpha = \lambda_2$, $\beta = \lambda_3$. Let $\beta_1(t)$ be a nonzero function satisfying

$$\beta_1' = -2\lambda_2\beta_1.$$

Since $\mu_2\beta_1 + 2\lambda_2\beta_1(\alpha - \lambda_2) = \mu_3\beta_1 + 2\lambda_2\beta_1(\beta - \lambda_3) = 0$, direct computation shows

$$D_X(\beta_1\xi + 2\beta_1\lambda_2 T) = D_Y(\beta_1\xi + 2\beta_1\lambda_2 T) = D_T(\beta_1\xi + 2\beta_1\lambda_2 T) = 0,$$

so the vector field $C := \beta_1(\xi + 2\lambda_2 T)$ is constant. Then

$$D_T T = \lambda_1 T + \xi = \frac{1}{\beta_1}C + (\lambda_1 - 2\lambda_2)T.$$

By integration, we show that the immersion $F : M \to \mathbb{R}^{n+m+2}$ satisfies

$$F = \gamma_0(t)C_0 + \gamma_1(t)C_1 + \gamma_2(t)C, \tag{4.12}$$

where C_0, C_1 do not depend on t and

$$\gamma_2'' = \gamma_2'(\lambda_1 - 2\lambda_2) + \frac{1}{\beta_1} \tag{4.13}$$

and γ_0 and γ_1 are independent solutions of

$$\gamma'' = \gamma'(\lambda_1 - 2\lambda_2). \tag{4.14}$$

We can take $\gamma_0 = 1$ and therefore $\gamma_1 \neq$ const. Then

$$X = D_X F = D_X C_0 + \gamma_1(t) D_X C_1$$

and, as before, $\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}F \in \mathcal{D}_2 \cap \mathcal{D}_3$ and vanishes. That implies that $\frac{\partial}{\partial y_j}C_0$ and $\frac{\partial}{\partial y_j}C_1$ depend only on $y_1, \ldots, y_m$.

Similarly, $\frac{\partial}{\partial x_i}C_0$ and $\frac{\partial}{\partial x_i}C_1$ depend only on $x_1, \ldots, x_n$. Therefore, we can write

$$\begin{aligned} F = {} & A_0(x_1, \ldots, x_n) + B_0(y_1, \ldots, y_m) \\ & + \gamma_1(t)(A_1(x_1, \ldots, x_n) + B_1(y_1, \ldots, y_m)) + \gamma_2(t)C. \end{aligned} \tag{4.15}$$

Furthermore,

$$0 = (\lambda_2 - \alpha)X = D_X T = D_X(\gamma_1'(A_1 + B_1) + \gamma_2' C) = \gamma_1' D_X A_1.$$

As $\gamma_1 \neq$const, we get $D_X A_1 = 0$ for all $X \in \mathcal{D}_2$. Therefore, A_1 and similarly B_1 are constant vector fields. From

$$D_{X'} D_X A_0 = D_{X'} D_X F = \widehat{\nabla}^{\perp}_{X'} X + K^{\perp}(X, X') + h(X, X')\frac{C}{\beta_1},$$

we see that $D_{X'} D_X A_0 \in \mathcal{D}_2 + \operatorname{span}\{C\}$ and A_0 is an improper affine hypersphere whose affine normal is parallel to C.

Similarly, B_0 is also an improper affine hypersphere. We have obtained (2) of Theorem 1.4.

Case I-(3) Now we have $\mu_2 = 0, \alpha = \lambda_2$, $\mu_3^2 + (\beta - \lambda_3)^2 \neq 0$. From (4.9), we see

$$\mu_3 = 2\lambda_2(\lambda_3 - \beta), \tag{4.16}$$

which implies that if $\mu_3 = 0$, then $\lambda_3 - \beta = 0$. It follows that Case I-(3)-(i) cannot occur.

We now assume Case I-(3)-(ii). Let β_1 be a nontrivial solution of the equation

$$\beta_1' = -2\lambda_2\beta_1. \tag{4.17}$$

Similarly as before, we can check that the vector field $C = \beta_1(\xi+2\lambda_2 T)$ satisfies $D_T C = D_X C = D_Y C = 0, \forall\, X \in \mathcal{D}_2, Y \in \mathcal{D}_3$, and thus is constant. Similarly, F is given by (4.12), where $\gamma_0 = 1$ and γ_1 and γ_2 satisfy (4.14) and (4.13), respectively. Also, for the same reasons as before the position vector field F is of the form (4.15). It follows from $\gamma_1' D_X A_1 = D_X T = (\lambda_2 - \alpha)X = 0$ that A_1 is constant. Moreover, combining $\gamma_1'(t) D_Y B_1 = D_Y T = (\lambda_3 - \beta)Y$ with

$$Y = D_Y F = D_Y B_0 + \gamma_1(t) D_Y B_1,$$

we have

$$D_Y B_0 = \left(\frac{\gamma_1'}{\lambda_3 - \beta} - \gamma_1\right) D_Y B_1.$$

By (4.16) a direct computation shows that $\frac{\gamma_1'}{\lambda_3-\beta} - \gamma_1 = c$ is a constant. Then

$$D_{X'} D_X A_0 = D_{X'} D_X F = \widehat{\nabla}^{\perp}_{X'} X + K^{\perp}(X', X) + h(X', X)\frac{C}{\beta_1},$$

so A_0 is an improper affine hypersphere whose affine normal is parallel to C.

On the other hand, we have

$$(c + \gamma_1(t)) D_{Y'} D_Y B_1 = D_{Y'} D_Y F = \widehat{\nabla}^{\perp}_{Y'} Y + K^{\perp}(Y', Y) + h(Y', Y)((\lambda_3 + \beta)T + \xi). \tag{4.18}$$

Set $\kappa = \frac{1}{\gamma_1'(\lambda_3+\beta-2\lambda_2)}$. It follows that $\kappa' + \kappa(\lambda_3 + \beta) = 0$, moreover

$$\begin{aligned}
&D_X(\kappa((\lambda_3 + \beta)T + \xi)) = 0 = D_X B_1,\\
&D_Y(\kappa((\lambda_3 + \beta)T + \xi)) = \frac{\lambda_3 - \beta}{\gamma_1'} Y = D_Y B_1,\\
&D_T(\kappa((\lambda_3 + \beta)T + \xi)) = 0 = D_T B_1,
\end{aligned}$$

so up to translation we can put $B_1 = \kappa((\lambda_3 + \beta)T + \xi)$. Then from (4.18) it follows that B_1 is a proper affine hypersphere. We have obtained (3) of Theorem 1.4.

We have completed the proof of Theorem 1.4 for case $\widehat{\nabla}_T T = 0$.

5 Affine Hypersurfaces with $\widehat{\nabla}_T T \neq 0$

In this section, we assume $\widehat{\nabla}_T T \neq 0$, and we prove that this case cannot occur. Recall that $\lambda_2 \neq \lambda_3$ and $\lambda_2\lambda_3 \neq 0$. We claim that $\lambda_1 \neq \lambda_2 + \lambda_3$. Otherwise, if $\lambda_1 = \lambda_2 + \lambda_3$, taking account also $\lambda_1 + n\lambda_2 + m\lambda_3 = 0$ we obtain $\lambda_2 = -\frac{m+1}{n+1}\lambda_3$. On the other hand, by Lemma 3.2 (1) we have $X_0(\lambda_3) = a\lambda_3$ and $X_0(\lambda_2) = \frac{n+2}{n} a\lambda_2$, respectively, which is impossible.

From Lemma 3.2 (3) and Lemma 3.3 (3), we have

$$\begin{aligned}
&a(\mu_1 - \mu_2) \sum_{i,j} h(K(X_0, X_i), X_j)^2 = 0,\\
&b(\mu_1 - \mu_3) \sum_{j,\ell} h(K(Y_0, Y_j), Y_\ell)^2 = 0.
\end{aligned} \tag{5.1}$$

If $\sum_{i,j} h(K(X_0, X_i), X_j)^2 = 0$, it follows that $K(X_0, X_i) = \lambda_2 h(X_0, X_i)T$. Then from the second equation of (3.14), we get $X_0(\lambda_2) = 0$, and from Lemma 3.2 (1) it follows $a\lambda_2 = 0$.

Similarly, $\sum_{j,\ell} h(K(Y_0, Y_j), Y_\ell)^2 = 0$ implies that $b\lambda_3 = 0$.

As $\widehat{\nabla}_T T = aX_0 + bY_0 \neq 0$, so we may, from now on take $a \neq 0$. Therefore, as $\lambda_2\lambda_3 \neq 0$ and thus $\sum_{i,j} h(K(X_0, X_i), X_j)^2 \neq 0$, by (5.1) it is sufficient to consider the following two subcases:

Case II-(1) $\mu_1 = \mu_2 \neq \mu_3$, $b = 0$; Case II-(2) $\mu_1 = \mu_2 = \mu_3$.

Case II-(1) Assume $\mu_1 = \mu_2 \neq \mu_3$ and $b = 0$. By Lemmas 3.4 and 3.5, we have

$$\begin{aligned} &h(\widehat{\nabla}_T X, Y) = h(\widehat{\nabla}_X T, Y) = h(\widehat{\nabla}_Y T, X) = 0, \\ &\beta\lambda_2 - \alpha\lambda_3 = \frac{1}{2}(\mu_1 - \mu_3). \end{aligned} \tag{5.2}$$

Also, the first equation of Lemma 3.3 (3) and the definition of β show

$$T(\mu_3) = (\mu_1 - \mu_3)(\lambda_3 - \beta). \tag{5.3}$$

Then the first equation of (3.11) and fourth equation of (3.13) imply that

$$h(\widehat{\nabla}_X Y, X') = 0, \quad h(\widehat{\nabla}_Y X, Y') = \frac{a\lambda_3}{\lambda_2 - \lambda_3} h(X, X_0) h(Y, Y'). \tag{5.4}$$

Also, the third equation (3.9) yields

$$h(\widehat{\nabla}_X X_1, T) = -\frac{1}{\lambda_1 - 2\lambda_2}(T(\lambda_2) h(X, X_1) - a h(K(X, X_1), X_0)). \tag{5.5}$$

Together with (5.4) and the third equation of (3.13), for unit vector field $Y \in \mathcal{D}_3$ we also get

$$\begin{aligned} &\widehat{\nabla}_T Y = \sum_{j=0}^{m-1} h(\widehat{\nabla}_T Y, Y_j) Y_j, \quad \widehat{\nabla}_Y T = -\beta Y, \\ &\widehat{\nabla}_Y Y = \beta T - \frac{a\lambda_3}{\lambda_2 - \lambda_3} X_0 + \sum_j h(\widehat{\nabla}_Y Y, Y_j) Y_j, \\ &\widehat{\nabla}_Y X_0 = \frac{a\lambda_3}{\lambda_2 - \lambda_3} Y + \sum_i h(\widehat{\nabla}_Y X_0, X_i) X_i, \quad \widehat{\nabla}_X Y = \sum_j h(\widehat{\nabla}_X Y, Y_j) Y_j. \end{aligned}$$

For unit vector field $Y \in \mathcal{D}_3$, the Codazzi equation

$$h((\widehat{\nabla}_X K)(Y, Y) - (\widehat{\nabla}_Y K)(X, Y), X') = \frac{1}{2}(\mu_1 - \mu_3) h(X, X')$$

gives further

$$\begin{aligned} \frac{1}{2}(\mu_1 - \mu_3) h(X, X') &= h(\widehat{\nabla}_X K(Y, Y), X') + h(K(X, \widehat{\nabla}_Y Y), X') \\ &= -h(K(Y, Y), \widehat{\nabla}_X X') + h(K(X, X'), \widehat{\nabla}_Y Y) \\ &= \frac{\lambda_3}{\lambda_1 - 2\lambda_2}(T(\lambda_2) h(X, X') - a h(K(X, X_0), X')) \\ &\quad - \frac{\lambda_2}{\lambda_1 - 2\lambda_3}\left(T(\lambda_3) + \frac{1}{2}(\mu_3 - \mu_1)\right) h(X, X') + \frac{a\lambda_3}{\lambda_3 - \lambda_2} h(K(X, X_0), X'). \end{aligned}$$

From this, we directly have

$$f h(X, X') + g h(K(X, X'), X_0) = 0, \tag{5.6}$$

where

$$\begin{aligned} &f = \frac{\lambda_3 T(\lambda_2)}{\lambda_1 - 2\lambda_2} - \frac{\lambda_2 T(\lambda_3)}{\lambda_1 - 2\lambda_3} - \left(\frac{\lambda_2}{\lambda_1 - 2\lambda_3} - 1\right)\frac{\mu_3 - \mu_1}{2}, \\ &g = \frac{a\lambda_3}{(\lambda_3 - \lambda_2)(\lambda_1 - 2\lambda_2)}(\lambda_1 - \lambda_2 - \lambda_3). \end{aligned}$$

Taking $X = X' = X_i$ in (5.6) and summing over the orthonormal basis of $\mathcal{D}_2$, we have $nf + gh(n\lambda_2 T, X_0) = 0$ and therefore we get $f = 0$.

Since $\sum_{i,j} h(K(X_0, X_i), X_j)^2 \neq 0$, we see that $h(K(X, X'), X_0)$ is not zero for some X, X'. We then get by (5.6) that $g = 0$ and hence we have $\lambda_3(\lambda_1 - \lambda_2 - \lambda_3) = 0$. This is a contradiction. We conclude that Case II-(1) cannot occur.

Case II-(2) In this case, the hypersurface is an affine hypersphere. Then the Codazzi equation

$$h((\widehat{\nabla}_Y K)(X, X'), Y') = h((\widehat{\nabla}_X K)(Y, X'), Y')$$

straightforwardly implies that

$$h(K(X, X'), \widehat{\nabla}_Y Y') = h(K(Y, Y'), \widehat{\nabla}_X X'). \tag{5.7}$$

Furthermore,

$$h(K(X, X'), \widehat{\nabla}_Y Y') = h(\widehat{\nabla}_Y Y', T) h(K(X, X'), T) + \sum_i h(\widehat{\nabla}_Y Y', X_i) h(K(X, X'), X_i).$$

The fourth equation of (3.13) yields

$$\begin{aligned} h(\widehat{\nabla}_Y Y', X_i) &= \frac{1}{\lambda_2 - \lambda_3} h(K(Y, Y'), \widehat{\nabla}_T X_i) \\ &= \frac{1}{\lambda_2 - \lambda_3} (h(\widehat{\nabla}_T X_i, T) h(K(Y, Y'), T) + \sum_j h(\widehat{\nabla}_T X_i, Y_j) h(K(Y, Y'), Y_j)), \end{aligned}$$

and, along with the third equation of (3.13), we obtain

$$\begin{aligned} & h(K(X, X'), \widehat{\nabla}_Y Y') \\ &= -\frac{\lambda_2}{\lambda_1 - 2\lambda_3} h(X, X') h(Y, Y') T(\lambda_3) + \frac{b\lambda_2}{\lambda_1 - 2\lambda_3} h(X, X') h(K(Y, Y_0), Y') \\ &\quad - \frac{a\lambda_3}{\lambda_2 - \lambda_3} h(Y, Y') h(K(X, X'), X_0) \\ &\quad + \frac{1}{\lambda_2 - \lambda_3} \sum_{i,j} h(K(X, X'), X_i) h(K(Y, Y'), Y_j) h(\widehat{\nabla}_T X_i, Y_j). \end{aligned} \tag{5.8}$$

Similarly, we obtain

$$\begin{aligned} & h(K(Y, Y'), \widehat{\nabla}_X X') \\ &= -\frac{\lambda_3}{\lambda_1 - 2\lambda_2} h(Y, Y') h(X, X') T(\lambda_2) + \frac{a\lambda_3}{\lambda_1 - 2\lambda_2} h(Y, Y') h(K(X, X_0), X') \\ &\quad - \frac{b\lambda_2}{\lambda_3 - \lambda_2} h(X, X') h(K(Y, Y'), Y_0) \\ &\quad + \frac{1}{\lambda_3 - \lambda_2} \sum_{i,j} h(K(Y, Y'), Y_j) h(K(X, X'), X_i) h(\widehat{\nabla}_T Y_j, X_i). \end{aligned} \tag{5.9}$$

Putting (5.8) and (5.9) into (5.7), we get

$$\begin{aligned} & h(X, X') h(Y, Y') \left(\frac{T(\lambda_2)\lambda_3}{\lambda_1 - 2\lambda_2} - \frac{T(\lambda_3)\lambda_2}{\lambda_1 - 2\lambda_3} \right) \\ &= (\lambda_1 - \lambda_2 - \lambda_3) \left(\frac{a\lambda_3 h(Y, Y') h(K(X, X'), X_0)}{(\lambda_2 - \lambda_3)(\lambda_1 - 2\lambda_2)} - \frac{b\lambda_2 h(X, X') h(K(Y, Y'), Y_0)}{(\lambda_3 - \lambda_2)(\lambda_1 - 2\lambda_3)} \right). \end{aligned}$$

In particular, if we take respectively, $X'\perp X$ and $X=X'=X_i$ for various i we get

$$\begin{aligned} &a\lambda_3 h(Y,Y')h(K(X,X'),X_0)\frac{\lambda_1-\lambda_2-\lambda_3}{(\lambda_2-\lambda_3)(\lambda_1-2\lambda_2)}=0,\\ &a\lambda_3 h(K(X_i,X_i),X_0)=a\lambda_3 h(K(X_j,X_j),X_0). \end{aligned} \tag{5.10}$$

Summarizing the second equation of (5.10) over an orthonormal basis, we get $nh(K(X,X),X_0) = 0$, which along with the first equation of (5.10) implies that $h(K(X,X'),X_0)=0$ and similarly $h(K(Y,Y'),Y_0)=0$.

Now, a straightforward computation from the Codazzi equation

$$h((\widehat{\nabla}_T K)(X_0,X_0),X_0)-h((\widehat{\nabla}_{X_0} K)(T,X_0),X_0)=0$$

shows that

$$\begin{aligned} 0 &= Th(K(X_0,X_0),X_0)+h(K(X_0,X_0),\widehat{\nabla}_{X_0}T)-h(K(X_0,X_0),\widehat{\nabla}_T X_0)\\ &\quad -X_0(\lambda_2)h(X_0,X_0)-2h(K(X_0,X_0),\widehat{\nabla}_T X_0)\\ &= -X_0(\lambda_2)-3h(K(X_0,X_0),\widehat{\nabla}_T X_0)\\ &= -X_0(\lambda_2)-3h(K(X_0,X_0),T)h(\widehat{\nabla}_T X_0,T). \end{aligned}$$

Thus $X_0(\lambda_2)=3\lambda_2 a$. By Lemma 3.2 (1), we have $X_0(\lambda_2)=\frac{n+2}{n}\lambda_2 a$. These give a contradiction $\lambda_2=0$, and this case is impossible.

Combining the conclusions in Sections 4 and 5, we complete the proof of Theorem 1.4.

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