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Multiple Positive Solutions for a Nonlinear Elliptic Equation Involving Hardy–Sobolev–Maz'ya Term

Shuang Jie PENG Jing YANG
School of Mathematics and Statistics, Central China Normal University, Wuhan 430079*, P. R. China*
E-mail: *sjpeng@mail.ccnu.edu.cn* *yyangecho@163.com*

Abstract In this paper, we study the existence and nonexistence of multiple positive solutions for the following problem involving Hardy–Sobolev–Maz'ya term:

$$-\Delta u-\lambda\frac{u}{|y|^2}=\frac{|u|^{p_t-1}u}{|y|^t}+\mu f(x),\quad x\in\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N(N\geq 2)$, $0\in\Omega$, $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$ and $p_t=\frac{N+2-2t}{N-2}$ $(0\leq t\leq 2)$. For $f(x)\in C^1(\overline{\Omega})\backslash\{0\}$, we show that there exists a constant $\mu^*>0$ such that the problem possesses at least two positive solutions if $\mu\in(0,\mu^*)$ and at least one positive solution if $\mu=\mu^*$. Furthermore, there are no positive solutions if $\mu\in(\mu^*,+\infty)$.

Keywords Hardy–Sobolev–Maz'ya inequality, Mountain Pass Lemma, positive solutions, subsolution and supersolution

MR(2010) Subject Classification 05B05, 05B25, 20B25

1 Introduction and Main Results

Let $p_t=\frac{N+2-2t}{N-2}$, $0\leq t<2$ and Ω be a bounded domain containing the origin in $\mathbb{R}^N(N\geq 2)$ with smooth boundary. We are concerned with the existence of multiple positive solutions for the following Hardy–Sobolev–Maz'ya equation:

$$\begin{cases}-\Delta u-\lambda\dfrac{u}{|y|^2}=\dfrac{|u|^{p_t-1}u}{|y|^t}+\mu f(x), & u>0, x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.1$_\mu$}$$

where $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $2\leq k<N$, $0\leq\lambda<\frac{(k-2)^2}{4}$ when $k>2$, $\lambda=0$ when $k=2$, $\mu>0$ is a parameter and $f(x)\in C^1(\bar{\Omega})\backslash\{0\}$. The main interest of this kind of problem, in addition to the presence of the singular potential $1/|y|^2$ related to Hardy's inequality, is the following well-known Hardy–Sobolev–Maz'ya inequality (see [6])

$$\left(\int_{\mathbb{R}^N}\frac{|u|^{p_t+1}}{|y|^t}dx\right)^{\frac{2}{p_t+1}}\leq(S_t^\lambda)^{-1}\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu\frac{u^2}{|y|^2}\right)dx,\quad\forall u\in C_0^\infty(\mathbb{R}^N),\tag{1.1}$$

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where $\lambda < (\frac{k-2}{2})^2$ and S_t^λ is the best constant. Owing to this inequality, it is easy to see that $(\int_\Omega(|\nabla u|^2 - \lambda\frac{u^2}{|y|^2})dx)^{\frac{1}{2}}$ is equivalent to $(\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$ in $H_0^1(\Omega)$.

From the mathematical point of view, the Hardy–Sobolev–Maz'ya term $\frac{u}{|y|^2}$ and $\frac{|u|^{p_t-1}u}{|y|^t}$ are critical: indeed they have the same homogeneity as the Laplacian and do not belong to the Katos class, hence they can cause the non-compactness of the corresponding functional. We also mention that the equation $(1.1)_\mu$ was proposed as a model for describing the dynamics of galaxies. Various equations similar to $(1.1)_\mu$ have been proposed to model several phenomena of interest in astrophysics. We recall here Eddington's and Matukuma's equations, which have attracted much interest in recent years (see [2, 22–24]). In [5] various astrophysical models are introduced and discussed, including some generalizations of Matukuma's equation.

After the pioneer work of Brezis and Nirenberg [7], there are many results for the problem $(1.1)_\mu$ with $\lambda = 0$ and $t = 0$, which is called Brizis–Nirenberg type problem. See for example [3, 11, 12, 26] and the references therein. Deng in [14] considered the problem $(1.1)_\mu$ with $0 \le f(x) \in C(\Omega) \cap C^{1+\alpha}(\bar{\Omega})$ and proved the existence of multiple positive solutions when $\lambda = 0, t = 0$. With similar argument, Deng and Peng in [15] obtained multiple positive solutions for $(1.1)_\mu$ with Neumann boundary condition in the case $\lambda = 0$, $t = 0$. For more other results, we refer the readers to [18–20] and the reference therein.

For the case $t > 0$, if $k = N$, there are a lot of results for the existence of positive and sign-changing solutions for $(1.1)_\mu$, we refer the readers to [8–10, 21] and the references therein. Chern and Lin in [13] studied the problem $(1.1)_\mu$ with $\mu = 0$ and $0 \in \partial\Omega$ and obtained the least energy solutions. If $2 \le k < N, \mu = 0$, Bhakta and Sandeep in [6] investigated the existence and non-existence of nontrivial solutions for $(1.1)_\mu$. If $\Omega = \mathbb{R}^N$, $\lambda = \mu = 0$, $t = 1$, the positive extremals of $(1.1)_\mu$ have been completely identified in [17]. And for more results, one can refer to [25, 27] and the reference therein.

In this paper, we intend to use the arguments similar to [14–16] to prove the existence of multiple positive solutions for $(1.1)_\mu$. We know that every nontrivial solution for $(1.1)_\mu$ is singular at $\{y = 0\}$ if $\lambda \ne 0$. So, unlike [14], we cannot apply directly the standard barrier method to obtain the existence of the minimal solution for $(1.1)_\mu$. To overcome this obstacle, we make a transformation of u by letting

$$v(x) = |y|^{-a}u(x) \quad \text{for } x \in \Omega, \tag{1.2}$$

where $a = \frac{\sqrt{(k-2)^2-4\lambda}-(k-2)}{2}$.

A straightforward computation yields

$$\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \frac{u^2}{|y|^2}dx = \int_\Omega |y|^{2a}|\nabla v|^2 dx.$$

Letting $H_0^1(\Omega, |y|^{2a}dx)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|v\|_a^2 := \int_\Omega |y|^{2a} \cdot|\nabla v|^2 dx$, then we can find

$$v \in H_0^1(\Omega, |y|^{2a}dx) \quad \text{if and only if} \quad u \in H_0^1(\Omega)$$

and from (1.1), we infer that

$$\left(\int_{\mathbb{R}^N} \frac{|y|^{a(p_t+1)}|v|^{p_t+1}}{|y|^t}dx\right)^{\frac{2}{p_t+1}} \le (S_{2a,t})^{-1}\int_{\mathbb{R}^N} |y|^{2a}|\nabla v|^2 dx \tag{1.3}$$

for some constant $S_{2a,t} > 0$.

Also by (1.2), we know that u is a positive solution of $(1.1)_\mu$ if and only if v is a positive solution of

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla v) = \dfrac{|y|^{a(p_t+1)}|v|^{p_t-1}v}{|y|^t} + \mu|y|^a f(x), & v>0,\ x\in\Omega, \\ v=0, & x\in\partial\Omega. \end{cases} \tag{$(1.2)_\mu$}$$

Applying the same argument with [14–16], to get the existence of multiple positive solutions for the problem $(1.2)_\mu$, we will first obtain a minimal positive solution to $(1.2)_\mu$ by the standard barrier method. Then using the minimal positive solution, we can transfer the problem $(1.2)_\mu$ to another equivalent problem and the second positive solution can be obtained by a variant of Mountain Pass Lemma (see [1]).

Notice that, unlike [14], in this paper, $f(x)$ is allowed to change sign, which makes the problem more interesting since we know that, in this case, the maximum principle fails. So it is obvious that to obtain positive solutions to $(1.1)_\mu$, we should impose suitable conditions on $f(x)$. In what follows, we assume that the following linear problem is solvable

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla\phi) = |y|^a f(x), & \phi\ge 0, x\in\Omega, \\ \phi=0, & x\in\partial\Omega, \end{cases} \tag{1.4}$$

and assume that $f(x)\in\mathcal{U}$ defined as

$$\mathcal{U} = \{f(x)\in C^1(\bar\Omega)\backslash\{0\} :\ (1.4) \text{ has positive solutions}\}.$$

Finally, from [6], we find that if $t = 2-\frac{N-2}{N-k+\sqrt{(k-2)^2-4\lambda}}$, the following form of the extremal function $U(y,z)$ achieves the best constant S_t^λ in (1.1):

$$U(y,z) = c(\lambda,N,k)\frac{|y|^{\frac{\sqrt{(k-2)^2-4\mu}-(k-2)}{2}}}{\big((1+|y|)^2+|z|^2\big)^{\frac{1}{p_t-1}}},$$

where $c(\lambda,N,k)$ is a constant depending on μ,N,k. Then it follows from (1.2) that

$$V(y,z) = |y|^{-a}U(y,z) = c(\lambda,N,k)\frac{1}{((1+|y|)^2+|z|^2)^{\frac{1}{p_t-1}}}$$

attains the best constant $S_{2a,t}$ in (1.3) if $t = 2-\frac{N-2}{N-k+\sqrt{(k-2)^2-4\lambda}}$ and $p_t = 1+\frac{2}{N-k+\sqrt{(k-2)^2-4\lambda}}$. Moreover, to guarantee $t>0$, throughout this paper, we set $N\ge 2k-2-2\sqrt{(k-2)^2-4\mu}$.

Our main results of this paper as follows.

Theorem 1.1 *For $p_t = 1+\frac{2}{N-k+\sqrt{(k-2)^2-4\lambda}}$, there exists a constant $\mu^*>0$ such that*

(i) *the problem $(1.2)_\mu$ possesses a minimal positive solution v_μ if $\mu\in(0,\mu^*]$ and there are no positive solutions for $(1.2)_\mu$ if $\mu>\mu^*$.*

(ii) *v_μ is increasing with respect to μ if $\mu\in(0,\mu^*)$ for all $x\in\Omega$.*

(iii) *v_μ is bounded uniformly in $H_0^1(\Omega,|y|^{2a}dx)$.*

(iv) *v_μ is bounded uniformly in Ω.*

Theorem 1.2 *If $N\ge\max\{k+1-\sqrt{(k-2)^2-4\lambda}, 2k-2-2\sqrt{(k-2)^2-4\lambda}\}$ and $\mu\in(0,\mu^*)$, then the problem $(1.2)_\mu$ possesses at least two positive solutions v_μ and V_μ satisfying $v_\mu<V_\mu$.*

As we have mentioned, in the present paper, we encounter some new difficulties mainly due to the singularity of solutions to $(1.1)_\mu$. To exclude the singularity, we need to estimate the precise singularity of the solutions which helps us carry out an important transformation (see (1.2)). Therefore, we can transform our problem $(1.1)_\mu$ into a new problem with divergent form. Moreover, in order to use the comparison theorem and obtain some needed estimates which are essential to verify a compactness condition in finding the second positive solution, we also need to prove that the solutions to the transformed problem are in $L^\infty(\Omega)$.

In this paper, H^* denotes the dual space of H, $L^p(\Omega, |y|^\beta dx)$ denotes the usual weighted L^p space with the norm $\|u\|_{p,\beta,\Omega} = (\int_\Omega |y|^\beta |u|^p dx)^{\frac{1}{p}}$ and we denote positive constants (possibly different) by C.

Our paper is organized as follows. In Section 2, we discuss the existence and nonexistence of the minimal solution for different $\mu > 0$ by means of the standard barrier method. In Section 3, we show the existence of the second positive solution for problem $(1.2)_\mu$ by using the Mountain Pass Lemma without (PS) condition.

2 The Existence of Minimal Solution

In order to obtain the positive minimal solution to $(1.2)_\mu$, we first establish the following strong maximum principle, which is similar to Proposition 3.1 in [4], so we omit the proof here.

Lemma 2.1 *Suppose that $\Omega \subset \mathbb{R}^N$, $\partial\Omega$ is continuous, $0 \in \Omega$. If $v \in C^1(\Omega\backslash\{y = 0\})$, $v \geq 0$ and $v \not\equiv 0$ satisfying*

$$-\mathrm{div}(|y|^{2a}\nabla v) \geq 0, \quad x \in \Omega,$$

then $v > 0$ in Ω.

Lemma 2.2 *Suppose that v is a positive solution to $(1.2)_\mu$, then v is bounded in Ω.*

Proof Recall that if v is a positive solution of $(1.2)_\mu$, then $u = |y|^a v$ is a positive solution of $(1.1)_\mu$. Define $\bar{v} = v + 1$, and for $m > 0$, let

$$v_m = \begin{cases} \bar{v}, & \text{if } v < m, \\ 1 + m, & \text{if } v \geq m. \end{cases} \tag{2.1}$$

Now for $s > 1$, take $\eta(x) = \bar{v}v_m^{2(s-1)} - 1$. It is easy to see that $\eta \in H_0^1(\Omega, |y|^{2a}dx)$. Since $v(x)$ solves $(1.2)_\mu$, we have

$$\begin{aligned}\int_\Omega |y|^{2a}\nabla v\nabla\eta dx &= \int_\Omega |y|^{2a}\big(v_m^{2(s-1)}|\nabla v|^2 + 2(s-1)v_m^{2(s-1)}|\nabla v_m|^2\big)dx \\ &= \int_\Omega \frac{|y|^{a(p_t+1)}|v|^{p_t-1}}{|y|^t} v\eta dx + \mu\int_\Omega |y|^a f(x)\eta dx.\end{aligned} \tag{2.2}$$

Setting $w_m = v_m^{s-1}\bar{v}$, together with (2.2), we can obtain

$$\begin{aligned}\int_\Omega |y|^{2a}|\nabla w_m|^2 dx &= \int_\Omega |y|^{2a}|(v_m^{s-1}\nabla v + (s-1)v_m^{s-1}\nabla v_m)|^2 dx \\ &\leq 2\bigg(\int_\Omega |y|^{2a}v_m^{2(s-1)}|\nabla v|^2 dx + (s-1)^2\int_\Omega |y|^{2a}v_m^{2(s-1)}|\nabla v_m)|^2 dx\bigg) \\ &\leq Cs\int_\Omega \frac{|y|^{a(p_t+1)}|v|^{p_t-1}}{|y|^t} v\eta dx + Cs\mu\int_\Omega |y|^a f(x)\eta dx.\end{aligned} \tag{2.3}$$

From [28, Proposition 2.2], $u \in L^p(\Omega, |y|^{-t}dx)$, $\forall p < \frac{2(k-t)}{k-2-\sqrt{(k-2)^2-4\lambda}}$. So choosing

$$\frac{N-t}{2-t} < q < \frac{(k-t)(N-2)}{(2-t)(k-2-\sqrt{(k-2)^2-4\lambda})}$$

such that

$$(p_t-1)q < \frac{2(k-t)}{k-2-\sqrt{(k-2)^2-4\lambda}} \quad \text{and} \quad 2 < \frac{2q}{q-1} < p_t+1,$$

we have, for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_\Omega \frac{|y|^{a(p_t+1)}|v|^{p_t-1}}{|y|^t} v\eta dx \\
&\quad \le \int_\Omega \frac{|y|^{a(p_t+1)}||y|^{-a}u|^{p_t-1}w_m^2}{|y|^t} dx \\
&\quad \le \|u\|^{p_t-1}_{p_t+1,t,\Omega} \||y|^a w_m\|^2_{\frac{2q}{q-1},t,\Omega} \\
&\quad \le C\big(\||y|^a w_m\|^\theta_{r_1,t,\Omega}\||y|^a w_m\|^{1-\theta}_{r_2,t,\Omega}\big)^2 \\
&\quad \le C\bigg(\varepsilon\big(\||y|^a w_m\|^\theta_{r_1,t,\Omega}\big)^{\alpha_1} + (\alpha_1\varepsilon)^{-\frac{\alpha_2}{\alpha_1}}\frac{1}{\alpha_2}\big(\||y|^a w_m\|^{1-\theta}_{r_2,t,\Omega}\big)^{\alpha_2}\bigg)^2,
\end{aligned} \tag{2.4}$$

where $\frac{\theta}{r_1} + \frac{1-\theta}{r_2} = \frac{q-1}{2q}$, $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$, $r_1 = p_t + 1$, $r_2 = 2$, $\frac{\theta}{r_1}\alpha_1 = \frac{1}{p_t+1}$, $\frac{1-\theta}{r_2}\alpha_2 = \frac{1}{2}$, and which implies that $\theta = \frac{N-t}{q(2-t)}$, $\alpha_1 = \frac{q(2-t)}{N-t}$, $\alpha_2 = \frac{q(2-t)}{q(2-t)-(N-t)}$. Thus, one has

$$\begin{aligned}
&\int_\Omega \frac{|y|^{a(p_t+1)}|v|^{p_t-1}}{|y|^t} v\eta dx \\
&\quad \le C\varepsilon^2\bigg(\int_\Omega \frac{|y|^{a(p_t+1)}|w_m|^{p_t+1}}{|y|^t}dx\bigg)^{\frac{2}{p_t+1}} + C\varepsilon^{-\frac{2(N-t)}{(2-t)q-(N-t)}}\int_\Omega \frac{|y|^{2a}|w_m|^2}{|y|^t}dx \\
&\quad \le C\varepsilon^2\int_\Omega |y|^{2a}|\nabla w_m|^2 dx + C\varepsilon^{-\frac{2(N-t)}{(2-t)q-(N-t)}}\int_\Omega \frac{|y|^{a(p_t+1)}|w_m|^2}{|y|^t}dx.
\end{aligned} \tag{2.5}$$

For $f(x) \in C^1(\bar\Omega)\backslash\{0\}$, proceeding as we prove (2.4) and (2.5), we can deduce

$$\begin{aligned}
\mu\int_\Omega |y|^a f(x)\eta dx &\le C\int_\Omega |y|^a w_m^2 dx \\
&\le C\bigg(\int_\Omega |y|^{-aq}dx\bigg)^{\frac{1}{q}}\bigg(\int_\Omega ||y|^a w_m|^{\frac{2q}{q-1}}dx\bigg)^{\frac{q-1}{q}} \\
&\le C\bigg(\int_\Omega \frac{||y|^a w_m|^{\frac{2q}{q-1}}}{|y|^t}|y|^t dx\bigg)^{\frac{q-1}{q}} \\
&\le C\bigg(\int_\Omega \frac{||y|^a w_m|^{\frac{2q}{q-1}}}{|y|^t}dx\bigg)^{\frac{q-1}{q}} \\
&\le C\varepsilon^2\int_\Omega |y|^{2a}|\nabla w_m|^2dx + C\varepsilon^{-\frac{2(N-t)}{(2-t)q-(N-t)}}\int_\Omega \frac{|y|^{a(p_t+1)}|w_m|^2}{|y|^t}dx,
\end{aligned} \tag{2.6}$$

since $-aq > -k$ and then $\big(\int_\Omega |y|^{-aq}dx\big)^{\frac{1}{q}} < +\infty$.

Then it follows from (2.3), (2.5) and (2.6) that

$$\int_\Omega |y|^{2a}|\nabla w_m|^2dx \le Cs\varepsilon^2\int_\Omega |y|^{2a}|\nabla w_m|^2dx + Cs\varepsilon^{-\frac{2(N-t)}{(2-t)q-(N-t)}}\int_\Omega \frac{|y|^{a(p_t+1)}w_m^2}{|y|^t}dx. \tag{2.7}$$

Taking $\varepsilon = \frac{1}{\sqrt{2Cs}}$ in (2.7), from (1.3), we infer that

$$\Big(\int_\Omega \frac{|y|^{a(p_t+1)} w_m^{p_t+1}}{|y|^t} dx\Big)^{\frac{2}{p_t+1}} \le C s^\alpha \int_\Omega \frac{|y|^{a(p_t+1)} w_m^2}{|y|^t} dx, \tag{2.8}$$

where $\alpha = \frac{(2-t)q}{(2-t)q-(N-t)}$.

Using $v_m \le \bar{v}$ and setting $\gamma = 2s$ and $\chi = \frac{N-t}{N-2}$, (2.8) becomes

$$\Big(\int_\Omega \frac{|y|^{a(p_t+1)} v_m^{\gamma\chi}}{|y|^t} dx\Big)^{\frac{1}{\gamma\chi}} \le (Cs^\alpha)^{\frac{1}{\gamma}} \Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma}}{|y|^t} dx\Big)^{\frac{1}{\gamma}}.$$

Passing to the limit as $m \to +\infty$, we get

$$\Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma\chi}}{|y|^t} dx\Big)^{\frac{1}{\gamma\chi}} \le (Cs^\alpha)^{\frac{1}{\gamma}} \Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma}}{|y|^t} dx\Big)^{\frac{1}{\gamma}}. \tag{2.9}$$

For $i = 0, 1, \ldots$, define $\gamma_0 = 2$ and $\gamma_i = 2\chi_i$. Then $\chi\gamma_i = \gamma_{i+1}$ and hence from (2.9), with $\gamma = \gamma_i$, we have

$$\Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma_{i+1}}}{|y|^t} dx\Big)^{\frac{1}{\gamma_{i+1}}} \le C^{\frac{1}{\gamma_i}} \Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma_i}}{|y|^t} dx\Big)^{\frac{1}{\gamma_i}},$$

which implies

$$\Big(\int_\Omega \bar{v}^{\gamma_{i+1}} dx\Big)^{\frac{1}{\gamma_{i+1}}} \le C\Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma_{i+1}}}{|y|^t} dx\Big)^{\frac{1}{\gamma_{i+1}}} \le C^{\sum \frac{1}{\gamma_i}} \Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^{\gamma_0}}{|y|^t} dx\Big)^{\frac{1}{\gamma_0}}.$$

Letting $i \to +\infty$, we get

$$\sup_\Omega \bar{v} \le C\Big(\int_\Omega \frac{|y|^{a(p_t+1)} \bar{v}^2}{|y|^t} dx\Big)^{\frac{1}{2}} < +\infty. \qquad \square$$

Now we give the following lemma which can be proved by using the standard variational method and then we omit its proof here.

Lemma 2.3 *The first eigenvalue η_1 of the following eigenvalue problem*

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla\varphi) = \eta_1 \dfrac{|y|^{a(p_t+1)}}{|y|^t}\varphi, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega \end{cases} \tag{2.10}$$

is positive and can be achieved by a function $\varphi_1(x) > 0$.

Lemma 2.4 *Let $\phi(x)$ be the nonnegative solution of* (1.4)*, and $v_1(x)$ be the solution of the problem*

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla v_1) = |y|^{a(p_t+1)-t}, & x \in \Omega, \\ v_1 = 0, & x \in \partial\Omega. \end{cases} \tag{2.11}$$

Then $\varphi_1(x)$, $\phi(x)$ and $v_1(x)$ are bounded uniformly in Ω.

Proof Similar to the proof of Lemma 2.2, it is clear that φ_1, $\phi(x)$ are bounded uniformly in Ω. Then we just need to show that $v_1(x)$ is bounded in Ω. To see this, letting $\eta(x)$ be mentioned as in Lemma 2.2, we have

$$\int_\Omega |y|^{a(p_t+1)-t}\eta dx \le C\int_\Omega \frac{|y|^{a(p_t+1)} w_m^2}{|y|^t} dx$$

$$\leq C\Big(\int_\Omega \frac{|y|^{a(p_t-1)q}}{|y|^t}dx\Big)^{\frac{1}{q}}\Big(\int_\Omega \frac{||y|^a w_m|^{\frac{2q}{q-1}}}{|y|^t}dx\Big)^{\frac{q-1}{q}}$$
$$\leq C\Big(\int_\Omega \frac{||y|^a w_m|^{\frac{2q}{q-1}}}{|y|^t}dx\Big)^{\frac{q-1}{q}},$$

since $a(p_t-1)q-t > a(p_t-1)\frac{(k-t)(N-2)}{(2-t)(k-2-\sqrt{(k-2)^2-4\lambda})}-t=-k$. Now proceeding as we done to prove (2.6), we can check that $v_1(x)$ is bounded in Ω. □

Lemma 2.5 *There exists a positive constant $\mu_*>0$ such that the problem $(1.2)_\mu$ has at least one positive solution for $\mu\in(0,\mu_*)$.*

Proof Let $\phi(x)$ be the nonnegative solution of (1.4) and set $\underline{V}=\mu\phi(x)$. Then we have

$$-\mathrm{div}(|y|^{2a}\nabla\underline{V})-|y|^{a(p_t+1)-t}|\underline{V}|^{p_t-1}\underline{V}-\mu|y|^a f(x)=-\mu^{p_t}|y|^{a(p_t+1)-t}\phi^{p_t}\leq 0.$$

Hence, $\underline{V}$ is a subsolution of $(1.2)_\mu$ for all $\mu>0$. We want to find a supersolution $W_0(x)$ of $(1.2)_\mu$ with $W_0(x)\geq\underline{V}$ in Ω. To this end, letting $v_1(x)$ be the solution of the problem (2.11), then by the strong maximum principle, $v_1>0$ in Ω. Setting $\bar V=Mv_1(x)$, we have

$$\begin{aligned}&-\mathrm{div}(|y|^{2a}\nabla\bar V)-|y|^{a(p_t+1)-t}\bar V^{p_t}-\mu|y|^a f(x)\\&=M|y|^{a(p_t+1)-t}-M^{p_t}|y|^{a(p_t+1)-t}v_1^{p_t}-\mu|y|^a f(x)\\&=|y|^a\big(M|y|^{ap_t-t}-M^{p_t}|y|^{ap_t-t}v_1^{p_t}-\mu f(x)\big).\end{aligned}$$

Choose $M=M_0>0$ satisfying

$$M_0\geq M_0^{p_t}\max_{\bar\Omega}|v_1|^{p_t}+M_0^{p_t}\max_{\bar\Omega}||y|^{t-ap_t}f(x)|.$$

So if $\mu\leq M_0^{p_t}$, we have

$$-\mathrm{div}(|y|^{2a}\nabla\bar V_0)-|y|^{a(p_t+1)-t}\bar V_0^{\,p_t}-\mu|y|^a f(x)\geq 0,$$

where $\bar V_0=M_0v_1(x)$. Notice that $\phi(x)$ and $v_1(x)$ satisfy respectively

$$-\mathrm{div}(|y|^{2a}\nabla\phi)=|y|^a f(x)$$

and

$$-\mathrm{div}(|y|^{2a}\nabla v_1)=|y|^a|y|^{ap_t-t}.$$

Since $ap_t-t<0$, there exists a constant $c>0$ such that $|y|^{ap_t-t}>cf(x)$ and then by the strong maximum principle, we have $v_1(x)\geq c\phi(x)$. Hence, we may choose μ_0 small enough such that $\mu_0\phi(x)\leq M_0v_1(x)$. By the iteration method, we can deduce that problem $(1.2)_\mu$ has a solution $v(x)$ satisfying $\underline{V}\leq v(x)\leq\bar V_0$ for all $\mu\leq\mu_0$. Moreover, $v>0$ in Ω. In fact, letting $W(x)=v(x)-\underline{V}(x)\not\equiv 0$, then

$$\begin{cases}-\mathrm{div}(|y|^{2a}\nabla W)=\dfrac{|y|^{a(p_t+1)}v^{p_t}}{|y|^t}\geq 0,\\ W|_{\partial\Omega}=0.\end{cases}$$

Hence, the strong maximum principle implies that $v>\underline{V}(x)\geq 0$ in Ω. Now, define

$$\mu_*=\sup\{\mu_0>0:\ \text{the problem }\ (1.2)_\mu\text{ has at least one solution for each }\mu\in(0,\mu_0)\}.$$

As a result, we complete the proof. □

The following lemma shows that μ_* is bounded.

Lemma 2.6 *There exists a constant $C > 0$ such that $(1.2)_\mu$ has no positive solutions if $\mu > C$.*

Proof Let η_1, φ_1 be mentioned as in Lemma 2.3. Assume that v is a positive solution of $(1.2)_\mu$, then we have

$$\mu \int_\Omega |y|^a f(x)\varphi_1 dx = \int_\Omega |y|^{a(p_t+1)-t}\big(\eta_1 v - v^{p_t}\big)\varphi_1 dx.$$

From $p_t > 1$, we can choose $C_1 > 0$ such that $C_1 \geq \eta_1 v - v^{p_t}$ for all $v > 0$. Thus,

$$\mu \int_\Omega |y|^a f(x)\varphi_1 dx \leq C_1 \int_\Omega |y|^{a(p_t+1)-t}\varphi_1 dx < +\infty,$$

since $a(p_t + 1) - t > -k$. So taking $C = \frac{C_1 \int_\Omega |y|^{a(p_t+1)-t}\varphi_1 dx}{\int_\Omega |y|^a f(x)\varphi_1 dx} < +\infty$, we obtain that $\mu < C$. This completes the proof. □

Lemma 2.7 *Problem $(1.2)_\mu$ has a positive solution for all $\mu \in (0, \Lambda)$ if it has a positive solution when $\mu = \Lambda > 0$.*

Proof Let $\widetilde{v}(x)$ be a positive solution of $(1.2)_\mu$ for $\mu = \Lambda > 0$. Set $\widetilde{v} = \Lambda w_1$. Then w_1 satisfies

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla w_1) = \Lambda^{p_t-1}\dfrac{|y|^{a(p_t+1)}w_1^{p_t}}{|y|^t} + |y|^a f(x), & x \in \Omega, \\ w_1|_{\partial\Omega} = 0, \quad w_1 > 0 \text{ in } \Omega. \end{cases}$$

Hence, for any $0 < \mu < \Lambda$, w_1 is a supersolution of the problem

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla w) = \mu^{p_t-1}\dfrac{|y|^{a(p_t+1)}|w|^{p_t-1}w}{|y|^t} + |y|^a f(x), & x \in \Omega, \\ w|_{\partial\Omega} = 0, \quad w > 0 \text{ in } \Omega. \end{cases} \tag{2.12}$$

On the other hand, the problem (1.4) has a nonnegative solution $\phi(x)$ satisfying

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla \phi) = |y|^a f(x) \leq \mu^{p_t-1}\dfrac{|y|^{a(p_t+1)}\phi^{p_t}}{|y|^t} + |y|^a f(x), & x \in \Omega, \\ \phi|_{\partial\Omega} = 0, \quad \phi \geq 0 \ \text{ in } \Omega. \end{cases}$$

So we know that $\phi(x)$ is a subsolution of (2.12). By the comparison principle, we have

$$0 \leq \phi \leq w_1 \quad \text{for all } x \in \Omega.$$

Thus, there exists a positive solution w_μ of (2.12) for all $\mu \in (0, \Lambda)$. Then $v = \mu w_\mu$ is a positive solution of the problem $(1.2)_\mu$ for all $\mu \in (0, \Lambda)$. □

Theorem 2.8 *There exists a positive constant $\mu^* < +\infty$ such that $(1.2)_\mu$ has at least one positive solution if $\mu \in (0, \mu^*)$ and $(1.2)_\mu$ has no positive solutions if $\mu > \mu^*$.*

Proof Lemma 2.5 implies that there exists a positive constant μ_* such that the problem $(1.2)_\mu$ has at least one positive solution for $\mu \in (0, \mu_*)$. Let

$$\mu^* = \sup\{\mu > 0 : \ (1.2)_\mu \text{ has a positive solution}\}.$$

It follows from Lemmas 2.6 and 2.7 that $0 < \mu^* < +\infty$ and for all $\mu \in (0, \mu^*)$, $(1.2)_\mu$ has at least one positive solution. Moreover, the definition of μ^* implies that $(1.2)_\mu$ has no solution when $\mu > \mu^*$. □

Proposition 2.9 *Assume that $\phi(x)$ is the nonnegative solution of (1.4), then the problem $(1.2)_\mu$ has at least one minimal positive solution v_μ for $\mu \in (0, \mu^*)$ satisfying*

(i) $v_\mu > \mu\phi(x)$ *for all* $x \in \Omega$,

(ii) v_μ *is increasing with respect to* μ *for all* $x \in \Omega$.

Proof Set $v_\mu(x) = \mu w_\mu(x)$. Then w_μ satisfies

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla w_\mu) = \mu^{p_t-1}\dfrac{|y|^{a(p_t+1)}w_\mu^{p_t}}{|y|^t} + |y|^a f(x), \quad x \in \Omega, \\ w_\mu|_{\partial\Omega} = 0, \quad w_\mu \geq 0 \,\text{in}\, \Omega. \end{cases}$$

Setting $\Phi = w_\mu - \phi$, we have

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla\Phi) = \mu^{p_t-1}\dfrac{|y|^{a(p_t+1)}w_\mu^{p_t}}{|y|^t} \geq 0, \quad x \in \Omega, \\ \Phi|_{\partial\Omega} = 0. \end{cases}$$

Then by the strong maximum principle, $\Phi = w_\mu - \phi > 0$ in Ω.

Notice that $\mu\phi$ is a subsolution of $(1.2)_\mu$, and all nonnegative supersolution of $(1.2)_\mu$ must be larger than or equal to $\mu\phi$. Hence, we can find a minimal solution of $(1.2)_\mu$ by a monotone iteration starting with $\mu\phi$.

To prove the second statement, we first let $\mu_1, \mu_2 \in (0, \mu^*)$ with $\mu_1 < \mu_2$ and the corresponding minimal solutions of $(1.2)_\mu$ are v_{μ_1} and v_{μ_1}. Set $v_{\mu_1} = \mu_1 w_{\mu_1}$ and $v_{\mu_2} = \mu_2 w_{\mu_2}$. Then w_{μ_2} must be a supersolution and $\phi(x)$ be a subsolution of (2.12) when $\mu = \mu_1$. By the monotone iteration again, we have $\phi \leq w_{\mu_1} \leq w_{\mu_2}$, that is, $v_{\mu_1} < v_{\mu_2}$. □

Let v_μ be the minimal positive solution of $(1.2)_\mu$ given by Proposition 2.9. Now consider the following eigenvalue problem

$$\begin{cases} -\mathrm{div}(|y|^{2a}\nabla\Psi) = d\dfrac{p_t|y|^{a(p_t+1)}v_\mu^{p_t-1}}{|y|^t}\Psi, \quad \text{in}\ \Omega, \\ \Psi_{\partial\Omega} = 0. \end{cases} \tag{2.13}$$

Then we have

Lemma 2.10 *The first eigenvalue*

$$d := \inf\left\{\int_\Omega |y|^{2a}|\nabla\Psi|^2 : \Psi \in H_0^1(\Omega, |y|^{2a}dx),\ \int_\Omega p_t|y|^{a(p_t+1)-t}v_\mu^{p_t-1}\Psi^2 = 1\right\} \tag{2.14}$$

of (2.13) *can be obtained by a function $\Psi_1 > 0$ if $\mu \in (0, \mu^*)$. Moreover, $d > 1$.*

Proof We can prove that the minimal problem (2.14) can be achieved by a function Ψ_1 if $\mu \in (0, \mu^*)$ by standard variational argument. So we only need to show that $d > 1$. Suppose that $\mu^* > \bar\mu > \mu > 0$, $v_{\bar\mu}$, v_μ are the minimal positive solution of $(1.2)_{\bar\mu}$, $(1.2)_\mu$ respectively. Then we have $v_{\bar\mu} > v_\mu > 0$ and

$$\begin{aligned} &-\mathrm{div}(|y|^{2a}\nabla(v_{\bar\mu} - v_\mu)) - (\bar\mu - \mu)|y|^a f(x) \\ &= \frac{|y|^{a(p_t+1)}v_{\bar\mu}^{p_t}}{|y|^t} - \frac{|y|^{a(p_t+1)}v_\mu^{p_t}}{|y|^t} \\ &= \frac{|y|^{a(p_t+1)}}{|y|^t}\Big(p_t v_\mu^{p_t-1}(v_{\bar\mu} - v_\mu) + \frac{1}{2}p_t(p_t-1)[v_\mu + \theta(v_{\bar\mu} - v_\mu)]^{p_t-2}(v_{\bar\mu} - v_\mu)^2\Big) \end{aligned}$$

$$> \frac{|y|^{a(p_t+1)}}{|y|^t} p_t v_\mu^{p_t-1}(v_{\bar\mu} - v_\mu),$$

where $0 < \theta < 1$. Note that from (1.4),

$$\int_\Omega |y|^a f(x)\Psi_1 dx = \int_\Omega -\mathrm{div}(|y|^{2a}\nabla\phi)\Psi_1 dx = \int_\Omega -\mathrm{div}(|y|^{2a}\nabla\Psi_1)\phi dx$$
$$= \int_\Omega d\frac{p_t|y|^{a(p_t+1)}v_\mu^{p_t-1}}{|y|^t}\Psi_1\phi dx \geq 0,$$

which implies

$$\int_\Omega (-\mathrm{div}|y|^{2a}\nabla\Psi_1)(v_{\bar\mu} - v_\mu)dx > \int_\Omega \frac{p_t|y|^{a(p_t+1)}}{|y|^t} v_\mu^{p_t-1}(v_{\bar\mu} - v_\mu)\Psi_1 dx.$$

Hence,

$$\int_\Omega d\frac{p_t|y|^{a(p_t+1)}v_\mu^{p_t-1}}{|y|^t}(v_{\bar\mu} - v_\mu)\Psi_1 dx > \int_\Omega \frac{p_t|y|^{a(p_t+1)}}{|y|^t} v_\mu^{p_t-1}(v_{\bar\mu} - v_\mu)\Psi_1 dx,$$

which implies $d > 1$. □

Lemma 2.11 $(1.2)_{\mu^*}$ *possesses a minimal positive solution.*

Proof For any $\mu \in (0, \mu^*)$, suppose that v_μ is the minimal positive solution of $(1.2)_\mu$. Since

$$\int_\Omega |y|^{2a}|\nabla v_\mu|^2 dx = \int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx + \mu\int_\Omega |y|^a f(x)v_\mu dx. \tag{2.15}$$

It follows from Lemma 2.10 that

$$\int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx + \mu\int_\Omega |y|^a f(x)v_\mu dx = \int_\Omega |y|^{2a}|\nabla v_\mu|^2 dx$$
$$\geq p_t\int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx.$$

Thus, we have

$$\begin{aligned}
&(p_t - 1)\int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx \\
&\leq \mu\int_\Omega |y|^a f(x)v_\mu dx \\
&\leq \mu^*\left(\delta\int_\Omega \left||y|^{\frac{a(p_t+1)-t}{p_t+1}}v_\mu\right|^{p_t+1} dx + C_\delta\int_\Omega \left||y|^{\frac{t}{p_t+1}}f\right|^{\frac{p_t+1}{p_t}} dx\right) \\
&\leq C_1\mu^*\delta\int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx + C_2C_\delta\mu^* dx,
\end{aligned} \tag{2.16}$$

where δ is a small positive constant satisfying $p_t - 1 - C_1\mu^*\delta > 0$ and C_δ is a positive constant only dependent on δ. So there exists a positive constant C_3 independent of μ such that

$$\int_\Omega \frac{|y|^{a(p_t+1)}v_\mu^{p_t+1}}{|y|^t}dx \leq C_3.$$

So (2.15) and (2.16) yield

$$\int_\Omega |y|^{2a}|\nabla v_\mu|^2 dx \leq C \tag{2.17}$$

for some positive constant C independent of μ.

Now, we will prove the existence of the minimal positive solution of $(1.2)_{\mu^*}$. Suppose that $\{\mu_j\}_{j\geq 1}$ is an increasing sequence in $(0,\mu^*)$ with $\lim_{j\to+\infty}\mu_j = \mu^*$ and the corresponding sequence of solutions is $\{v_{\mu_j}\}_{j\geq 1}\subset H_0^1(\Omega,|y|^{2a}dx)$. From (2.17), we can choose a subsequence still denoted by $\{v_{\mu_j}\}$ such that

$$\begin{aligned} v_{\mu_j} &\rightharpoonup \bar{v}, \quad \text{in } H_0^1(\Omega,|y|^{2a}dx),\\ v_{\mu_j} &\rightharpoonup \bar{v}, \quad \text{in } L^{p_t+1}(\Omega,|y|^{a(p_t+1)-t}dx) \end{aligned}$$

and

$$v_{\mu_j}^{p_t} \rightharpoonup \bar{v}^{p_t}, \quad \text{in } (L^{\frac{p_t+1}{p_t}}(\Omega,|y|^{a(p_t+1)-t}dx))^*,$$

as $j\to+\infty$. Thus, for any $\bar{\varphi}\in C_0^\infty(\Omega)$,

$$\begin{aligned} 0 &= \int_\Omega |y|^{2a}\nabla v_{\mu_j}\nabla\bar{\varphi} - \int_\Omega \frac{|y|^{a(p_t+1)}v_{\mu_j}^{p_t}}{|y|^t}\bar{\varphi} - \mu_j\int_\Omega |y|^a f(x)\bar{\varphi}\\ &\to \int_\Omega |y|^{2a}\nabla\bar{v}\nabla\bar{\varphi} - \int_\Omega \frac{|y|^{a(p_t+1)}\bar{v}^{p_t}}{|y|^t}\bar{\varphi} - \mu^*\int_\Omega |y|^a f(x)\bar{\varphi}, \end{aligned}$$

as $j\to+\infty$. Hence, $\bar{v}$ is a solution of $(1.2)_{\mu^*}$. Note that $\mu\phi$ is always a subsolution of $(1.2)_{\mu^*}$. Using the methods of monotone iteration and strong maximum principle, we conclude that there exists a minimal positive solution v_{μ^*} of $(1.2)_{\mu^*}$. □

3 Existence of the Second Solution

Let v_μ be the minimal positive solution of $(1.2)_\mu$ for $\mu\in(0,\mu^*]$. To find the second positive solution of $(1.2)_\mu$, we study the following problem

$$\begin{cases} -\text{div}(|y|^{2a}\nabla w) = \dfrac{(w+v_\mu)^{p_t}-v_\mu^{p_t}}{|y|^{t-a(p_t+1)}}, & w>0, x\in\Omega,\\ w=0, & x\in\partial\Omega. \end{cases} \tag{$3.1)_\mu$$

Obviously, we can get another positive solution $V_\mu = w+v_\mu$ for the problem $(1.2)_\mu$ if $(3.1)_\mu$ has a positive solution w. Now we prove that $(3.1)_\mu$ has a positive solution for $\mu\in(0,\mu^*)$ by using a variant of the Mountain Pass Lemma. To this end, set

$$h(x,w) = \frac{(w+v_\mu)^{p_t}-w^{p_t}-v_\mu^{p_t}}{|y|^{t-a(p_t+1)}}$$

and

$$\alpha(x) = \frac{p_t v_\mu^{p_t-1}}{|y|^{t-a(p_t+1)}}.$$

The values of $h(x,w)$ for $w<0$ are irrelevant and we may define

$$h(x,w) = \alpha(x)w \quad \text{for } w<0, x\in\Omega.$$

We will prove that

$$\begin{cases} -\text{div}(|y|^{2a}\nabla w) = h(x,w) + \dfrac{w^{p_t}}{|y|^{t-a(p_t+1)}}, & w>0, x\in\Omega,\\ w=0, & x\in\partial\Omega \end{cases} \tag{$3.2)_\mu$$

possesses a positive solution.

We first give a compact embedding result whose proof is standard and hence omitted.

Lemma 3.1 *The embedding*

$$H_0^1(\Omega, |y|^{2a}dx) \hookrightarrow L^q(\Omega, |y|^{a(p_t+1)-t}dx)$$

is compact if $0 < q < p_t + 1$.

Define the energy functional corresponding to $(3.2)_\mu$ by

$$J(w) = \frac{1}{2}\int_\Omega |y|^{2a}|\nabla w|^2 dx - \frac{1}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w^+)^{p_t+1}dx - \int_\Omega H(x,w)dx,$$

where $H(x,w) = \int_0^w h(x,s)ds$. By Lemma 3.1, this functional is well defined in $H_0^1(\Omega, |y|^{2a}dx)$ and $J \in C^1(H_0^1(\Omega, |y|^{2a}dx), \mathbb{R})$. Hence, if $w \in H_0^1(\Omega, |y|^{2a}dx)$ is a critical point of $J(w)$, then

$$\int_\Omega |y|^{2a}|\nabla w^-|^2 dx - \int_\Omega \alpha(x)(w^-)^2 dx = \langle J'(w), w^-\rangle = 0,$$

which combined with Lemma 2.10 implies that $w^- = 0$ and hence the strong maximum principle yields that w is a positive solution of $(3.1)_\mu$. Thus in order to obtain a positive solution of $(3.1)_\mu$, it suffices to find a nonzero critical point of $J(w)$.

Set

$$c_* = \inf_{\gamma\in\Gamma}\max_{t\in(0,1)} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega, |y|^{2a}dx)), \gamma(0) = 0, J(\gamma(1)) < 0\}$. It is clear that there exists a constant t_0 large enough such that $J(t_0 w) < 0$ for any $w \in H_0^1(\Omega, |y|^{2a}dx)$ and $w^+ \not\equiv 0$. We claim that c_* is larger than zero. In fact,

$$\begin{aligned} J(w) &= \int_\Omega \left(\frac{1}{2}|y|^{2a}|\nabla w|^2 - \frac{1}{p_t+1}|y|^{a(p_t+1)-t}(w^+)^{p_t+1} - H(x,w)\right)dx \\ &= \frac{1}{2}\int_\Omega (|y|^{2a}|\nabla w|^2 - \alpha(x)w^2)dx - \frac{1}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w^+)^{p_t+1}dx \\ &\quad - \int_\Omega \left(H(x,w) - \frac{1}{2}\alpha(x)w^2\right)dx. \end{aligned}$$

By Lemma 2.10,

$$\int_\Omega (|y|^{2a}|\nabla w|^2 - \alpha(x)w^2) \ge \left(1 - \frac{1}{d}\right)\int_\Omega |y|^{2a}|\nabla w|^2 \ge \beta_1\|w\|_a^2 \tag{3.1}$$

for some positive constant β_1. Note that

$$\lim_{w\to\infty} \frac{h(x,w)}{|y|^{a(p_t+1)-t}w^{p_t}} = 0, \quad \text{uniformly for } x \in \Omega \tag{3.2}$$

and

$$\lim_{w\to 0} \frac{h(x,w)}{\alpha(x)w} = 1, \quad \text{uniformly for } x \in \Omega. \tag{3.3}$$

Then for any $\varepsilon > 0$, we have

$$\int_\Omega \left(H(x,w) - \frac{1}{2}\alpha(x)w^2\right)dx \le \frac{\varepsilon}{2}\int_\Omega \alpha(x)w^2 dx + \frac{(C_\varepsilon+1)}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}|w|^{p_t+1}dx,$$

which implies

$$J(w) \ge \frac{1}{2}\beta_1\|w\|_a^2 - \frac{1}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w^+)^{p_t+1}dx - \frac{C\varepsilon}{2}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}w^2 dx$$

$$- \frac{C_\varepsilon}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}|w|^{p_t+1}dx$$
$$\geq \frac{1}{2}\beta_1\|w\|_a^2 - \frac{C\varepsilon}{2}\|w\|_a^2 - \frac{C_\varepsilon}{p_t+1}\|w\|_a^{p_t+1}.$$

Setting $\varepsilon = \frac{\beta_1}{2C}$, we have

$$J(w) \geq \frac{1}{4}\beta_1\|w\|_a^2 - C\|w\|_a^{p_t+1}.$$

So we can choose $\rho > 0$ small such that

$$J(w)\big|_{\partial B_\rho(0)} \geq c > 0.$$

Therefore, $c_* > 0$ holds.

The following lemma shows that J satisfies the Palais–Smale condition $(\mathrm{PS})_c$ at level c provided that c is in a suitable range.

Lemma 3.2 *Suppose that $\mu \in (0,\mu^*)$, $c_* < (\frac{1}{2}-\frac{1}{p_t+1})(S_{2a,t})^{\frac{p_t+1}{p_t-1}}$. Then $(3.1)_\mu$ possesses a positive solution.*

Proof Using the Mountain Pass Lemma without (PS) condition (see [1]), there exists a sequence of $\{w_n\}_{n\geq 1} \subset H_0^1(\Omega, |y|^{2a}dx)$ with $J(w_n) \to c_*$ in $H_0^1(\Omega,|y|^{2a}dx)$ and $J'(w_n) \to 0$ in $(H_0^1(\Omega,|y|^{2a}dx))^*$. Then we deduce

$$\frac{1}{2}\int_\Omega |y|^{2a}|\nabla w_n|^2dx - \frac{1}{p_t+1}\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w_n^+)^{p_t+1}dx - \int_\Omega H(x,w_n)dx = c_* + o(1) \tag{3.4}$$

and

$$\int_\Omega |y|^{2a}|\nabla w_n|^2dx = \int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w_n^+)^{p_t+1}dx + \int_\Omega h(x,w_n)w_ndx + o(1)\|w_n\|_a. \tag{3.5}$$

So we have

$$\left(\frac{1}{2}-\frac{1}{p_t+1}\right)\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w_n^+)^{p_t+1}dx$$
$$\leq \int_\Omega \left(H(x,w_n) - \frac{1}{2}h(x,w_n)w_n\right)dx + c_* + o(1)(1+\|w_n\|_a).$$

Now we shall prove that $\{w_n\}$ is bounded. It follows from (3.2) that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|h(x,w)w| \leq \varepsilon|y|^{a(p_t+1)-t}w^{p_t+1} + C_\varepsilon|y|^{a(p_t+1)-t}$$

for all $x \in \Omega$ and $w \geq 0$. Then we have

$$\left|\int_\Omega h(x,w_n)w_ndx\right| \leq \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}w_n^{p_t+1}dx + C_\varepsilon\int_\Omega |y|^{a(p_t+1)-t}dx$$
$$\leq \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}w_n^{p_t+1}dx + C_\varepsilon$$

and

$$\left|\int_\Omega H(x,w_n)dx\right| \leq \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}w_n^{p_t+1}dx + C_\varepsilon.$$

Taking ε suitably, we can deduce

$$\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w_n^+)^{p_t+1}dx \leq C + o(1)\|w_n\|_a.$$

Combining the above estimates together, we see that $\{w_n\}$ is bounded in $H_0^1(\Omega,|y|^{2a}dx)$. Up to a subsequence, we may assume that as $n\to+\infty$,

$$\begin{aligned}
&w_n \rightharpoonup w, \quad \text{in } H_0^1(\Omega,|y|^{2a}dx),\\
&w_n \to w, \quad \text{in } L^2(\Omega,|y|^{a(p_t+1)-t}dx),\\
&w_n \to w, \quad \text{a.e. } \Omega,\\
&(w_n^+)^{p_t} \rightharpoonup (w^+)^{p_t}, \quad \text{in } (L^{\frac{p_t+1}{p_t}}(\Omega,|y|^{a(p_t+1)-t}dx))^*
\end{aligned}$$

and

$$h(x,w_n) \rightharpoonup h(x,w), \quad \text{in } (L^{p_t+1}(\Omega,|y|^{a(p_t+1)-t}dx))^*.$$

Therefore, we know that w solves

$$-\mathrm{div}(|y|^{2a}\nabla w) = \frac{(w^+)^{p_t}}{|y|^{t-a(p_t+1)}} + h(x,w).$$

If $w\not\equiv 0$, our result holds true. So now it suffices to prove that $w\not\equiv 0$. Suppose that $w\equiv 0$. Using (3.2) and (3.3), for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$\begin{aligned}
\left|\int_\Omega h(x,w_n)w_n dx\right| &\le \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t} w_n^{p_t+1}dx + C_\varepsilon\int_\Omega \alpha(x)w_n^2 dx\\
&\le \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t} w_n^{p_t+1}dx + C_\varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t} w_n^2 dx
\end{aligned}$$

and

$$\left|\int_\Omega H(x,w_n)dx\right| \le \varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t} w_n^{p_t+1}dx + C_\varepsilon\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t} w_n^2 dx.$$

Then using Lemma 3.1, we see that as $n\to+\infty$,

$$\int_\Omega h(x,w_n)w_n dx \to 0 \tag{3.6}$$

and

$$\int_\Omega H(x,w_n)dx \to 0. \tag{3.7}$$

By extracting a subsequence of $\{w_n\}$, we may assume that

$$\int_\Omega |y|^{2a}|\nabla w_n|^2 dx \to l$$

for some constant $l\ge 0$. Combining (3.5) and (3.6), we deduce

$$\int_\Omega \frac{|y|^{a(p_t+1)}}{|y|^t}(w_n)^{p_t+1}dx \to l.$$

Hence, by (3.4),

$$c_* = \left(\frac{1}{2}-\frac{1}{p_t+1}\right)l \ge \left(\frac{1}{2}-\frac{1}{p_t+1}\right)(S_{2a,t})^{\frac{p_t+1}{p_t-1}},$$

which yields a contradiction to $c_* < (\frac{1}{2}-\frac{1}{p_t+1})(S_{2a,t})^{\frac{p_t+1}{p_t-1}}$. □

Set

$$c^* = \inf_{w\in H_0^1(\Omega,|y|^{2a}dx)}\Big\{\sup_{s>0} J(sw); w\ge 0, w\not\equiv 0\Big\}.$$

Then $c_* \le c^*$. So, next we will verify that there exists a function $w \ge 0$ such that

$$c^* \le \sup_{s>0} J(sw) < \Big(\frac{1}{2} - \frac{1}{p_t+1}\Big)(S_{2a,t})^{\frac{p_t+1}{p_t-1}}. \tag{3.8}$$

Set

$$V_\varepsilon(x) := \varepsilon^{\frac{2-N-2a}{2}} V\Big(\frac{x}{\varepsilon}\Big),$$

where $V(x)$ is given in the introduction. Then $V_\varepsilon(x)$ also achieves $S_{2a,t}$ with

$$\int_{\mathbb{R}^N} |y|^{2a}|\nabla V_\varepsilon|^2 dx = \int_{\mathbb{R}^N} \frac{|y|^{a(p_t+1)}}{|y|^t} V_\varepsilon^{p_t+1} dx = (S_{2a,t})^{\frac{p_t+1}{p_t-1}} := K_1.$$

For $\theta > 0$ and $\theta < \max_{x\in\Omega} v_\mu(x)$, suppose that $\mathcal{A} := \{x \in \Omega : v_\mu > \theta\} \ne \emptyset$. Choosing $\rho > 0$ small enough such that $B_{2\rho}(0) \subset \mathcal{A} \subseteq \Omega \subset B_R(0)$. Let $v_\varepsilon(x) := \psi(x)V_\varepsilon(x)$, where $\psi \in C_0^\infty(\Omega)$ is a cut-off function satisfying $\psi = 1$ on $B_\rho(0)$ and $\psi = 0$ on $\Omega\backslash B_{2\rho}(0)$. Then we have the following estimates.

Lemma 3.3 *If $N \ge k+1-\sqrt{(k-2)^2-4\lambda} := \beta$ and ε is sufficiently small, then*

$$K_1(\varepsilon) := \int_\Omega \frac{|y|^{a(p_t+1)}|v_\varepsilon|^{p_t+1}}{|y|^t} dx = K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1}) \tag{3.9}$$

and

$$K_2(\varepsilon) := \int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2 dx = K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}). \tag{3.10}$$

Moreover,

$$K_3(\varepsilon) := \int_\Omega \frac{|y|^{a(p_t+1)}|v_\varepsilon|^2}{|y|^t} dx \le \begin{cases} C\varepsilon, & N > \beta, \\ C\varepsilon|\ln\varepsilon|, & N = \beta. \end{cases} \tag{3.11}$$

Proof Note that

$$\int_\Omega \frac{|y|^{a(p_t+1)}|v_\varepsilon|^{p_t+1}}{|y|^t} dx = \int_{\mathbb{R}^N} \frac{|y|^{a(p_t+1)}|V_\varepsilon|^{p_t+1}}{|y|^t} dx - \int_{\mathbb{R}^N\backslash B_\rho(0)} \frac{|y|^{a(p_t+1)}|V_\varepsilon|^{p_t+1}}{|y|^t} dx + \int_{\Omega\backslash B_\rho(0)} \frac{|y|^{a(p_t+1)}|V_\varepsilon|^{p_t+1}}{|y|^t} \psi^{p_t+1} dx.$$

By direct computation, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N\backslash B_\rho(0)} |y|^{a(p_t+1)-t} V_\varepsilon^{p_t+1} dx \\
&= \int_{\mathbb{R}^N\backslash B_{\frac{\rho}{\varepsilon}}(0)} |\zeta_y|^{a(p_t+1)-t} |V(\zeta)|^{p_t+1} d\zeta \\
&\le C \int_{\mathbb{R}^N\backslash B_{\frac{\rho}{\varepsilon}}(0)} \frac{|\zeta_y|^{a(p_t+1)-t}}{[(1+|\zeta_y|)^2 + |\zeta_z|^2]^{\frac{p_t+1}{p_t-1}}} d\zeta \\
&\le C \int_{\frac{\rho}{\varepsilon}}^{+\infty} \int_{\frac{\rho}{\varepsilon}}^{+\infty} \frac{r^{a(p_t+1)-t+k-1} s^{N-k-1}}{[(1+r)^2+s^2]^{\frac{p_t+1}{p_t-1}}} ds dr \\
&\le C \int_{\frac{\rho}{\varepsilon}}^{+\infty} \frac{r^{a(p_t+1)-t+k-1}}{(1+r)^{\frac{2(p_t+1)}{p_t-1}-N+k}} dr \int_0^{+\infty} \frac{t^{N-k-1}}{(1+t^2)^{\frac{p_t+1}{p_t-1}}} dt \\
&\le C \int_{\frac{\rho}{\varepsilon}}^{+\infty} \frac{1}{r^{N-k+\sqrt{(k-2)^2-4\lambda}+2}} dr \int_0^{+\infty} \frac{t^{N-k-1}}{(1+t^2)^{\frac{p_t+1}{p_t-1}}} dt
\end{aligned}$$

$$\leq C\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1},$$

where $x=(y,z)=\varepsilon\zeta=\varepsilon(\zeta_y,\zeta_z)$. Similarly, we have

$$\int_{\Omega\setminus B_\rho(0)}\frac{|y|^{a(p_t+1)}|V_\varepsilon|^{p_t+1}}{|y|^t}\psi^{p_t+1}dx\leq C\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1}.$$

Hence, (3.9) holds.

Observe that V_ε satisfies

$$-\mathrm{div}(|y|^{2a}\nabla V_\varepsilon)=|y|^{a(p_t+1)-t}V_\varepsilon^{p_t},\quad x\in\mathbb{R}^N. \tag{3.12}$$

Multiplying (3.12) by ψ^2V_ε, we have

$$\int_\Omega 2|y|^{2a}\nabla\psi V_\varepsilon\psi\nabla V_\varepsilon dx=\int_\Omega|y|^{a(p_t+1)-t}V_\varepsilon^{p_t+1}\psi^2dx-\int_\Omega|y|^{2a}\psi^2|\nabla V_\varepsilon|^2dx$$

and then

$$\int_\Omega|y|^{2a}|\nabla v_\varepsilon|^2dx=\int_\Omega|y|^{2a}|\nabla\psi|^2V_\varepsilon^2dx+\int_\Omega|y|^{a(p_t+1)-t}\psi^2V_\varepsilon^{p_t+1}dx. \tag{3.13}$$

Similar to the proof of (3.9), we deduce

$$\int_\Omega\frac{|y|^{a(p_t+1)}|v_\varepsilon|^{p_t+1}}{|y|^t}\psi^2dx=\int_\Omega\frac{|y|^{a(p_t+1)}|V_\varepsilon|^{p_t+1}}{|y|^t}dx+O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1})$$

and

$$\int_\Omega|y|^{2a}|\nabla\psi|^2V_\varepsilon^2dx=O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}).$$

Thus, from (3.13), (3.10) holds.

Now, it remains to prove (3.11). Applying the same argument with (3.9), we have

$$\begin{aligned}\int_\Omega\frac{|y|^{a(p_t+1)}|v_\varepsilon|^2}{|y|^t}dx&\leq C\int_{B_{2\rho}(0)}|y|^{a(p_t+1)-t}\left|\varepsilon^{\frac{2-N-2a}{2}}V\left(\frac{x}{\varepsilon}\right)\right|^2dx\\&\leq C\varepsilon^{2-2a+a(p_t+1)-t}\int_0^{\frac{2\rho}{\varepsilon}}\frac{r^{a(p_t+1)-t+k-1}}{(1+r)^{\frac{4}{p_t-1}-N+k}}dr\int_0^{+\infty}\frac{t^{N-k-1}}{(1+t^2)^{\frac{2}{p_t-1}}}dt\\&\leq C\varepsilon\int_0^{\frac{2\rho}{\varepsilon}}\frac{1}{(1+r)^{N-k+\sqrt{(k-2)^2-4\lambda}}}dr\\&\leq\begin{cases}C\varepsilon, & N>\beta,\\ C\varepsilon|\ln\varepsilon|, & N=\beta,\end{cases}\end{aligned}$$

where $\beta=k+1-\sqrt{(k-2)^2-4\lambda}$. □

Lemma 3.4 *There exists a constant s_ε such that*

$$\sup_{s>0}J(sv_\varepsilon)=J(s_\varepsilon v_\varepsilon). \tag{3.14}$$

Moreover, $0<C_1<s_\varepsilon<C_2<+\infty$ for some constants C_1 and C_2 independent of ε.

Proof Denote

$$\Phi(s):=J(sv_\varepsilon)=\frac{s^2}{2}\int_\Omega|y|^{2a}|\nabla v_\varepsilon|^2dx-\frac{s^{p_t+1}}{p_t+1}\int_\Omega|y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx-\int_\Omega H(x,sv_\varepsilon)dx.$$

It follows from (3.1) that

$$\begin{aligned}\Phi'(s) &= s\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx - s^{p_t}\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx - \int_\Omega h(x, sv_\varepsilon)v_\varepsilon dx \\ &= s\int_\Omega \big(|y|^{2a}|\nabla v_\varepsilon|^2 - \alpha(x)v_\varepsilon^2\big)dx \\ &\quad - \int_\Omega \frac{[(sv_\varepsilon + v_\mu)^{p_t} - v_\mu^{p_t} - p_t v_\mu^{p_t-1}sv_\varepsilon]v_\varepsilon}{|y|^{t-a(p_t+1)}}dx \\ &\geq \beta_1 s\|v_\varepsilon\|_a^2 - \int_\Omega \frac{\big(\delta s v_\varepsilon^2 + C_\delta s^{p_t}v_\varepsilon^{p_t+1}\big)}{|y|^{t-a(p_t+1)}}dx \\ &\geq \beta_1 s\|v_\varepsilon\|_a^2 - \delta s C\|v_\varepsilon\|_a^2 - C_\delta s^{p_t}\int_\Omega \frac{v_\varepsilon^{p_t+1}}{|y|^{t-a(p_t+1)}}dx\end{aligned}$$

for any $\delta > 0$. Take δ be small enough such that $0 < \delta < \frac{\beta_1}{2C}$. Then from Lemma 3.3, we have

$$\Phi'(s) \geq \frac{\beta_1}{2}sK_1 - C_\delta s^{p_t}K_1 + o(\varepsilon) > 0$$

for $s > 0$ and $\varepsilon > 0$ small.

On the other hand,

$$\begin{aligned}\Phi'(s) &= s\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx - s^{p_t}\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx \\ &\quad - \int_\Omega \frac{[(sv_\varepsilon + v_\mu)^{p_t} - v_\mu^{p_t} - s^{p_t}v_\varepsilon^{p_t}]v_\varepsilon}{|y|^{t-a(p_t+1)}}dx \\ &\leq s\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx - s^{p_t}\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx \\ &\to -\infty, \quad \text{as } s \to +\infty.\end{aligned}$$

Thus $\Phi'(s) < 0$ for s large enough. Since $\Phi(0) = 0$, there exists a constant $s_\varepsilon > 0$ such that $\Phi'(s_\varepsilon) = 0$ and $s_\varepsilon > 0$ satisfies (3.14).

Now we shall prove that there exist some positive constants $C_1 < C_2$ such that $C_1 < s_\varepsilon < C_2$ for all $\varepsilon > 0$.

In fact, since $\Phi'(s_\varepsilon) = 0$ and $s_\varepsilon > 0$, we have

$$\begin{aligned}&\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx - s_\varepsilon^{p_t-1}\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx \\ &\quad - \frac{1}{s_\varepsilon}\int_\Omega \frac{[(s_\varepsilon v_\varepsilon + v_\mu)^{p_t} - v_\mu^{p_t} - s_\varepsilon^{p_t}v_\varepsilon^{p_t}]v_\varepsilon}{|y|^{t-a(p_t+1)}}dx = 0. \end{aligned} \tag{3.15}$$

Since $(sv_\varepsilon + v_\mu)^{p_t} - v_\mu^{p_t} - s^{p_t}v_\varepsilon^{p_t} \geq 0$ for all $x \in \Omega$, we obtain

$$\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx \geq s_\varepsilon^{p_t-1}\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx.$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}s_\varepsilon^{p_t-1} &\leq \frac{\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2dx}{\int_\Omega |y|^{a(p_t+1)-t}v_\varepsilon^{p_t+1}dx} \\ &= \frac{K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}})}{K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1})}\end{aligned}$$

$$
\begin{aligned}
&\le 1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}) \\
&\le 1 + o(\varepsilon)
\end{aligned}
$$

for all $N \ge \beta$.

On the other hand, using (3.2) and (3.3), there exists a constant $C_\delta > 0$ such that

$$h(x,w) \le \delta |y|^{a(p_t+1)-t} w^{p_t} + C_\delta |y|^{a(p_t+1)-t} w$$

for any $\delta > 0$. Then from (3.15), we find

$$
\begin{aligned}
&\int_\Omega |y|^{2a} |\nabla v_\varepsilon|^2 dx - s_\varepsilon^{p_t-1} \int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^{p_t+1} dx \\
&\quad \le \frac{1}{s_\varepsilon} \int_\Omega \big(\delta |y|^{a(p_t+1)-t} s_\varepsilon^{p_t} v_\varepsilon^{p_t} dx + C_\delta |y|^{a(p_t+1)-t} s_\varepsilon v_\varepsilon\big) v_\varepsilon dx \\
&\quad \le \delta s_\varepsilon^{p_t-1} \int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^{p_t+1} dx + C_\delta \int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^2 dx \\
&\quad \le \delta(1+o(\varepsilon))(K_1 + o(\varepsilon)) + O(\varepsilon|\ln\varepsilon|) \\
&\quad \le \delta K_1 + O(\varepsilon|\ln\varepsilon|).
\end{aligned}
$$

Thus,

$$s_\varepsilon^{p_t-1} \ge \frac{\int_\Omega |y|^{2a}|\nabla v_\varepsilon|^2 dx}{\int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^{p_t+1} dx} - \frac{\delta K_1 + O(\varepsilon|\ln\varepsilon|)}{\int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^{p_t+1} dx} = 1 - \delta + O(\varepsilon|\ln\varepsilon|).$$

Taking $\delta = 1/2$, we conclude the existence of positive constants $C_1 < C_2$ independent of ε such that $C_1 < s_\varepsilon < C_2$ for all $\varepsilon > 0$ small. □

In order to prove our main results, we introduce the following result.

Lemma 3.5 ([14, Lemma 3.5]) *If $p > 1$, then there exist a small constant ϵ and a large $B > 0$ independent of x such that*

$$(w + v_\mu)^p - v_\mu^p - w^p \ge w^\epsilon \quad \textit{for all } w \ge B, x \in \mathcal{A}. \tag{3.16}$$

Since

$$(w + v_\mu)^p - v_\mu^p - w^p \ge 0 \tag{3.17}$$

for any $w \ge 0$ and $x \in \Omega, p > 1$. Then applying (3.16) and (3.17), we have for any $x \in \mathcal{A}$,

$$g(x,w) = (w + v_\mu)^p - v_\mu^p - w^p \ge B^\epsilon \chi_I(w) \ge 0,$$

where $\chi_I(s)$ denotes a characteristic function of $I = (B, +\infty)$. Thus,

$$G(x,w) = \int_0^w g(x,s) ds \ge \beta_0 > 0 \tag{3.18}$$

for some constant $\beta_0 > 0$ if $w \ge B_1 > B$ and $x \in \mathcal{A}$.

Now we come to prove our main result.

Proof of Theorem 1.2 It follows from Lemma 3.2 that, to prove Theorem 1.2, we only need to show that (3.8) holds. Using Lemmas 3.3 and 3.4, we find

$$
\begin{aligned}
\sup_{s\ge 0} J(sv_\varepsilon) &= \frac{s_\varepsilon^2}{2} \int_\Omega |y|^{2a} |\nabla v_\varepsilon|^2 dx - \frac{s_\varepsilon^{p_t+1}}{p_t+1} \int_\Omega |y|^{a(p_t+1)-t} v_\varepsilon^{p_t+1} dx - \int_\Omega H(x, s_\varepsilon v_\varepsilon) dx \\
&= \frac{s_\varepsilon^2}{2} \big(K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}})\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{s_\varepsilon^{p_t+1}}{p_t+1}\big(K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1})\big) - \int_\Omega H(x, s_\varepsilon v_\varepsilon)dx \\
&\le \left(\frac{s_\varepsilon^2}{2} - \frac{s_\varepsilon^{p_t+1}}{p_t+1}\right)K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}) - \int_\Omega H(x, s_\varepsilon v_\varepsilon)dx \\
&\le \left(\frac{1}{2} - \frac{1}{p_t+1}\right)K_1 + O(\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}) - \int_\Omega H(x, s_\varepsilon v_\varepsilon)dx.
\end{aligned}
$$

So if we can prove

$$
\lim_{\varepsilon\to 0^+} \frac{\int_\Omega H(x, s_\varepsilon v_\varepsilon)}{\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}} = +\infty,
$$

we have done. In fact, setting $\bar{g}(x,w) = (w+v_\mu)^{p_t} - v_\mu^{p_t} - w^{p_t}$, then $H(x,w) = |y|^{a(p_t+1)-t}\bar{G}(x, w)$, where $\bar{G}(x,w) = \int_0^w g(x,s)ds$. So we have

$$
\begin{aligned}
\int_\Omega H(x, s_\varepsilon v_\varepsilon)dx &= \int_\Omega |y|^{a(p_t+1)-t}\bar{G}(x, s_\varepsilon v_\varepsilon)dx \\
&= \int_{B_2\rho(0)} |y|^{a(p_t+1)-t}\bar{G}\left(x, s_\varepsilon \varepsilon^{\frac{2-N-2a}{2}}\psi V\left(\frac{x}{\varepsilon}\right)\right)dx \\
&\ge \int_{B_{\frac{\rho}{\varepsilon}}(0)} |\varepsilon\zeta_y|^{a(p_t+1)-t}\bar{G}(\varepsilon\zeta, C_1\varepsilon^{\frac{2-N-2a}{2}} V(\zeta))\varepsilon^N d\zeta \\
&\ge \int_{B_{\frac{\rho}{\varepsilon}}(0)} |\varepsilon\zeta_y|^{a(p_t+1)-t}\bar{G}\left(\varepsilon\zeta, C_1\varepsilon^{\frac{2-N-2a}{2}} \frac{1}{[(1+|\zeta_y|)^2 + |\zeta_z|^2]^{\frac{1}{p_t-1}}}\right)\varepsilon^N d\zeta,
\end{aligned}
$$

where $x = \varepsilon\zeta$, and $\zeta = (\zeta_y, \zeta_z)$.

Set

$$
F := \left\{\zeta \in B_{\frac{\rho}{\varepsilon}}(0) : C_1\varepsilon^{\frac{2-N-2a}{2}} \frac{1}{[(1+|\zeta_y|)^2 + |\zeta_z|^2]^{\frac{1}{p_t-1}}} > B_1\right\}.
$$

Note that $|\zeta| \le C\varepsilon^{\frac{(2-N-2a)}{4}(p_t-1)}$ if $\zeta \in F$.

So from (3.18), we deduce

$$
\begin{aligned}
\int_\Omega H(x, s_\varepsilon v_\varepsilon)dx &\ge \beta_0\varepsilon^{a(p_t+1)-t+N}\int_F |\zeta_y|^{a(p_t+1)-t}d\zeta \\
&\ge C\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}+1}\int_0^{C\varepsilon^{\frac{(2-N-2a)}{4}(p_t-1)}} r^{a(p_t+1)-t+N-1}dr \\
&= C\varepsilon^{\frac{N-k+\sqrt{(k-2)^2-4\lambda}+1}{2}}.
\end{aligned}
$$

As a result,

$$
\lim_{\varepsilon\to 0^+} \frac{\displaystyle\int_\Omega H(x, s_\varepsilon v_\varepsilon)}{\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}} \ge \lim_{\varepsilon\to 0^+} \frac{C\varepsilon^{\frac{N-k+\sqrt{(k-2)^2-4\lambda}+1}{2}}}{\varepsilon^{N-k+\sqrt{(k-2)^2-4\lambda}}} = +\infty,
$$

since $N - k + \sqrt{(k-2)^2 - 4\lambda} > 1$.

Therefore, we complete the proof.

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