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Gaussian Fluctuations of Eigenvalues in Log-gas Ensemble: Bulk Case I

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Abstract We study the central limit theorem of the k-th eigenvalue of a random matrix in the log-gas ensemble with an external potential $V = q_{2m}x^{2m}$. More precisely, let $P_n(dH) = C_n \mathrm{e}^{-n\mathrm{Tr}V(H)}dH$ be the distribution of $n \times n$ Hermitian random matrices, $\rho_V(x)dx$ the equilibrium measure, where C_n is a normalization constant, $V(x) = q_{2m}x^{2m}$ with $q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}$, and $m \geq 1$. Let $x_1 \leq \cdots \leq x_n$ be the eigenvalues of H. Let $k := k(n)$ be such that $\frac{k(n)}{n} \in [a, 1-a]$ for n large enough, where $a \in (0, \frac{1}{2})$. Define

$$G(s) := \int_{-1}^{s} \rho_V(x)dx, \quad -1 \leq s \leq 1,$$

and set $t := G^{-1}(k/n)$. We prove that, as $n \to \infty$,

$$\frac{x_k - t}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2} n\rho_V(t)}} \to N(0,1)$$

in distribution. Multi-dimensional central limit theorem is also proved. Our results can be viewed as natural extensions of the bulk central limit theorems for GUE ensemble established by J. Gustavsson in 2005.

Keywords Bulk case, central limit theorem, the Costin–Lebowitz–Soshnikov theorem, eigenvalues, log-gas ensemble

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1 Introduction and Formulation of Results

Let $\mathscr{H}_n$ be the space of $n \times n$ Hermitian matrices $H = (H_{ij})_{1 \leq i,j \leq n}$. The log-gas ensemble, called also the unitary invariant ensemble or the general β-ensemble in the literature, is defined by the probability distribution on $\mathscr{H}_n$ of the form:

$$P_n(dH) = C_n \mathrm{e}^{-nTrV(H)}dH, \tag{1.1}$$

where C_n is a normalization constant, $V(x)$ is a real analytic on $\mathbb{R}$ with $\frac{V(x)}{\log(x^2+1)} \to \infty$ as $|x| \to \infty$, and

$$dH = \prod_{1 \leq i < j \leq n} d\mathrm{Re}H_{ij} d\mathrm{Im}H_{ij} \prod_{i=1}^{n} dH_{ii}.$$

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The distribution (1.1) naturally induces the following distribution density function of the ordered real eigenvalues $x_1 \le \cdots \le x_n$ of H

$$\mathscr{R}_{n,n}(x_1,\dots,x_n)=\frac{1}{Z_n}\prod_{1\le i<j\le n}|x_i-x_j|^2\mathrm{e}^{-n\sum_{i=1}^n V(x_i)}, \tag{1.2}$$

with Z_n a normalization constant. The probability density function $\mathscr{P}_n(x_1,\dots,x_n)$ of n unordered real eigenvalues $\{x_i\}_{i=1}^n$ of H is given by

$$\mathscr{P}_n(x_1,\dots,x_n)=\frac{1}{n!}\mathscr{R}_{n,n}(x_1,\dots,x_n).$$

Following [1], we define k-point correlation functions $\mathscr{R}_{n,k}$ for $1 \le k \le n-1$ by

$$\mathscr{R}_{n,k}(x_1,\dots,x_k)=\frac{n!}{(n-k)!}\underbrace{\int\cdots\int}_{n-k}\mathscr{P}_n(x_1,\dots,x_k,x_{k+1},\dots,x_n)dx_{k+1}\cdots dx_n. \tag{1.3}$$

By (5.40) and (8.7) in [1], a key property of $\mathscr{R}_{n,k}$ is that, for $1 \le k \le n$,

$$\mathscr{R}_{n,k}(x_1,\dots,x_k)=\det(\mathscr{K}_n(x_i,x_j))_{1\le i,j\le k}. \tag{1.4}$$

Here $\mathscr{K}_n(x,y)$ is the reproducing kernel with respect to the weighted measure $\mathrm{e}^{-nV(x)}dx$ defined by

$$\mathscr{K}_j(x,y)=\sum_{i=0}^{j-1}h_i(x;n)h_i(y;n)\mathrm{e}^{-n\frac{V(x)+V(y)}{2}},\quad j=1,2,\dots,$$

where $h_i(x;n)$ is the i-th orthogonal polynomial with respect to $\mathrm{e}^{-nV(x)}dx$, namely,

$$h_i(x;n)=\gamma_i^{(n)}x^i+\cdots,\quad \gamma_i^{(n)}>0,$$
$$\int_{\mathbb{R}}h_i(x;n)h_j(x;n)\mathrm{e}^{-nV(x)}dx=\delta_{ij},\quad i,j=0,1,2,\dots.$$

Due to the formula (1.4), the log-gas ensemble is a determinantal point field. This allows us to apply the central limit theorems proved by Soshnikov in [10] and [11] to study the problem addressed in this paper.

A typical example of the log-gas ensemble is the Gaussian Unitary Ensemble (GUE), where $V(x)=2x^2$. In recent years, more general type of the log-gas models with external potentials V has been extensively studied in the literature. In particular, the following cases have been studied in [1–4, 7]

(I) Monomial weight

$$V(x)=q_{2m}x^{2m},\quad m\ge 1,\ q_{2m}=\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}. \tag{1.5}$$

(II) Freud-type weight (generalization of (I))

$$V=V_n(x)=\frac{1}{n}Q(c_nx+d_n), \tag{1.6}$$

where $Q(x)=\sum_{k=0}^{2m}q_kx^k, m\in\mathbb{N}^+$, $c_n=\frac{\beta_n-\alpha_n}{2}$, $d_n=\frac{\beta_n+\alpha_n}{2}$, and α_n,β_n are the n-th Mhasker–Rakhmanov–Saff numbers (see [3] for more details).

(III) Uniform convex, i.e., there exists $c>0$, such that

$$\inf_{\mathbb{R}} V''\ge c>0. \tag{1.7}$$

In the study of the asymptotic statistics of the eigenvalues, most of the interest has centered around two aspects: the global properties and the local properties. One famous global phenomenon is associated with the equilibrium measure $d\mu^V$, which is the unique minimizer of the variational problem (see [1, 8])

$$\inf_{\mu\in\mathscr{M}_1(\mathbb{R})} I_V(\mu),$$

where

$$I_V(\mu) = \iint_{\mathbb{R}^2} \log|s-t|^{-1} d\mu(s)d\mu(t) + \int_{\mathbb{R}} V(t)d\mu(t)$$

and $\mathscr{M}_1(\mathbb{R}) = \{\mu : \int_{\mathbb{R}} d\mu = 1\}$.

In GUE case, μ^V is the celebrated Wigner law $d\mu^V(x) = \frac{2}{\pi}\sqrt{1-x^2}\chi_{(-1,1)}(x)dx$. In the case when V satisfies (1.5), (1.6) or (1.7), it follows from [2–4] that, $d\mu^V$ is absolutely continues to the Lebesgue measure with density ρ_V, which is supported on a single interval $[b,a]$ and strictly positive on (b,a).

Under the assumption $\frac{V(x)}{\log(x^2+1)} \to \infty$ as $|x| \to \infty$, Johanssson [8] proved that, for any bounded and continues function $\phi : \mathbb{R}^k \to \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}} \phi(x)\mathscr{R}_{n,1}(x)dx = \int_{\mathbb{R}} \phi(t)d\mu^V(t),$$

and

$$\lim_{n\to\infty} \frac{1}{n^k} \int_{\mathbb{R}^k} \phi(x_1,\ldots,x_k)\mathscr{R}_{n,k}(x_1,\ldots,x_k)dx_1\cdots dx_k = \int_{\mathbb{R}^k} \phi(t_1,\ldots,t_k)d\mu^V(t_1)\cdots d\mu^V(t_k).$$

Let

$$d\mu_n(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}.$$

Then μ_n weakly converges to μ^V as $n \to \infty$.

The interest of this paper lies in the local fluctuation properties of the spectrum and we are concerned with the asymptotic behavior of fluctuations around the k-th eigenvalue x_k.

When k is fixed, Tracy and Widom [14] have shown that in the GUE case, after being suitably centered and scaled, x_k converges in distribution to the Tracy–Widom distribution, which is a non-Gaussian distribution.

On the other hand, Gustavsson [6] considered another situation where k tends to infinity as $n \to \infty$ under some condition and showed that in the GUE case x_k is normally distributed in the limit, after being suitably centered and scaled. This work was later extended to other ensembles by many authors, see e.g. [9] for GOE and GSE cases, [12] for the complex covariance matrices and [13] for non-Gaussian Wigner matrices.

Inspired by the universality phenomena, it is natural and interesting to raise the question, whether in the general log-gas ensemble x_k converges in distribution to the Normal distribution, if k tends to infinity under some condition? The purpose of this paper is to study this question for V of type (I). The cases (II) and (III) will be studied in a forthcoming paper.

We now state the main results of this paper, which can be viewed as a natural generalization of Theorems 1.1 and 1.3 in the bulk case in [6].

Theorem 1.1 *Let* $V(x) = q_{2m}x^{2m}$ *with* $q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}$, *and* $m \geq 1$. *For* $a \in (0, \frac{1}{2})$, *set*

$$G(s) = \int_{-1}^{s} \rho_V(x)dx, \quad -1 \leq s \leq 1,$$

where $\rho_V(x)dx$ *is the equilibrium measure and* $t = t(k,n) = G^{-1}(k/n)$, $k = k(n) \in [an, (1-a)n]$. *Denote* x_k *the* k*-th ordered eigenvalue. Then*

$$\frac{x_k - t}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2}n\rho_V(t)}} \to N(0,1)$$

in distribution as $n \to \infty$.

Theorem 1.2 *Under the same condition and notation as in Theorem* 1.1, *let* $\{x_{k_i}\}_{i=1}^m$ *be the eigenvalues such that* $0 < k_i - k_{i+1} \sim n^{\theta_i}, 0 < \theta_i \leq 1$, *and* $k_i \in [a_i n, (1-a_i)n]$, *where* $a_i \in (0, \frac{1}{2})$. *Set* $s_i = s_i(k_i, n) = G^{-1}(\frac{k_i}{n})$, *and let*

$$X_i = \frac{x_{k_i} - s_i}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2}n\rho_V(s_i)}}, \quad i = 1, \dots, m.$$

Then

$$P_n(X_1 \leq \xi_1, \dots, X_m \leq \xi_m) \to \Phi_\Lambda(\xi_1, \dots, \xi_m)$$

as $n \to \infty$, *where* Φ_Λ *is the distribution function of the* m*-dimensional Normal Distribution with mean zero and correlation matrix* Λ, $\Lambda_{i,i} = 1$ *and* $\Lambda_{i,j} = 1 - \max_{i \leq k < j \leq m} \theta_k$, $1 \leq i < j \leq m$.

Remark 1.3 In the GUE case, Gustavsson [6] pointed out that x_k and x_m are asymptotically independent if $|k - m| \sim n$. Here, the same situation also holds for the log-gas ensemble with $V(x) = q_{2m}x^{2m}$.

As a byproduct, we have the following result concerning the zeros of orthogonal polynomials, which is analogous to Remark 4 in [6].

Theorem 1.4 *Let* $V(x) = q_{2m}x^{2m}$ *with* $q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}$, *and* $m \geq 1$. *Let* $z_{1,n} < \cdots < z_{k,n}$ *be the zeros of the* n*-th orthogonal polynomials* $h_j(x;n)$ *with respect to the weight* $\mathrm{e}^{-nV(x)}$. *Let* ρ_V, x_k, t, $\{x_{k_i}\}_{i=1}^m$ *and* $\{s_i\}_{i=1}^m$ *be as in Theorems* 1.1 *and* 1.2. *Then as* $n \to \infty$,

$$\frac{x_k - z_{k,n}}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2}n\rho_V(t)}} \to N(0,1).$$

Set

$$\overline{X_i} = \frac{x_{k_i} - z_{k_i,n}}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2}n\rho_V(s_i)}}, \quad i = 1, \dots, m.$$

Then as $n \to \infty$,

$$P_n[\overline{X_1} \leq \xi_1, \dots, \overline{X_m} \leq \xi_m] \to \Phi_\Lambda(\xi_1, \dots, \xi_m),$$

where Φ_Λ *and* Λ *are as in Theorem* 1.2.

The rest of this paper is organized as follows: In Section 2, we set up some preliminaries. In Section 3, we prove the crucial asymptotic estimate of the expectation and the variance, from which we prove the main theorems. To avoid the technical complexity, we postpone some detail proofs of the asymptotical estimates to Appendix.

2 Preliminaries

Let $V(x)=q_{2m}x^{2m}$, $q_{2m}=\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}, m\geq 1$. It is well known that (cf. [4, (1.8)], [1, (6.151), (1.157)])

$$d\mu^V(x)=\rho_V(|x|)\chi_{(-1,1)}(x)dx, \tag{2.1}$$

where

$$\rho_V(x)=\frac{2m}{\pi}x^{2m-1}\int_1^{\frac{1}{x}}\frac{u^{2m-1}}{\sqrt{(u^2-1)}}du, \tag{2.2}$$

$$=\frac{mq_{2m}}{\pi}\sqrt{1-x^2}h(x),\quad x\in(0,1), \tag{2.3}$$

with $h(x)=x^{2m-2}+\sum_{j=1}^{m-1}x^{2m-2-2j}A_j$, and $A_j=\prod_{k=1}^{j}\frac{2k-1}{2k}$.

Set

$$F(x)=\left|\int_x^1\frac{2m}{\pi}y^{2m-1}dy\int_1^{\frac{1}{y}}\frac{u^{2m-1}}{\sqrt{|u^2-1|}}du\right|.$$

Then, for $x\in(-1,1)$,

$$F(x)=\int_x^1\rho_V(y)dy, \tag{2.4}$$

and $F(x)=1-G(x)$ with G defined as in Theorem 1.1.

Using the explicit expression of $\rho_V(x)$, we can prove the following lemma. For the proof, see Appendix.

Lemma 2.1 *Let $0<\delta<1$. For $x\in[-1+\delta,1-\delta]$, it holds that*

(i) *$|(F'(x))^{-1}|$ and $|F''(x)|$ are bounded,*

(ii) $\pi F(x)+\frac{\pi}{2m}x\rho_V(x)=\arccos x$.

From the relation (1.4), we may obtain the asymptotic behavior of $\mathscr{R}_{n,k}$ from the asymptotic behavior of $\mathscr{K}_n(x,y)$. Thus, applying the asymptotic analysis of the orthogonal polynomials, we have the following estimates of $\mathscr{K}_n(x,y)$. For the proof, see Appendix.

Throughout this paper, C is a constant which may change from one line to another and $f=\mathcal{O}(g)$ means $|f/g|$ is bounded.

Lemma 2.2 (i) *Take any $t\in(-1,1)$ and define $\Gamma_1^1=\{(x,y):t\leq x\leq t+\frac{1-t}{\log n},t-\frac{1+t}{\log n}\leq y\leq t-\frac{1}{n}\}$. Then, for $(x,y)\in\Gamma_1^1$*

$$\mathscr{K}_n(x,y)=\frac{\sin[\pi n(F(y)-F(x))]+\mathcal{O}(\frac{1}{\log n})}{\pi(x-y)}. \tag{2.5}$$

(ii) *For any $\delta>0$ sufficiently small and $x,y\in[-1+\delta,1-\delta]$, it holds that*

$$\mathscr{K}_n^2(x,y)=\mathcal{O}\left(\frac{1}{(x-y)^2}\right). \tag{2.6}$$

(iii) *For any $t\in(-1,1)$ and $(x,y)\in\{(x,y):t\leq x\leq t+\frac{1}{n},t-\frac{1}{n}\leq y\leq t\}$, it holds that*

$$\mathscr{K}_n(x,y)=\mathcal{O}(n).$$

We conclude this section with the estimates of $h_n(x;n)$. For the proofs, see Appendix.

Lemma 2.3 *There exists a $\delta_0 > 0$ such that for all $0 < \delta \le \delta_0$, the following statements hold:*

(i) *if $-1+\delta < x < 1-\delta$,*

$$|h_n(x;n)\mathrm{e}^{-\frac{n}{2}V(x)}| \le C;$$

(ii) *if $1-\delta < x < 1$,*

$$|h_n(x;n)\mathrm{e}^{-\frac{n}{2}V(x)}| \le C\left[\frac{1}{(1-x)^{\frac{1}{4}}}+1\right];$$

(iii) *if $1 < x < 1+\delta$*

$$|h_n(x;n)\mathrm{e}^{-\frac{n}{2}V(x)}| \le C\left[\frac{1}{(x-1)^{\frac{1}{4}}}+1\right];$$

(iv) *if $x > 1+\delta$*

$$|h_n(x;n)\mathrm{e}^{-\frac{n}{2}V(x)}| \le C\mathrm{e}^{-n\pi F(x)},$$

where C is a constant independent of n.

3 Proofs of Main Theorems

The proofs of Theorems 1.1 and 1.2 depend crucially on the following two lemmas concerning the asymptotic analysis of the expectation and the variance.

Lemma 3.1 *Fix $\xi \in \mathbb{R}$, set*

$$a_n = \frac{\sqrt{\log n}}{\sqrt{2\pi^2}n\rho_V(t)}$$

and

$$I_n = [t + a_n\xi, \infty),$$

where $t = t(k,n) = G^{-1}(\frac{k}{n})$ with k as in Theorem 1.1. Denote by $\#I_n$ the number of eigenvalues in I_n. Then it holds that

$$n - k - E(\#I_n) = \frac{\sqrt{\log n}}{\sqrt{2\pi^2}}\xi + \mathcal{O}(1). \tag{3.1}$$

Proof Since $\frac{k}{n} \in [a,(1-a)] \subset (0,1)$ and $G : [-1,1] \to [0,1]$ is a strictly increasing function, thus

$$\sup_n |t(k,n)| = \sup\left|G^{-1}\left(\frac{k}{n}\right)\right| < 1.$$

Then it follows from Lemma 2.1 that

$$a_n = \mathcal{O}\left(\frac{\sqrt{\log n}}{n}\right) \tag{3.2}$$

and for n large enough,

$$|t + a_n\xi| < 1. \tag{3.3}$$

From the definition of $\mathscr{R}_{n,1}(x)$ and (1.4), it follows that

$$E(\#I_n) = \int_{I_n}\mathscr{R}_{n,1}(x)dx = \int_{t+a_n\xi}^{\infty}\mathscr{K}_n(x,x)dx = \frac{n}{2} - \int_0^{t+a_n\xi}\mathscr{K}_n(x,x)dx, \tag{3.4}$$

where in the last step we used $\int_{\infty}^{\infty}\mathscr{K}_n(x,x)dx = n$ and $\mathscr{K}_n(-x,-x) = \mathscr{K}_n(x,x)$.

Moreover, from the formula (4.2) in [5], we have the asymptotic formula of $\mathscr{K}_n(x,x)$, for $x \in [-1+\delta, 1-\delta]$,

$$\frac{1}{n}\mathscr{K}_n(x,x) = \rho_V(x) + \frac{1}{4\pi n}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)\cos\left(n\int_x^1 \rho_V(s)ds\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{3.5}$$

Therefore, taking (3.5) into (3.4), since $|(\frac{1}{x-1} + \frac{1}{x+1})\cos(n\int_x^1 \rho_V(s)ds)| \le \frac{2}{\delta}$ for $x \in [-1+\delta, 1-\delta]$ and $\int_{-1}^0 \rho_V(x)dx = \frac{1}{2}$, taking Taylor expansion we have

$$\begin{aligned} E(\#I_n) &= \frac{n}{2} - n\int_0^{t+a_n\xi} \rho_V(x)dx + \mathcal{O}(1) \\ &= n - n\int_{-1}^t \rho_V(x)dx - n\int_t^{t+a_n\xi} \rho_V(x)dx + \mathcal{O}(1) \\ &= n - k - n\rho_V(t)a_n\xi + n\mathcal{O}(a_n^2) + \mathcal{O}(1) \\ &= n - k - \frac{\sqrt{\log n}}{\sqrt{2\pi^2}}\xi + \mathcal{O}\left(\frac{\log n}{n}\right) + \mathcal{O}(1), \end{aligned}$$

which concludes the proof of Lemma 3.1. □

Lemma 3.2 *Let* $\{t_n\}_{n=1}^\infty$ *be a sequence such that* $\sup_n |t_n| < 1$. *Set* $I_n = [t_n, \infty), n = 1, 2, \dots$ *and denote by* $\#I_n$ *the number of eigenvalues in* I_n. *Then*

$$\mathrm{Var}(\#I_n) = \frac{1}{2\pi^2}\log n + \mathcal{O}(\log\log n). \tag{3.6}$$

Proof Let us first notice that, since $\sup_n |t_n| < 1$, there exists a $\delta_1 > 0$ such that $\inf_n(1-t_n) > \delta_1 > 0$, and $\inf_n(1+t_n) > \delta_1 > 0$. We choose $\delta > 0$ sufficiently small such that $0 < \delta < \delta_1$. Then for sufficiently large $n, t_n, t_n \pm \frac{1}{n}$ and $t_n \pm \frac{1-t_n}{\log n}$ all lie in $[-1+\delta, 1-\delta]$. From now on, we will assume that n is sufficiently large so that all the above statements hold.

By the definition of $\mathscr{R}_{n,1}$ and $\mathscr{R}_{n,2}$, using (1.4) and $\mathscr{K}_n(x,x) = \int_{\mathbb{R}} \mathscr{K}_n^2(x,y)dy$, we have

$$\begin{aligned} \mathrm{Var}(\#I_n) &= E[(\#I_n)^2] - [E(\#I_n)]^2 \\ &= \sum_{k=1}^n E[\chi_{I_n}(x_k)] + \sum_{j\neq k} E[\chi_{I_n}(x_k)\chi_{I_n}(x_j)] - \left[\sum_{k=1}^n E(\chi_{I_n}(x_k))\right]^2 \\ &= \int_{\mathbb{R}} \chi_{I_n}(x)\mathscr{R}_{n,1}(x)dx + \iint_{\mathbb{R}\times\mathbb{R}} \chi_{I_n}(x)\chi_{I_n}(y)\mathscr{R}_{n,2}(x,y)dxdy - \left(\int_{\mathbb{R}} \chi_{I_n}(x)\mathscr{R}_{n,1}(x)\right)^2 \\ &= \int_{\mathbb{R}} \chi_{I_n}(x)\mathscr{K}_n(x,x)dx - \iint_{\mathbb{R}\times\mathbb{R}} \chi_{I_n}(x)\chi_{I_n}(y)\mathscr{K}_n^2(x,y)dxdy \\ &= \iint_{\mathbb{R}\times\mathbb{R}} \chi_{I_n}(x)\chi_{I_n^c}(y)\mathscr{K}_n^2(x,y)dxdy, \end{aligned}$$

thus

$$\mathrm{Var}(\#I_n) = \int_{t_n}^\infty \int_{-\infty}^{t_n} \mathscr{K}_n^2(x,y)dxdy.$$

Following [6], we divide the integration domain $\Omega_n = \{(x,y) : t_n \le x < \infty, -\infty < y \le t_n\}$ into two sets $\Gamma \cup (\Omega_n \setminus \Gamma)$ with $\Gamma = \{(x,y) : t_n \le x \le 1-\delta, -1+\delta \le y \le t_n\}$.

Using the asymptotic formulas of $\mathscr{K}_n(x,y)$ in Lemma 2.2, we will show that

$$\iint_\Gamma \mathscr{K}_n^2(x,y)dxdy = \iint_{\Gamma_1^1} \mathscr{K}_n^2(x,y)dxdy + \mathcal{O}(\log\log n)$$

$$= \frac{1}{2\pi^2}\log n + \mathcal{O}(\log\log n), \tag{3.7}$$

where Γ_1^1 is as in Lemma 2.2 (i).

Indeed, from Lemma 2.2 (ii) and (iii), direct calculations show that

$$\iint_{\Gamma/\Gamma_1^1} \frac{1}{(x-y)^2} dxdy = \mathcal{O}(\log\log n).$$

For the remaining domain Γ_1^1, using Lemma 2.2 (i), we have

$$\begin{aligned}\iint_{\Gamma_1^1} \mathscr{K}_n^2(x,y)dxdy &= \iint_{\Gamma_1^1} \frac{\sin^2[\pi n(F(y)-F(x))] + \mathcal{O}(\frac{1}{\log n})}{\pi^2(x-y)^2}dxdy \\ &= \iint_{\Gamma_1^1} \frac{1-\cos[2\pi n(F(y)-F(x))]}{2\pi^2(x-y)^2}dxdy + \mathcal{O}(1) \\ &= \frac{1}{2\pi^2}\log n - \iint_{\Gamma_1^1} \frac{\cos[2\pi n(F(y)-F(x))]}{2\pi^2(x-y)^2}dxdy + \mathcal{O}(\log\log n). \end{aligned}\tag{3.8}$$

Since $\cos[2\pi n(F(y)-F(x))] = \frac{\frac{d}{dy}\{\sin[2\pi n(F(y)-F(x))]\}}{2\pi n F'(y)}$, using integration by parts for the second term, we obtain

$$\iint_{\Gamma_1^1} \frac{\cos[2\pi n(F(y)-F(x))]}{2\pi^2(x-y)^2}dxdy = I_1 - I_2,$$

where

$$I_1 = \int_{t_n}^{t_n+\frac{1-t_n}{\log n}} \frac{\sin[2\pi n(F(y)-F(x))]}{4\pi^3 n F'(y)(x-y)^2}\Bigg|_{y=t_n-\frac{1+t_n}{\log n}}^{y=t_n-\frac{1}{n}} dx,$$

$$I_2 = \iint_{\Gamma_1^1} \frac{\sin[2\pi n(F(y)-F(x))]}{4\pi^3 n} \frac{\partial}{\partial y}\left\{\frac{1}{F'(y)(x-y)^2}\right\} dy.$$

From Lemma 2.1, $|(F'(x))^{-1}|$ and $|F''(x)|$ are bounded in Γ, it follows that

$$I_1 \le \frac{C}{n}\int_{t_n}^{t_n+\frac{1-t_n}{\log n}} \left[\frac{1}{(x-t_n+\frac{1}{n})^2} + \frac{1}{(x-t_n+\frac{t_n+1}{\log n})^2}\right]dx = \mathcal{O}(1), \tag{3.9}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial y}\left\{\frac{1}{F'(y)(x-y)^2}\right\}\right| &= |(-1)(F'(y))^{-2}F''(y)(x-y)^{-2} + 2(F'(y))^{-1}(x-y)^{-3}| \\ &\le C\left(\frac{1}{(x-y)^2} + \frac{1}{(x-y)^3}\right)\end{aligned}$$

with C independent of n, which yields

$$I_2 \le \frac{C}{4\pi^3 n}\iint_{\Gamma_1^1}\left(\frac{1}{(x-y)^2} + \frac{1}{(x-y)^3}\right)dxdy = \mathcal{O}(1). \tag{3.10}$$

Consequently, taking (3.9) and (3.10) into (3.8) yields

$$\iint_{\Gamma_1^1} \mathscr{K}_n^2(x,y)dxdy = \frac{1}{2\pi^2}\log n + \mathcal{O}(\log\log n),$$

which completes the proof of (3.7).

We next show that

$$\iint_{\Omega_n/\Gamma} \mathscr{K}_n^2(x,y)dxdy = \mathcal{O}(1). \tag{3.11}$$

Indeed, since $\inf_n(1 \pm t_n) > \delta_1 > 0$ and $\delta_1 - \delta > 0$, we have for $(x, y) \in \Omega_n/\Gamma$,

$$|x - y| \geq \min\{\delta_1 - \delta, 2 - 2\delta, \delta_1 + \delta\} = \delta_1 - \delta > 0.$$

Taking this into the Christoffel–Darboux identity (cf. [1]) yields

$$\begin{aligned}\mathscr{K}_n^2(x,y) \leq C\big\{&\big[h_n(x;n)h_{n-1}(y;n)\mathrm{e}^{-n\frac{V(x)+V(y)}{2}}\big]^2 \\ &+ \big[h_n(y;n)h_{n-1}(x;n)\mathrm{e}^{-n\frac{V(x)+V(y)}{2}}\big]^2\big\}.\end{aligned}$$

By Lemma 2.3, we may obtain (3.11). Indeed, we may divide Ω_n/Γ into five subsets $\bigcup_{i=1}^5 J_i$, where

$$\begin{aligned}J_1 =& \{(x, y) : 1 - \delta \leq x \leq 1 + \delta, -1 + \delta \leq y \leq t_n\},\\ J_2 =& \{(x, y) : t_n \leq x \leq 1 - \delta, -1 - \delta \leq y \leq -1 + \delta\},\\ J_3 =& \{(x, y) : 1 - \delta \leq x \leq 1 + \delta, -1 - \delta \leq y \leq -1 + \delta\},\\ J_4 =& \{(x, y) : 1 + \delta \leq x < \infty, -\infty < y \leq t_n\},\\ J_5 =& \{(x, y) : t_n \leq x < 1 + \delta, -\infty < y \leq -1 - \delta\},\end{aligned}$$

and denote $\widetilde{I}_i = \int_{J_i} \mathscr{K}_n^2(x, y)dxdy$, $i = 1, \ldots, 5$.

Now, $\forall (x, y) \in J_1$,

$$|h_n(x;n)h_{n-1}(y;n)\mathrm{e}^{-n\frac{V(x)+V(y)}{2}}| \leq C\left[\frac{1}{|1-x|^{\frac{1}{4}}} + 1\right],$$

which yields

$$\widetilde{I}_1 = \int_{J_1} \mathscr{K}_n^2(x, y)dxdy = \mathcal{O}(1). \tag{3.12}$$

Similarly, computations show that

$$\widetilde{I}_2 + \widetilde{I}_3 = \mathcal{O}(1). \tag{3.13}$$

Next, $\forall (x, y) \in J_4$

$$|h_n(x;n)\mathrm{e}^{-n\frac{V(x)}{2}}| \leq C\mathrm{e}^{-n\pi F(x)},$$

and

$$|h_n(y;n)\mathrm{e}^{-n\frac{V(y)}{2}}| \leq C\left[\mathrm{e}^{-n\pi\widetilde{F}(y)}\chi_{(-\infty,-1-\delta)}(y) + \left(\frac{1}{|1+y|^{\frac{1}{4}}} + 1\right)\chi_{[-1-\delta,t_n]}(y)\right],$$

with $\widetilde{F}(y) = F(-y)$ for $y \in (-\infty, -1 - \delta)$.

Moreover, by the estimate in Appendix with $c = 1 - (1 + \delta)^{-2} > 0$,

$$F(x) \geq \frac{c^{m-\frac{1}{2}}}{2m\pi}x^{2m} - \frac{c^{m-\frac{1}{2}}}{2m\pi} - \frac{1}{\sqrt{c}\pi}\log x, \tag{3.14}$$

from which we get

$$\begin{aligned}\widetilde{I}_4 &\leq C\int_{1+\delta}^{\infty}\int_{-\infty}^{-1-\delta} \mathrm{e}^{-2n\pi F(x)}\mathrm{e}^{-2n\pi\widetilde{F}(y)}dxdy \\ &\quad + C\int_{1+\delta}^{\infty}\int_{-1-\delta}^{t_n}\left[\frac{1}{|1+y|^{\frac{1}{2}}} + 1\right]\mathrm{e}^{-2n\pi F(x)}dxdy \\ &= o(1).\end{aligned} \tag{3.15}$$

Similarly,

$$\widetilde{I}_5 = o(1). \tag{3.16}$$

Consequently, from (3.12), (3.13), (3.15) and (3.16), we obtain (3.11) and finish the proof of Lemma 3.2. □

Proof of Theorem 1.1 The proof follows directly from Lemmas 3.1 and 3.2. Indeed, take t, a_n and ξ as in Lemma 3.1, for the k-th eigenvalue x_k, it follows that

$$\begin{aligned} P_n\Big(\frac{x_k - t}{a_n} \le \xi\Big) &= P_n\Big(\frac{\#I_n - E(\#I_n)}{\sqrt{\mathrm{Var}(\#I_n)}} \le \frac{n-k-E(\#I_n)}{\sqrt{\mathrm{Var}(\#I_n)}}\Big) \\ &= P_n\Big(\frac{\#I_n - E(\#I_n)}{\sqrt{\mathrm{Var}(\#I_n)}} \le \xi + o(1)\Big), \end{aligned}$$

which tends to the normal distribution by Costin–Lebowitz–Soshnikov theorem in [10] (see also [6, Theorem 2.1]). □

Proof of Theorem 1.2 Thanks to the estimates (3.7) and (3.11), the proof follows similarly as the arguments in [6] and Soshinikov's central limit theorem in [11] (see also [6, Theorem 3.1]). For simplicity of exposition, we omit the details here. □

We conclude this section with the proof of Theorem 1.4.

Proof of Theorem 1.4 By [3, Theorem 2.3], there exist constants k_0 and $C > 0$ such that for all $k_0 < k < n - k_0$,

$$\Big|z_{k,n} - F^{-1}\Big(1 - \frac{k}{n} + \frac{1}{2n} + \frac{1}{2\pi n}\arcsin\Big(F^{-1}\Big(1 - \frac{k-1}{n}\Big)\Big)\Big)\Big| \le \frac{C}{[\alpha(1-\alpha)]^{\frac{4}{3}}}\frac{1}{n^2}, \tag{3.17}$$

where $\alpha = \frac{k}{n}$, $F(x) = \int_x^1 \rho_V(t)dt, x \in [-1,1]$ and F^{-1} denotes its inverse function.

By definition, $G^{-1}(x) = F^{-1}(1-x)$, $x \in [0,1]$. Moreover, for $k \in [an, (1-a)n]$, $\alpha = \frac{k}{n} \in [a, 1-a]$, by Lemma 2.1, $[\rho_V(G^{-1}(x))]^{-1}$ is bounded in $[a-\delta, 1-a+\delta]$ with δ sufficiently small. Hence,

$$\begin{aligned} &\Big|F^{-1}\Big(1 - \frac{k}{n} + \frac{1}{2n} + \frac{1}{2\pi n}\arcsin\Big(F^{-1}\Big(1 - \frac{k-1}{n}\Big)\Big)\Big) - G^{-1}\Big(\frac{k}{n}\Big)\Big| \\ &= \Big|G^{-1}\Big(\frac{k}{n} - \frac{1}{2n} - \frac{1}{2\pi n}\arcsin\Big(G^{-1}\Big(\frac{k-1}{n}\Big)\Big)\Big) - G^{-1}\Big(\frac{k}{n}\Big)\Big| \\ &= \mathcal{O}\Big(\frac{1}{n}\Big), \end{aligned}$$

which, together with (3.17), yields that

$$\Big|z_{k,n} - G^{-1}\Big(\frac{k}{n}\Big)\Big| = \mathcal{O}\Big(\frac{1}{n}\Big).$$

Thus

$$\frac{x_k - z_{k,n}}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n}\rho_V(t)}} = \frac{x_k - G^{-1}(\frac{k}{n})}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n}\rho_V(t)}} + \frac{G^{-1}(\frac{k}{n}) - z_{k,n}}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n}\rho_V(t)}} = \frac{x_k - G^{-1}(\frac{k}{n})}{\frac{\sqrt{\log n}}{\sqrt{2\pi^2 n}\rho_V(t)}} + \mathcal{O}\Big(\frac{1}{\sqrt{\log n}}\Big),$$

which concludes Theorem 1.4 by Theorems 1.1 and 1.2. □

4 Appendix

Proof of Lemma 2.1 This lemma is obvious in the case $m = 1$, where $\rho_V(x) = \frac{2}{\pi}\sqrt{1-x^2}$. Moreover, since $\rho_V(-x) = \rho_V(x)$, we need only to prove (i), (ii) for $m \geq 2$ and $x \in [0, 1-\delta]$.

(i) From (2.3), (2.4) and the fact that $h(x) \geq A_{m-1} > 0$, it follows that

$$|(F'(x))^{-1}| \leq \frac{\pi}{mq_{2m}} \frac{1}{A_{m-1}} \frac{1}{\sqrt{1-(1-\delta)^2}} < \infty.$$

Moreover, since $h(x)$ and $h'(x)$ are positive continuous functions on $[0, 1-\delta]$, they are bounded from above by some positive constant C on $[0, 1-\delta]$, thus

$$\begin{aligned}|F''(x)| &= |\rho_V'(x)| \\ &= \left|\frac{mq_{2m}}{\pi}\left[-\frac{x}{\sqrt{1-x^2}}h(x) + \sqrt{1-x^2}h'(x)\right]\right| \\ &\leq \frac{mq_{2m}}{\pi}\left[\frac{C}{\sqrt{1-(1-\delta)^2}} + C\right] < \infty\end{aligned}$$

and the first assertion (i) follows.

(ii) From (2.2), straightforward calculations show that

$$\frac{d}{dx}\left[\pi F + \frac{\pi}{2m}x\rho_V - \arccos x\right] = 0,$$

which concludes the proof of part (ii), due to the fact that $\pi F + \frac{\pi}{2m}x\rho_V - \arccos x$ is zero at $x = 1$. □

Proof of Lemma 2.3 The proof follows directly from the Plancherel–Rotach type asymptotics in [7, Theorem 1.16], the relation (4.4) and the asymptotic estimates of the Airy functions (see, e.g., [12, (2.60), (2.61), (3.6), (3.7)]). □

We next show the proof of Lemma 2.2. Before that, let us state the following lemma.

Lemma 4.1 *Let $\delta \in (0, 1)$. Set $\alpha_x = n\pi F(x) - \frac{1}{2}\arcsin x$ and $\theta_x = \arccos x$. Then for all $x, y \in [-1+\delta, 1-\delta]$,*

$$\mathscr{K}_n(x, y) = \frac{\cos\alpha_x \cos(\alpha_y - \theta_y) - \cos\alpha_y \cos(\alpha_x - \theta_x) + \mathcal{O}(\frac{1}{n})}{\pi(1-x^2)^{\frac{1}{4}}(1-y^2)^{\frac{1}{4}}(x-y)}. \tag{4.1}$$

Proof First, using Christoffel–Darboux identity, we have

$$\mathscr{K}_n(x, y) = \frac{\gamma_{n-1}^{(n)}}{\gamma_n^{(n)}} \frac{h_n(x; n)h_{n-1}(y; n) - h_n(y; n)h_{n-1}(x; n)}{x - y} \mathrm{e}^{-n\frac{V(x)+V(y)}{2}}. \tag{4.2}$$

For the asymptotics of $\frac{\gamma_{n-1}^{(n)}}{\gamma_n^{(n)}}$, we recall [7, Theorem 1.5] that

$$\frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2}n^{\frac{1}{2m}} + \mathcal{O}\left(\frac{n^{\frac{1}{2m}}}{n^2}\right), \tag{4.3}$$

where γ_n is defined as the leading coefficient of the n-th orthogonal polynomial $h_n(x)$ with respect to the measure $\mathrm{e}^{-V(x)}dx$, namely,

$$\begin{aligned}&h_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0, \\ &\int h_i(x)h_j(x)\mathrm{e}^{-V(x)}dx = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots.\end{aligned}$$

By definitions, it is straightforward to verify that

$$h_i(x;n) = n^{\frac{1}{4m}} h_i(n^{\frac{1}{2m}} x), \tag{4.4}$$

$$\gamma_i^{(n)} = n^{\frac{1}{2m}(\frac{1}{2}+i)}\gamma_i, \quad i = 0,1,2,\ldots. \tag{4.5}$$

Thus, it follows from (4.3) and (4.5) that

$$\frac{\gamma_{n-1}^{(n)}}{\gamma_n^{(n)}} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{4.6}$$

For the asymptotics of $h_n(x;n)$ and $h_{n-1}(x;n)$, we have by [7, Theorem 1.16, (ii)] (see also [1]) and the relation (4.4) that

$$h_n(x;n)\mathrm{e}^{-\frac{n}{2}V(x)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1-x^2)^{\frac{1}{4}}} \left[\cos(\alpha_x) + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{4.7}$$

Moreover, we claim that

$$h_{n-1}(x;n)\mathrm{e}^{-\frac{n}{2}V(x)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1-x^2)^{\frac{1}{4}}} \left[\cos(\alpha_x - \theta_x) + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{4.8}$$

Then, (4.1) follows from (4.2), (4.6), (4.7) and (4.8).

We are now left to prove (4.8). Indeed, notice that

$$h_{n-1}(x;n)\mathrm{e}^{-\frac{n}{2}V(x)} = \left(\frac{n}{n-1}\right)^{\frac{1}{4m}} h_{n-1}(x_n;n-1)\mathrm{e}^{-\frac{n-1}{2}V(x_n)}, \tag{4.9}$$

where $x_n = (\frac{n}{n-1})^{\frac{1}{2m}} x = x + \frac{1}{2m}\frac{1}{n-1}x + \mathcal{O}(\frac{1}{n^2})$.

Then, using again [7, Theorem 1.16, (ii)], we have

$$\begin{aligned} &h_{n-1}(x;n)\mathrm{e}^{-\frac{n}{2}V(x)} \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{n}{n-1}\right)^{\frac{1}{4m}} \frac{1}{(1-x_n^2)^{\frac{1}{4}}}\left[\cos\left((n-1)\pi F(x_n) - \frac{1}{2}\arcsin(x_n)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right]. \end{aligned} \tag{4.10}$$

Moreover, since $\frac{1}{(1-x_n^2)^{\frac{1}{4}}} = \frac{1}{(1-x^2)^{\frac{1}{4}}} + \mathcal{O}(\frac{1}{n})$, $\arcsin x_n = \arcsin x + \mathcal{O}(\frac{1}{n})$, and

$$\begin{aligned} F(x_n) &= \int_{x_n}^1 \rho_V(y)dy = \int_{x+\frac{1}{2m}\frac{1}{n-1}x+\mathcal{O}(\frac{1}{n^2})}^1 \rho_V(y)dy \\ &= F(x) - \frac{1}{2m(n-1)} x\rho_V(x) + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned}$$

taking the above asymptotic expansions into (4.10) and using Lemma 2.1 yield (4.8) and complete the proof of Lemma 4.1. □

Proof of Lemma 2.2 (i) follows from Lemma 4.1 using similar arguments as in [6] and (ii) follows directly from Lemma 4.1. Thus we only need to prove (iii).

Indeed, from [2, Lemma 6.1], for $t \in (-1,1)$ and ξ, η in compact subsects of $\mathbb{R}$, we have

$$\frac{1}{n\rho_V(t)}\mathscr{K}_n\left(t + \frac{\eta + \mathcal{O}(1)}{n\rho_V(t)},\ t + \frac{\xi + \mathcal{O}(1)}{n\rho_V(t)}\right) = \frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)} + \mathcal{O}(1),$$

where the error term is uniformly in terms of $t \in [-1,1]$. Applying this result to

$$x = t + \frac{\eta}{n\rho_V(t)}, \qquad y = t - \frac{\xi}{n\rho_V(t)},$$

where

$$0 \le \eta = n\rho_V(t)(x-t) \le n\rho_V(t)\frac{1}{n} = \rho_V(t),$$
$$0 \le \xi = n\rho_V(t)(t-y) \le n\rho_V(t)\frac{1}{n} = \rho_V(t),$$

we derive that

$$\begin{aligned}\mathscr{K}_n(x,y) &= \mathscr{K}_n\left(t+\frac{\eta}{n\rho_V(t)}, t-\frac{\xi}{n\rho_V(t)}\right)\\ &= \left[\frac{\sin\pi(\xi+\eta)}{\pi(\xi+\eta)} + \mathcal{O}(1)\right] n\rho_V(t)\\ &= \mathcal{O}(n).\end{aligned}$$

□

Proof of (3.14) Since for $x > 1+\delta$, $x^2-1 > cx^2$ with $c = 1-(1+\delta)^{-2} > 0$, we have

$$\begin{aligned}F(x) &= \int_1^x \frac{2m}{\pi} y^{2m-1} dy \int_{\frac{1}{y}}^1 \frac{u^{2m-1}}{\sqrt{1-u^2}} du\\ &\ge \int_1^x \frac{2m}{\pi}\frac{y^{2m-1}}{\sqrt{1-y^{-2}}} dy \int_{\frac{1}{y}}^1 u^{2m-1} du\\ &= \int_1^x \frac{1}{\pi}\frac{y^{2m}}{\sqrt{y^2-1}} dy - \int_1^x \frac{1}{\pi\sqrt{y^2-1}} dy\\ &\ge \int_1^x \frac{1}{\pi}\frac{(\sqrt{y^2-1})^{2m}}{\sqrt{y^2-1}} dy - \int_1^x \frac{1}{\sqrt{c}\pi y} dy\\ &= \int_1^x \frac{1}{\pi}(\sqrt{y^2-1})^{2m-1} dy - \frac{1}{\sqrt{c}\pi}\log x\\ &\ge \int_1^x \frac{1}{\pi}(\sqrt{cy^2})^{2m-1} dy - \frac{1}{\sqrt{c}\pi}\log x\\ &= \frac{c^{m-\frac{1}{2}}}{2m\pi}x^{2m} - \frac{c^{m-\frac{1}{2}}}{2m\pi} - \frac{1}{\sqrt{c}\pi}\log x,\end{aligned}$$

which completes the proof. □

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