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Measure of Noncompactness and Semilinear Nonlocal Functional Differential Equations in Banach Spaces

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Abstract This paper is concerned with the measure of noncompactness in the spaces of continuous functions and semilinear functional differential equations with nonlocal conditions in Banach spaces. The relationship between the Hausdorff measure of noncompactness of intersections and the modulus of equicontinuity is studied for some subsets related to the semigroup of linear operators in Banach spaces. The existence of mild solutions is obtained for a class of nonlocal semilinear functional differential equations without the assumption of compactness or equicontinuity on the associated semigroups of linear operators.

Keywords Measure of noncompactness, equicontinuity, differential equation, nonlocal condition, C_0-semigroup, mild solution

MR(2010) Subject Classification 35R20, 47D06, 47H09

1 Introduction

The concept of a measure of noncompactness was initiated by Kuratowski [21] and Darbo [12]. In recent decades measures of noncompactness play very important role in nonlinear analysis. They are often applied to the theories of differential and integral equations as well as to the operator theory and geometry of Banach spaces (cf. [1, 2, 5, 6]). For this reason, from 1970s there have been a lot of papers concerning that concept and its applications (cf. [13–18, 20, 22, 26–30]).

In the theory of measures of noncompactness, the Hausdorff measure of noncompactness [5] and the Kuratowski measure of noncompactness [21] play special role. Especially the Hausdorff measure of noncompactness is frequently used in many branches of nonlinear analysis and its applications. It is caused by the fact that it is defined in a natural way and has several useful properties.

Let us notice that in applications to differential and integral equations in abstract spaces, the measure of noncompactness in the space of continuous functions is very important. In this case, the Hausdorff measure of noncompactness can be expressed by formulae involving the Hausdorff measure of noncompactness of the intersections and the modulus of equi-continuity. In general cases, they are independent. When studying differential and integral equations in

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abstract spaces, there is the case that some sets of continuous functions are related to the semigroups of linear operators (cf. [13–18, 20, 22, 26–30]). If the associated semigroup is not equicontinuous, it is difficult to discuss the modulus of equi-continuity of such sets. However, in these cases, the Hausdorff measure of noncompactness of the intersections and the modulus of equi-continuity may have some relations.

In this paper, we study such relations. As application, we study some existence results for a class of semilinear functional differential equations with nonlocal conditions in Banach spaces. Precisely, we discuss the model

$$\frac{d}{dt}x(t) = Ax(t) + f(t, x_t), \quad t \in [0, b], \tag{1.1}$$

$$x_0 = \phi + g(x) \in C([-q, 0]; X), \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of linear operators defined on a Banach space X, $x \in C([-q, b]; X)$, and $x_t : [-q, 0] \to X$ defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-q, 0]$; $\phi \in C([-q, 0]; X)$, $f : [0, b] \times C([-q, 0]; X) \to X$ and $g : C([0, b]; X) \to C([-q, 0]; X)$ are appropriate given functions; $b, q > 0$ are constants.

The theory of differential and functional differential equations with nonlocal conditions was initiated by Byszewski and it has been extensively studied in the literature. We refer the readers to [3, 4, 8–11, 13–16, 23, 24, 26–30] and references therein. In this paper, we obtain the existence of mild solutions to (1.1)–(1.2), and the compactness of solution sets, without the assumption of compactness or equicontinuity on the associated semigroups. These results extend and improve the corresponding results in [4, 8–11, 23, 27].

2 Measure of Noncompactness

In this section we first recall the concept of the measure of noncompactness in Banach spaces and some lemmas. Then we study some relations between the Hausdorff measure of noncompactness of the intersections and the modulus of equi-continuity of some subsets in the space of continuous functions.

Definition 2.1 *Let E^+ be a positive cone of an ordered Banach space $(E, \leq)$. A function Φ on the collection of all bounded subsets of a Banach space X with values in E^+ is called a measure of noncompactness if $\Phi(\overline{\text{co}}B) = \Phi(B)$ for all bounded subset $B \subset X$, where $\overline{\text{co}}B$ stands for the closed convex hull of B.*

A measure of noncompactness Φ is said to be

(i) *monotone* if for all bounded subset B_1, B_2 of X, $B_1 \subset B_2$ implies $\Phi(B_1) \leq \Phi(B_2)$;

(ii) *nonsingular* if $\Phi(\{a\} \cup B) = \Phi(B)$ for every $a \in X$ and every nonempty subset $B \subset X$;

(iii) *regular* if $\Phi(B) = 0$ if and only if B is relatively compact in X.

One of the most important examples of measure of noncompactness is the *Hausdorff's measure of noncompactness* β_Y, which is defined by

$$\beta_Y(B) = \inf\{r > 0; B \textit{ can be covered with a finite number of balls of radius equal to } r\}$$

for bounded set B in a Banach space Y.

The following properties of Hausdorff's measure of noncompactness are well known.

Lemma 2.2 ([5]) *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded. The following properties are satisfied:*

(1) *B is pre-compact if and only if $\beta_Y(B) = 0$;*

(2) *$\beta_Y(B) = \beta_Y(\overline{B}) = \beta_Y(\mathrm{conv}B)$, where $\overline{B}$ and $\mathrm{conv}B$ mean the closure and convex hull of B respectively;*

(3) *$\beta_Y(B) \leq \beta_Y(C)$ when $B \subseteq C$;*

(4) *$\beta_Y(B + C) \leq \beta_Y(B) + \beta_Y(C)$, where $B + C = \{x + y; x \in B, y \in C\}$;*

(5) *$\beta_Y(B \cup C) \leq \max\{\beta_Y(B), \beta_Y(C)\}$;*

(6) *$\beta_Y(\lambda B) = |\lambda|\beta_Y(B)$ for any $\lambda \in \mathbb{R}$;*

(7) *If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then $\beta_Z(QB) \leq k\beta_Y(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space;*

(8) *$\beta_Y(B) = \inf\{d_Y(B, C); C \subseteq Y$ is precompact$\} = \inf\{d_Y(B, C); C \subseteq Y$ is finite valued$\}$, where $d_Y(B, C)$ means the nonsymmetric (or symmetric) Hausdorff distance between B and C in Y.*

(9) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to+\infty} \beta_Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.*

The map $Q : W \subseteq Y \to Y$ is said to be a β_Y-*contraction* if there exists a positive constant $k < 1$ such that $\beta_Y(Q(B)) \leq k\beta_Y(B)$ for any bounded closed subset $B \subseteq W$, where Y is a Banach space.

Lemma 2.3 ([5], Darbo–Sadovskii) *If $W \subseteq Y$ is bounded closed and convex, the continuous map $Q : W \to W$ is a β_Y-contraction, then the map Q has at least one fixed point in W.*

Lemma 2.4 ([19]) *Let $W \subset Y$ be bounded closed and convex and $Q : W \to W$ be a continuous β_Y-contraction. If the fixed point set of Q is bounded, then the set of fixed points of Q is compact.*

In this paper we denote by β the Hausdorff's measure of noncompactness of X and by β_c the Hausdorff's measure of noncompactness of $C([a, b]; X)$. To discuss the existence we need the following lemmas in this paper.

Lemma 2.5 ([5]) *If $W \subseteq C([a, b]; X)$ is bounded, then*

$$\beta(W(t)) \leq \beta_c(W)$$

for all $t \in [a, b]$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore, if W is equicontinuous on $[a, b]$, then $\beta(W(t))$ is continuous on $[a, b]$ and

$$\beta_c(W) = \sup\{\beta(W(t)), t \in [a, b]\}.$$

Lemma 2.6 ([18, 19]) *If $\{u_n\}_{n=1}^{\infty} \subset L^1(a, b; X)$ is uniformly integrable, then $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\beta\bigg(\bigg\{\int_a^t u_n(s)ds\bigg\}_{n=1}^{\infty}\bigg) \leq 2\int_a^t \beta(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.1}$$

Lemma 2.7 ([5]) *If $W \subseteq C([a, b]; X)$ is bounded and equicontinuous, then $\beta(W(s))$ is continuous and*

$$\beta\bigg(\int_a^t W(s)ds\bigg) \leq \int_a^t \beta(W(s))ds \tag{2.2}$$

for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

Now we consider another measure of noncompactness in the Banach space $C([a,b];X)$. For a bounded subset $B\in C([a,b];X)$, we define

$$\chi_1(B)=\sup_{t\in[a,b]}\beta(B(t)),$$

then χ_1 is well defined from the properties of Hausdorff's measure of noncompactness. We also define

$$\chi_2(B)=\sup_{t\in[a,b]}\mathrm{mod}_C(B(t)),$$

where $\mathrm{mod}_C(B(t))$ is the modulus of equicontinuity of the set of functions B at point t given by the formula

$$\mathrm{mod}_C(B(t))=\lim_{\delta\to0}\Big\{\sup_{x\in B}\{\sup\{\|x(t_1)-x(t_2)\|:t_1,t_2\in(t-\delta,t+\delta)\}\}\Big\}.$$

Define

$$\chi(B)=\chi_1(B)+\chi_2(B).$$

Then χ is a monotone and nonsingular measure of noncompactness in $C([a,b];X)$. Furthermore, χ is also regular by the famous Ascole–Arzela's theorem. Similar definitions with χ_1, χ_2 and χ can be found in [5].

The following property of regular measure of noncompactness is useful for our results.

Lemma 2.8 *Suppose that Φ is a regular measure of noncompactness in a Banach Y, and $\{B_n\}$ is a sequence of nonempty, closed and bounded subsets in Y satisfying $B_{n+1}\subset B_n$ for $n=1,2,\ldots$. If $\lim_{n\to\infty}\Phi(B_n)=0$, $B=\bigcap_{n\geq1}B_n\neq\emptyset$ and B is a compact subset in Y.*

In the rest of this paper X will represent a Banach space with norm $\|\cdot\|$. As usual, $C([a,b];X)$ denotes the Banach space of all continuous X-valued functions defined on $[a,b]$ with norm $\|x\|_{[a,b]}=\sup_{s\in[a,b]}\|x(s)\|$ for $x\in C([a,b];X)$, and $L([a,b];X)$ denotes the Banach space of all Bochner integrable functions defined on $[a,b]$ with norm $\|x\|_1=\int_a^b\|x(t)\|dt$.

Let $A:D(A)\subset X\to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T(t):t\geq0\}$ of linear operators on X. We always assume that $\|T(t)\|\leq M$ $(M\geq1)$ for every $t\in[0,b]$.

For more details of the semigroup theory we refer the readers to [25].

Consider a uniformly integrable subset $D\subset L([0,b];X)$, i.e., there exists a real integrable function $\eta\in L([0,b];\mathbb{R}^+)$ such that $\|x(t)\|\leq\eta(t)$ a.e. on $[0,b]$ for all $x\in D$. Let $W=\{y:[0,b]\to X;y(t)=T(t)x_0+\int_0^tT(t-s)x(s)ds,t\in[0,b],x\in D\}$, where $\{T(t):t\geq0\}$ is a strongly continuous semigroup of linear operators on X and $x_0\in X$. Then W is a bounded subset in $C([0,b];X)$.

Proposition 2.9 *There exists a constant $K>0$ such that $\chi_2(W)\leq K\chi_1(W)$.*

Proof Let $\chi_1(W)=\lambda$. Then $\beta(W(t))\leq\lambda$ for each $t\in[0,b]$, where $W(t)=\{x(t):x\in W\}$. To prove the proposition, it is sufficient to prove that for every $t_0\in[0,b]$, $\mathrm{mod}_C(W(t_0))\leq K\lambda$.

Take $\varepsilon>0$ arbitrary. First note that, since D is uniformly integrable on $[0,b]$, there is $\delta_1>0$ such that

$$\Big\|\int_{\tau_1}^{\tau_2}T(t-s)x(s)ds\Big\|<\varepsilon \tag{2.3}$$

for $0 < \tau_2 - \tau_1 < 2\delta_1$, $0 \le s \le t \le b$ and every $x \in D$. For every $y \in W$, there is an $x \in D$ such that for all $t \in [0, b]$,

$$y(t) = T(t)x_0 + \int_0^t T(t-s)x(s)ds.$$

If $0 < t' < t$, then we have

$$\begin{aligned} y(t) &= T(t-t')T(t')x_0 + T(t-t')\int_0^{t'} T(t'-s)x(s)ds + \int_{t'}^t T(t-s)x(s)ds \\ &= T(t-t')y(t') + \int_{t'}^t T(t-s)x(s)ds. \end{aligned}$$

It follows that for $t_1, t_2 \in (t_0 - \delta_1, t_0 + \delta_1)$ (we may assume that $t_0 - \delta_1 > 0$),

$$y(t_1) = T(t_1 - t_0 + \delta_1)y(t_0 - \delta_1) + \int_{t_0-\delta_1}^{t_1} T(t_1 - s)x(s)ds, \tag{2.4}$$

$$y(t_2) = T(t_2 - t_0 + \delta_1)y(t_0 - \delta_1) + \int_{t_0-\delta_1}^{t_2} T(t_2 - s)x(s)ds. \tag{2.5}$$

Now, on the basis of the definition of Hausdorff's measure of noncompactness and the fact that $\beta(W(t_0 - \delta_1)) \le \lambda$, we may find $y_1, y_2, \ldots, y_k$, such that

$$W(t_0 - \delta_1) \subset \bigcup_{i=1}^k B(y_i(t_0 - \delta_1), 2\lambda),$$

and hence there is an i, $1 \le i \le k$ such that

$$\|y(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| < 2\lambda. \tag{2.6}$$

On the other hand, on account of the strong continuity of $T(\cdot)$, there is a $\delta > 0$ (we may choose $\delta < \delta_1$) such that

$$\|T(\tau)y_i(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| < \varepsilon \tag{2.7}$$

for all $\tau \in (0, \delta)$ and $i = 1, 2, \ldots, k$. From (2.3)–(2.7), we obtain

$$\begin{aligned} \|y(t_1) - y(t_2)\| \le\ & \|T(t_1 - t_0 + \delta_1)y_i(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| \\ & + 2M\|y(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| \\ & + \|T(t_2 - t_0 + \delta_1)y_i(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| \\ & + \left\| \int_{t_0-\delta_1}^{t_1} T(t_1 - s)x(s)ds - \int_{t_0-\delta_1}^{t_2} T(t_2 - s)x(s)ds \right\| \\ \le\ & 4M\lambda + 4\varepsilon \end{aligned}$$

for $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$. Since $\varepsilon > 0$ is taken arbitrary, we get that

$$\|y(t_1) - y(t_2)\| \le 4M\lambda$$

for $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$, which implies that

$$\mathrm{mod}_C(W(t_0)) \le 4M\lambda$$

for every $t_0 \in [0, b]$. Accordingly,

$$\chi_2(W) \le 4M\lambda = 4M\chi_1(W).$$

3 Semilinear Nonlocal Functional Differential Equations

In this section, we discuss a class of semilinear nonlocal functional differential equations in Banach spaces. Using the theory of measure of noncompactness, the results we get in the last section (i.e., Proposition 2.9), and the famous Schauder fixed point theorem, we obtain the existence of mild solution to (1.1)–(1.2), and the compactness of solution set. We will not assume the compactness or equicontinuity on the associated semigroup.

In order to define the concept of mild solution for (1.1)–(1.2), by comparison with the abstract Cauchy initial value problem

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \quad x(0) = \overline{x} \in X,$$

whose properties are well known [25], we associate (1.1)–(1.2) to the integral equation

$$x(t) = T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t-s)f(s, x_s)ds, \quad t \in [0, b]. \tag{3.1}$$

Definition 3.1 *A continuous function $x : [-q, b] \to X$ is said to be a mild solution to the nonlocal problem* (1.1)–(1.2) *if $x_0 = \phi + g(x)$ and* (3.1) *is satisfied.*

Here we list the following hypotheses.

(Hf) (1) $f : [0, b] \times C([-q, 0]; X) \to X$ satisfies the Carathéodory-type condition, i.e., $f(\cdot, v) : [0, b] \to X$ is measurable for all $v \in C([-q, 0]; X)$ and $f(t, \cdot) : C([-q, 0]; X) \to X$ is continuous for a.e. $t \in [0, b]$;

(2) There exists a function $h : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(\cdot, s) \in L(0, b; \mathbb{R}^+)$ for every $s \geq 0$, $h(t, \cdot)$ is continuous and increasing for a.e. $t \in [0, b]$, and $\|f(t, v)\| \leq h(t, \|v\|_{[-q,0]})$ for a.e. $t \in [0, b]$ and all $v \in C([-q, 0]; X)$, and for each positive constant K, the following scalar equation

$$m(t) = MK + M\int_0^t h(s, m(s))ds, \quad t \in [0, b] \tag{3.2}$$

has at least one solution;

(3) There exists $\eta \in L(0, b; \mathbb{R}^+)$ such that

$$\beta(T(s)f(t, D)) \leq \eta(t) \sup_{-q \leq \theta \leq 0} \beta(D(\theta))) \tag{3.3}$$

for a.e. $t, s \in [0, b]$ and any bounded subset $D \subset C([-q, 0]; X)$.

(Hg) (1) $g : C([0, b]; X) \to C([-q, 0]; X)$ is continuous and compact;

(2) There exists a constant $N > 0$ such that

$$\|g(x)\|_{[-q,0]} \leq N \tag{3.4}$$

for all $x \in C([0, b]; X)$.

Remark 3.2 If the semigroup $\{T(t) : t \geq 0\}$ or the function f is compact (see, e.g., [10, 23, 24, 27]), or f satisfies Lipschitz-type condition (see, e.g., [3, 8, 9], then (Hf)(3) is automatically satisfied.

Now, we give an existence result under the above hypotheses. Some of the ideas are from [27].

Theorem 3.3 *Assume the hypotheses* (Hf) *and* (Hg) *are satisfied. Then for each $\phi \in C([-q, 0]; X)$, the solution set of the problem* (1.1)–(1.2) *is a nonempty compact subset of the space $C([-q, b]; X)$.*

Proof Let $m : [-q, b] \to \mathbb{R}^+$ be the function such that $m(t) = M(\|\phi\|_{[-q,0]} + N)$ for $t \in [-q, 0]$ and the restriction of m on $[0, b]$ be a solution of the scalar equation:

$$m(t) = M(\|\phi\|_{[-q,0]} + N) + M \int_0^t h(s, m(s))ds, \quad t \in [0, b]. \tag{3.5}$$

For each $x \in C([-q, b]; X)$, the restriction of x on $[0, b]$ $x|_{[0,b]} \in C([0, b]; X)$. For simplicity, we write $g(x|_{[0,b]})$ as $g(x)$.

Define a map $\Gamma : C([-q, b]; X) \to C([-q, b]; X)$ by

$$\Gamma x(t) = \begin{cases} \phi(t) + g(x)(t), & t \in [-q, 0], \\ T(t)(\phi(0) + g(x)(0)) + \int_0^t T(t - s)f(s, x_s)ds, & t \in [0, b] \end{cases}$$

for all $x \in C([-q, b] : X)$. It is easily seen that $x \in C([-q, b]; X)$ is a mild solution of the problem (1.1)–(1.2) if and only if x is a fixed point of Γ. We shall show that Γ has a fixed point by Schauder's fixed point theorem. To do this, we first see that Γ is continuous by the usual technique involving (Hf), (Hg) and Lebesgue's dominate convergence theorem.

We denote by $W_0 = \{x \in C([-q, b]; X) : \sup_{-q\le s\le t} \|x(s)\| \le m(t), \forall t \in [-q, b]\}$. Then $W_0 \subset C([-q, b]; X)$ is bounded and convex.

Define $W_1 = \overline{\mathrm{conv}}\Gamma W_0$, where $\overline{\mathrm{conv}}$ means the closure of the convex hull in $C([-q, b]; X)$. Then it is easily seen that $W_1 \subset C([-q, b]; X)$ is closed and convex. Furthermore, for every $x \in C([-q, b]; X)$, we have

$$\|\Gamma x(t)\| \le \|\phi\|_{[-q,0]} + N \le M(\|\phi\|_{[-q,0]} + N)$$

for $t \in [-q, 0]$ and

$$\|\Gamma x(t)\| \le M(\|\phi\|_{[-q,0]} + N) + M \int_0^t h(s, m(s))ds$$

for $t \in [0, b]$. Hence

$$\sup_{-q\le s\le t} \|\Gamma x(s)\| \le m(t)$$

for $t \in [-q, 0]$ and

$$\begin{aligned} \sup_{-q\le s\le t} \|\Gamma x(s)\| &\le M(\|\phi\|_{[-q,0]} + N) + M \sup_{-q\le s\le t} \int_0^s h(r, m(r))dr \\ &\le M(\|\phi\|_{[-q,0]} + N) + M \int_0^t h(s, m(s))ds \\ &= m(t). \end{aligned}$$

It then follows that W_1 is bounded and $W_1 \subset W_0$ is equicontinuous on $[-q, 0]$ by the compactness of g involving Ascoli–Arzela's theorem.

Define $W_{n+1} = \overline{\mathrm{conv}}\Gamma W_n$ for $n = 1, 2, \ldots$. From the above proof we have that $\{W_n\}_{n=1}^{\infty}$ is a decreasing sequence of equicontinuous on $[-q, 0]$, bounded closed convex and nonempty subsets in $C([-q, b]; X)$.

Now, for every $n \ge 1$ and $t \in [-q, 0]$, $\beta(W_n(t)) = 0$ by the compactness of g. For $t \in (0, b]$, $W_n(t)$ and $\Gamma W_n(t)$ are bounded subsets of X. Hence, for any $\varepsilon > 0$, there is a sequence $\{x_k\}_{k=1}^{\infty} \subset W_n$ such that (see, e.g., [7, p. 125])

$$\beta(\Gamma W_n(t)) \le 2\beta(\{\Gamma x_k(t)\}_{k=1}^{\infty}) + \varepsilon$$

$$\le 2\beta(T(t)g(\{x_k(t)\}_{k=1}^{\infty}) \\ +2\beta\bigg(\int_0^t T(t-s)f(s,\{x_{ks}\}_{k=1}^{\infty})ds\bigg)+\varepsilon,$$

where $x_{ks}=(x_k)_s$. From the compactness of g, Lemmas 2.2–2.7 and (Hf)(3), we have

$$\begin{aligned}\beta(W_{n+1}(t)) &= \beta(\Gamma W_n(t)) \\ &\le 2\beta(\int_0^t T(t-s)f(s,\{x_{ks}\}_{k=1}^{\infty})ds)+\varepsilon \\ &\le 2\int_0^t \beta(T(t-s)f(s,\{x_{ks}\}_{k=1}^{\infty}))ds+\varepsilon \\ &\le 2\int_0^t \eta(s)\sup_{-q\le\theta\le 0}\beta(\{x_k(\theta+s)\}_{k=1}^{\infty})ds+\varepsilon \\ &\le 2\int_0^t \eta(s)\sup_{-q\le\tau\le s}\beta(\{x_k(\tau)\}_{k=1}^{\infty})ds+\varepsilon \\ &\le 2\int_0^t \eta(s)\sup_{-q\le\tau\le s}\beta(W(\tau))ds+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, it follows from the above inequality that

$$\beta(W_{n+1}(t))\le 2\int_0^t \eta(s)\sup_{-q\le\tau\le s}\beta(W(\tau))ds \tag{3.6}$$

for all $t\in(0,b]$. Define functions $f_n:[-q,b]\to[0,+\infty)$ by

$$f_n(t)=\sup_{-q\le\tau\le t}\beta(W_n(\tau)).$$

Observing that $\beta(W_n(t))=0$ for $t\in[-q,0]$, we obtain from (3.6) that

$$f_{n+1}(t)=\sup_{0\le\tau\le t}\beta(W_{n+1}(t))\le 2\sup_{0\le\tau\le t}\int_0^{\tau}\eta(s)f_n(s)ds=2\int_0^t\eta(s)f_n(s)ds \tag{3.7}$$

for all $t\in[0,b]$.

Notice that $\chi_1(W_n)=\sup_{-q\le t\le b}\beta(W_n(t))=\sup_{-q\le t\le b}f_n(t)$. Now we consider $\chi_2(W_n)$. According to Proposition 2.9, there is a constant $K>0$ such that

$$\chi_2(W_n)\le K\chi_1(W_n). \tag{3.8}$$

Come back to consider f_n. Since W_n is decreasing for n, we know that $f(t)=\lim_{n\to\infty}f_n(t)$ exists for $t\in[-q,b]$ (notice that $f_n(t)=0$ for all $t\in[-q,0]$ and $n\ge 1$). Taking limit as $n\to\infty$ in (3.7), we have

$$f(t)\le 2\int_0^t\eta(s)f(s)ds \tag{3.9}$$

for $t\in[0,b]$. It follows that $f(t)=0$ for all $t\in[0,b]$, and therefore, for all $t\in[-q,b]$. This means that $\lim_{n\to\infty}\chi_1(W_n)=0$. Then the inequality (3.8) implies that $\lim_{n\to\infty}\chi_2(W_n)=0$, and hence $\lim_{n\to\infty}\chi(W_n)=0$. Using Lemma 2.8 we know that $W=\bigcap_{n=1}^{\infty}W_n$ is a convex compact and nonempty subset in $C([-q,b];X)$ and $\Gamma W\subset W$. According to the famous Schauder's fixed point theorem, there exists at least one fixed point $x\in W$ of Γ, which is the mild solution of (1.1)–(1.2). From the proof we can also see that all the fixed points of Γ are in W which is compact in $C([-q,b];X)$. The continuity of the map Γ implies the closeness of the fixed point set. Hence we conclude that the solution set of problem (1.1)–(1.2) is compact in the space $C([-q,b];X)$.

Since $\{T(t)\}$ is a C_0-semigroup, the condition (Hf)(3) can be replaced by

(Hf)(3′): There exists $\tilde{\eta} \in L(0,b;\mathbb{R}^+)$ such that

$$\beta(f(t,D)) \le \frac{\tilde{\eta}(t)}{M} \cdot \sup_{-q\le\theta\le 0} \beta(D(\theta)) \tag{3.10}$$

for a.e. $t,s \in [0,b]$ and any bounded subset $D \subset C([-q,0];X)$.

From Theorem 3.3, we can get the following obvious result:

Theorem 3.4 *Assume the hypotheses* (Hf)(1)(2)(3′) *and* (Hg) *are satisfied. Then for each* $\phi \in C([-q,0];X)$*, the solution set of the problem* (1.1)–(1.2) *is a nonempty compact subset of the space* $C([-q,b];X)$*.*

In some of the early related results in references and the two results above, it is supposed that the map g is uniformly bounded. We indicate here that this condition can be released. Indeed, the fact that g is compact implies that g is bounded on bounded subset. And the hypothesis (Hf)(2) may be difficult to be verified sometimes. Here we give an existence result under another growth condition of f when g is not uniformly bounded. Precisely, we replace the hypothesis (Hf)(2) by

(Hf)(2′): There exists a function $\alpha \in L(0,b;\mathbb{R}^+)$ and an increasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,v)\| \le \alpha(t)\Omega(\|v\|_{[-q,0]})$$

for a.e. $t \in [0,b]$ and all $v \in C([-q,0];X)$.

Theorem 3.5 *Suppose that the hypotheses* (Hf)(1)(2′)(3) *and* (Hg)(1) *are satisfied. If*

$$\limsup_{k\to\infty} \frac{M}{k}\left(\gamma(k) + \Omega(k)\int_0^b \alpha(s)ds\right) < 1, \tag{3.11}$$

where $\gamma(k) = \sup\{\|g(x)\|_{[-q,0]} : \|x\|_{[0,b]} \le k\}$*, then for each* $\phi \in C([-q,0];X)$*, the solution set of the problem* (1.1)–(1.2) *is a nonempty compact subset of the space* $C([-q,b];X)$*.*

Proof The inequality (3.11) implies that there exists a constant $k > 0$ such that

$$\|\phi\|_{[-q,0]} + \gamma(k) < k$$

and

$$M\left(\|\phi\|_{[-q,0]} + \gamma(k) + \Omega(k)\int_0^b \alpha(s)ds\right) < k.$$

As in the proof of Theorem 3.3, let $W_0 = \{x \in C([-q,b];X) : \|x\|_{[-q,b]} \le k\}$ and $W_1 = \overline{\mathrm{conv}}\Gamma W_0$. Then for any $x \in W_1$, we have

$$\|x(t)\| \le \|\phi\|_{[-q,0]} + \gamma(k) < k$$

for $t \in [-q,0]$, and

$$\|x(t)\| \le M(\|\phi\|_{[-q,0]} + \gamma(k)) + M\Omega(k)\int_0^b \alpha(s)ds < k$$

for $t \in [0,b]$. It follows that $W_1 \subset W_0$. So we can complete the proof similarly to Theorem 3.3.

Remark 3.6 In some previous papers the authors assumed that the space X is a separable Banach space and the semigroup $T(t)$ is equicontinuous (see, e.g., [27, 28]). We mention here that these assumptions are not necessary.

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