Acta Mathematica Sinica, English Series
Feb., 2015, Vol. 31, No. 2, pp. 331–344
Published online: January 15, 2015
DOI: 10.1007/s10114-015-2687-0
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Energy Decay in Thermoelasticity with Viscoelastic Damping of General Type

Muhammad I. MUSTAFA
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, P.O. Box 860, Dhahran 31261, Saudi Arabia
E-mail: *mmustafa@kfupm.edu.sa*

Abstract In this paper we consider an n-dimensional thermoelastic system with viscoelastic damping. We establish an explicit and general decay rate result without imposing restrictive assumptions on the behavior of the relaxation function at infinity. Our result allows a larger class of relaxation functions and generalizes previous results existing in the literature.

Keywords Thermoelasticity, viscoelastic damping, general decay, convexity

MR(2010) Subject Classification 35B37, 35L55, 74D05, 93D15, 93d20

1 Introduction

In this paper we are concerned with the following problem:

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div} u) + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + \beta\nabla\theta = 0, & \text{in } \Omega\times(0,\infty), \\ b\theta_t - k\Delta\theta + \beta\mathrm{div} u_t = 0, & \text{in } \Omega\times(0,\infty), \\ u(x,t) = \theta(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

a memory-type thermoelastic system associated with homogeneous Dirichlet boundary conditions and initial data in suitable function spaces. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\partial\Omega$, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature, and the relaxation function g is a positive nonincreasing function. The coefficients $b, k, \beta, \mu, \lambda$ are positive constants, where μ, λ are Lame moduli. In this work, we study the decay properties of the solutions of (1.1) for functions g of general-type decay.

Over the past few decades, there has been a lot of work on local existence, global existence, well-posedeness, and asymptotic behavior of solutions to some initial-boundary value problems in both one-dimensional and multi-dimensional thermoelasticity. In the absence of the viscoelastic term, it is well known (see [3, 5, 11]) that the one-dimensional linear thermoelastic system associated with various types of boundary conditions decays to zero exponentially. Irmscher and Racke [6] obtained explicit sharp exponential decay rates for solutions of the system of classical thermoelasticity in one dimension. They also considered the model of thermoelasticity with

Received November 8, 2012, revised April 13, 2014, accepted July 2, 2014
Supported by KFUPM (Grant No. #IN101029)

second sound and compared the results of both models with respect to the asymptotic behavior of solutions. Also, Rivera and Qin [14, 19] established the global existence, uniqueness and exponential stability of solutions to equations of one-dimensional nonlinear thermoelasticity with thermal memory subject to Dirichlet–Dirichlet or Dirichlet–Neumann boundary conditions.

In the multi-dimensional case the situation is much different. It was shown that the dissipation given by heat conduction is not strong enough to produce uniform rate of decay to the solution as in the one-dimensional case. We have the pioneering work of Dafermos [4], in which he proved an asymptotic stability result; but no rate of decay has been given. The uniform rate of decay for the solution in two or three dimensional space was obtained by Jiang et al. [8] in special situation like radial symmetry. Lebeau and Zuazua [9] proved that the decay rate is never uniform when the domain is convex. Thus, in order to solve this problem, additional damping mechanisms are necessary. In this aspect, Pereira and Menzala [18] introduced a linear internal damping effective in the whole domain, and established the uniform decay rate. A similar result was obtained by Liu [10] for a linear boundary velocity feedback acting on the elastic component of the system, and by Liu and Zuazua [12] for a nonlinear boundary feedback. Oliveira and Charao [17] improved the result in [18] by including a weak localized dissipative term effective only in a neighborhood of part of the boundary and proved an exponential decay result when the damping term is linear and a polynomial decay result for a nonlinear damping term. Recently, Mustafa [16] treated weak frictional damping of more general type and established an explicit decay result. For more literature on the subject, we refer the reader to books by Jiang and Racke [7] and Zheng [20].

Regarding viscoelastic damping, we mention the work of Rivera and Racke [15] who considered magneto-thermoelastic model with a boundary condition of memory type. If g is the relaxation function and k is the resolvent kernel of $\frac{-g'}{g(0)}$, they showed that the energy of the solution decays exponentially (polynomially) when k and $(-k')$ decay exponentially (polynomially). In [13], Messaoudi and Al-Shehri considered a wider class of kernels k that are not necessarily decaying exponentially or polynomially and proved a more general energy decay result. In this context, we refer to the work [1] by Alabau-Boussouira and Cannarsa in which the authors considered the following viscoelastic problem:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

where g is a positive function satisfying

$$g'(t) \leq -\chi\left(g(t)\right), \tag{1.2}$$

where χ is a nonnegative function, with $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex on $(0, k_0]$, for some $k_0 > 0$. They also required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \quad \liminf_{s\to 0^+} \frac{\frac{\chi(s)}{s}}{\chi'(s)} > \frac{1}{2}$$

and proved an energy decay result. In addition to these assumptions, if

$$\limsup_{s\to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1 \quad \text{and} \quad g'(t) = -\chi\left(g(t)\right) \tag{1.3}$$

then, in this case, an explicit rate of decay is given.

Our aim in this work is to investigate (1.1) for relaxation functions satisfying (1.2) and obtain a general relation between the decay rate of the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity. We provide an explicit energy decay formula that allows a larger class of functions g than that of [1] and the usual exponential and polynomial decay rates are only special cases of our result. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. The paper is organized as follows. In Section 2, we present some notation and material needed for our work. Some technical lemmas and the proof of our main result are given in Section 3.

2 Preliminaries

We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant. We also consider the following assumption:

(A) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad \mu - \int_0^{+\infty} g(s)ds = l > 0 \tag{2.1}$$

and there exists a positive function $H \in C^1(\mathbb{R}_+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$g'(t) \le -H(g(t)), \quad \forall t > 0.$$

For completeness we state, without proof, the following existence and regularity result.

Proposition 2.1 *Let* $(u_0, u_1, \theta_0) \in H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega)$ *be given. Assume that* (A) *is satisfied. Then the problem* (1.1) *has a unique global* (*weak*) *solution*

$$u \in C(\mathbb{R}_+; H_0^1(\Omega)^n) \cap C^1(\mathbb{R}_+; L^2(\Omega)^n),$$
$$\theta \in C(\mathbb{R}_+; L^2(\Omega)).$$

Moreover, if

$$(u_0, u_1, \theta_0) \in (H^2(\Omega)^n \cap H_0^1(\Omega)^n) \times H_0^1(\Omega)^n \times (H^2(\Omega) \cap H_0^1(\Omega)),$$

then the solution satisfies

$$u \in C(\mathbb{R}_+; H^2(\Omega)^n \cap H_0^1(\Omega)^n) \cap C^1(\mathbb{R}_+; H_0^1(\Omega)^n) \cap C^2(\mathbb{R}_+; L^2(\Omega)^n),$$
$$\theta \in C(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)).$$

Remark 2.2 This result can be proved using standard arguments such as the Galerkin method.

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_\Omega \left(|u_t|^2 + \left(\mu - \int_0^t g(s)ds\right)|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}u)^2 + b\theta^2 \right) dx + \frac{1}{2}(g\circ \nabla u)(t),$$

where $|\nabla u|^2 = \sum_{i=1}^n |\nabla u_i|^2$ and

$$(g\circ v)(t) = \int_\Omega \int_0^t g(t-s)\,|v(t)-v(s)|^2\,dsdx.$$

By multiplying the first equation in (1.1) by u_t and the second equation by θ, adding the resulting equations, and integrating over Ω, we obtain

$$E'(t) = -k\int_\Omega |\nabla\theta|^2\,dx + \frac{1}{2}(g'\circ\nabla u) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2\,dx. \tag{2.2}$$

Then, the hypothesis (A) implies that the energy is a nonincreasing function of t.

Our main stability result is the following

Theorem 2.3 *Assume that* (A) *holds. Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of* (1.1) *satisfies*

$$E(t) \le k_3 H_1^{-1}(k_1 t + k_2), \quad \forall t \ge 0, \tag{2.3}$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)}ds \quad \text{and} \quad H_0(t) = H(D(t))$$

provided that D *is a positive* C^1 *function, with* $D(0) = 0$, *for which* H_0 *is strictly increasing and strictly convex* C^2 *function on* $(0, r]$ *and*

$$\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))}ds < +\infty. \tag{2.4}$$

Moreover, if $\int_0^1 H_1(t)dt < +\infty$ *for some choice of* D, *then we have the improved estimate*

$$E(t) \le k_3 G^{-1}(k_1 t + k_2), \quad \text{where } G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)}ds. \tag{2.5}$$

In particular, this last estimate is valid for the special case $H(t) = ct^p$ *where* $1 \le p < \frac{3}{2}$.

Remark 2.4 1) Using the properties of H, one can show that the function H_1 is strictly decreasing and convex on $(0, 1]$, with $\lim_{t\to 0} H_1(t) = +\infty$. Therefore, Theorem 2.2 ensures

$$\lim_{t\to+\infty} E(t) = 0.$$

2) Our result is obtained under very general hypotheses on the relaxation function g that allow to deal with a much larger class of functions g that guarantee the uniform stability of (1.1) with an explicit formula for the decay rates of the energy.

3) The usual exponential and polynomial decay rate estimates, already proved for g satisfying (2.1) and $g' \le -kg^p$, $1 \le p < \frac{3}{2}$, are special cases of our result. We will provide a "simpler" proof for these special cases.

4) Our result allows relaxation functions which are not necessarily of exponential or polynomial decay. For instance, if

$$g(t) = a\exp(-t^q)$$

for $0 < q < 1$ and a is chosen so that g satisfies (2.1), then $g'(t) = -H(g(t))$ where for $t \in (0, r],\ r < a$,

$$H(t) = \frac{qt}{[\ln(\frac{a}{t})]^{\frac{1}{q}-1}},$$

which satisfies the hypothesis (A). Also, by taking $D(t) = t^\alpha$, (2.4) is satisfied for any $\alpha > 1$. Therefore, we can use Theorem 2.2 and do some calculations (see Appendix) to deduce that the energy decays at the same rate of g, that is,

$$E(t) \leq c\exp(-\omega t^q).$$

One can show that this example does not satisfy (1.6), and so no explicit rate of decay for this case is given in [1].

5) The well-known Jensen's inequality will be of essential use in establishing our main result. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and h are integrable functions on Ω, $h(x) \geq 0$, and $\int_\Omega h(x)dx = k > 0$, then Jensen's inequality states that

$$F\left[\frac{1}{k}\int_\Omega f(x)h(x)dx\right] \leq \frac{1}{k}\int_\Omega F[f(x)]h(x)dx.$$

6) By (A), we easily deduce that $\lim_{t\to+\infty} g(t) = 0$. This implies that $\lim_{t\to+\infty}(-g'(t))$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty}(-g'(t)) = 0$. Hence, there is $t_1 > 0$ large enough such that $g(t_1) > 0$ and

$$\max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \geq t_1. \tag{2.6}$$

As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$, then $g(t) > 0$ for any $t \in [0, t_1]$ and

$$0 < g(t_1) \leq g(t) \leq g(0), \quad \forall t \in [0, t_1].$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b, \quad \forall t \in [0, t_1]$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -H(g(t)) \leq -a = -\frac{a}{g(0)}g(0) \leq -\frac{a}{g(0)}g(t),$$

which gives, for some positive constant d,

$$g'(t) \leq -dg(t), \quad \forall t \in [0, t_1]. \tag{2.7}$$

3 Proof of Main Result

In this section we prove Theorem 2.2. For this purpose, we establish several lemmas.

Lemma 3.1 Under the assumption (A), the functional

$$K_1(t) := \int_\Omega u \cdot u_t dx$$

satisfies, along the solution of (1.1), the estimate

$$K_1'(t) \leq -\frac{l}{2}\int_\Omega |\nabla u|^2 dx - (\mu+\lambda)\int_\Omega (\mathrm{div} u)^2 dx + \int_\Omega |u_t|^2 dx + c\int_\Omega |\nabla\theta|^2 dx + c(g\circ\nabla u)(t). \tag{3.1}$$

Proof Direct computations, using (1.1) and (2.1), yield

$$\begin{aligned} K_1'(t) &= \int_\Omega (|u_t|^2 + \mu u \cdot \Delta u + (\mu+\lambda) u \cdot \nabla(\mathrm{div} u) - \beta u \cdot \nabla\theta) dx \\ &\quad - \int_\Omega \int_0^t g(t-s) u(t) \cdot \Delta u(s) ds dx \\ &\le \int_\Omega (|u_t|^2 - l|\nabla u|^2 - (\mu+\lambda)(\mathrm{div} u)^2 - \beta u \cdot \nabla\theta) dx \\ &\quad - \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot (\nabla u(s) - \nabla u(t)) ds dx. \end{aligned}$$

By Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} K_1'(t) &\le \int_\Omega (|u_t|^2 - l\,|\nabla u|^2 - (\mu+\lambda)(\mathrm{div} u)^2) dx + \delta \int_\Omega |u|^2\, dx + \frac{\beta^2}{4\delta} \int_\Omega |\nabla\theta|^2\, dx \\ &\quad + \delta \left(\int_0^t g(s) ds\right) \int_\Omega |\nabla u|^2\, dx + \frac{1}{4\delta} \int_\Omega \int_0^t g(t-s)\, |\nabla u(s) - \nabla u(t)|^2\, ds dx \\ &\le \int_\Omega (|u_t|^2 - l\,|\nabla u|^2 - (\mu+\lambda)(\mathrm{div} u)^2) dx + \delta c \int_\Omega |\nabla u|^2\, dx + \frac{\beta^2}{4\delta} \int_\Omega |\nabla\theta|^2\, dx \\ &\quad + \delta\mu \int_\Omega |\nabla u|^2\, dx + \frac{1}{4\delta} (g \circ \nabla u)(t), \end{aligned}$$

which by choosing δ small enough, gives (3.1). □

Lemma 3.2 *Under the assumption* (A), *the functional*

$$K_2(t) := -\int_\Omega u_t(t) \cdot \int_0^t g(t-s)(u(t) - u(s)) ds dx$$

satisfies for any $0 < \delta < 1$, *along the solution of* (1.1), *the estimate*

$$\begin{aligned} K_2'(t) &\le -\left(\int_0^t g(s) ds - \delta\right) \int_\Omega |u_t|^2\, dx + \delta \int_\Omega |\nabla u|^2\, dx \\ &\quad + \frac{c}{\delta} (g \circ \nabla u)(t) - \frac{c}{\delta} (g' \circ \nabla u)(t) + c \int_\Omega |\nabla\theta|^2\, dx. \end{aligned} \tag{3.2}$$

Proof By exploiting the equation (1.1) and integrating by parts, we have

$$\begin{aligned} K_2'(t) &= \mu \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot (\nabla u(t) - \nabla u(s)) ds dx \\ &\quad + (\mu+\lambda) \int_\Omega \int_0^t g(t-s)(\mathrm{div}[u(t)])(\mathrm{div}[u(t) - u(s)]) ds dx \\ &\quad + \beta \int_\Omega \int_0^t g(t-s) \nabla\theta(t) \cdot (u(t) - u(s)) ds dx \\ &\quad - \int_\Omega \left(\int_0^t g(t-s) \nabla u(s) ds\right) \cdot \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds\right) dx \\ &\quad - \int_\Omega \int_0^t g'(t-s) u_t(t) \cdot (u(t) - u(s)) ds dx - \left(\int_0^t g(s) ds\right) \int_\Omega |u_t|^2\, dx. \end{aligned} \tag{3.3}$$

Using Cauchy–Schwarz and Young's inequalities, we obtain

$$\mu \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot (\nabla u(t) - \nabla u(s)) ds dx$$

$$
\begin{aligned}
&-\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&=\left(\mu-\int_0^t g(s)ds\right)\int_\Omega\int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(t)-\nabla u(s))dsdx\\
&\quad+\int_\Omega\left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2 dx\\
&\leq \frac{\delta}{2}\int_\Omega |\nabla u|^2\,dx+\frac{c}{\delta}(g\circ\nabla u)(t).
\end{aligned}\tag{3.4}
$$

Also, the use of Young's and Poincaré's inequalities gives

$$
\begin{aligned}
&(\mu+\lambda)\int_\Omega\int_0^t g(t-s)(\mathrm{div}[u(t)])(\mathrm{div}[u(t)-u(s)])dsdx\\
&\quad+\beta\int_\Omega\int_0^t g(t-s)\nabla\theta(t)\cdot(u(t)-u(s))dsdx-\int_\Omega\int_0^t g'(t-s)u_t(t)\cdot(u(t)-u(s))dsdx\\
&\leq\frac{\delta}{2}\int_\Omega|\nabla u|^2\,dx+\delta\int_\Omega|u_t|^2\,dx+\frac{c}{\delta}(g\circ\nabla u)(t)-\frac{c}{\delta}(g'\circ\nabla u)(t)+c\int_\Omega|\nabla\theta|^2\,dx.
\end{aligned}\tag{3.5}
$$

Combining (3.3)–(3.5), (3.2) is established. □

Proof of Theorem 2.3 For $N_1, N_2 > 1$, define

$$\mathcal{L}(t) := N_1E(t)+K_1(t)+N_2K_2(t)$$

and let $g_1=\int_0^{t_1} g(s)ds>0$, where t_1 was introduced in (2.6). By combining (2.2), (3.1), (3.2) and taking $\delta=\frac{l}{4N_2}$, we obtain, for all $t\geq t_1$,

$$
\begin{aligned}
\mathcal{L}'(t)\leq&-\frac{l}{4}\int_\Omega|\nabla u|^2\,dx-\left(N_2g_1-\frac{l}{4}-1\right)\int_\Omega|u_t|^2\,dx-(\mu+\lambda)\int_\Omega(\mathrm{div}u)^2dx\\
&-(kN_1-c-cN_2)\int_\Omega|\nabla\theta|^2\,dx+\left(\frac{4c}{l}N_2^2+c\right)(g\circ\nabla u)(t)\\
&+\left(\frac{1}{2}N_1-\frac{4c}{l}N_2^2\right)(g'\circ\nabla u)(t).
\end{aligned}\tag{3.6}
$$

At this point, we choose N_2 large enough so that

$$\gamma_1 := N_2g_1-\frac{l}{4}-1>0.$$

Then N_1 is large enough so that

$$\gamma_2 := kN_1-c-cN_2>0$$

and

$$\frac{1}{2}N_1-\frac{4c}{l}N_2^2>0.$$

So, we arrive at

$$\mathcal{L}'(t)\leq-\int_\Omega\left[\frac{l}{4}|\nabla u|^2\,dx+\gamma_1|u_t|^2+(\mu+\lambda)(\mathrm{div}u)^2+\gamma_2|\nabla\theta|^2\right]dx+c(g\circ\nabla u)(t),$$

which, using Poincaré's inequality, yields

$$\mathcal{L}'(t)\leq -mE(t)+c(g\circ\nabla u)(t),\quad \forall t\geq t_1.\tag{3.7}$$

On the other hand, we find that

$$\begin{aligned}|\mathcal{L}(t) - N_1E(t)| &\le |K_1(t)| + N_2\,|K_2(t)| \\ &\le \int_\Omega |u\cdot u_t|\,dx + N_2\int_\Omega \left|u_t(t)\cdot\int_0^t g(t-s)(u(t)-u(s))ds\right|dx \\ &\le \frac{1}{2}\int_\Omega |u|^2\,dx + \frac{1+N_2}{2}\int_\Omega |u_t|^2\,dx + \frac{N_2}{2}\int_\Omega \left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2 dx \\ &\le c\left[\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u_t|^2\,dx + (g\circ\nabla u)(t)\right] \le cE(t).\end{aligned}$$

Therefore, we can choose N_1 even larger (if needed) so that

$$\mathcal{L}(t) \sim E(t). \tag{3.8}$$

Now, we use (2.2) and (2.7) to conclude that, for any $t \ge t_1$,

$$\begin{aligned}\int_0^{t_1} g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dxds &\le -\frac{1}{d}\int_0^{t_1} g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dxds \\ &\le -cE'(t).\end{aligned} \tag{3.9}$$

Next, we take $F(t) = \mathcal{L}(t) + cE(t)$, which is clearly equivalent to $E(t)$, and use (3.7) and (3.9), to get, for all $t \ge t_1$,

$$F'(t) \le -mE(t) + c\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dxds. \tag{3.10}$$

(I) $H(t) = ct^p$ and $1 \le p < \frac{3}{2}$.

Case 1 $p = 1$: Estimate (3.10) yields

$$F'(t) \le -mE(t) + c(g'\circ\nabla u)(t) \le -mE(t) - cE'(t), \quad \forall t \ge t_1,$$

which gives

$$(F + cE)'(t) \le -mE(t), \quad \forall t \ge t_1.$$

Hence, using the fact that $F + cE \sim E$, we easily obtain

$$E(t) \le c'\mathrm{e}^{-ct} = c'G^{-1}(t).$$

Case 2 $1 < p < \frac{3}{2}$: One can easily show that $\int_0^{+\infty} g^{1-\delta_0}(s)ds < +\infty$ for any $\delta_0 < 2 - p$. Using this fact and (2.2) and choosing t_1 even larger if needed, we deduce that, for all $t \ge t_1$,

$$\begin{aligned}\eta(t) := &\int_{t_1}^t g^{1-\delta_0}(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dxds \\ &\le 2\int_{t_1}^t g^{1-\delta_0}(s)\int_0^1 (|\nabla u(t)|^2 + |\nabla u(t-s)|^2)dxds \\ &\le cE(0)\int_{t_1}^t g^{1-\delta_0}(s)ds < 1.\end{aligned} \tag{3.11}$$

Then, Jensen's inequality, (2.2), the hypothesis (A), and (3.11) lead to

$$\int_{t_1}^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dxds$$

$$
\begin{aligned}
&= \int_{t_1}^{t} g^{\delta_0}(s) g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \\
&= \int_{t_1}^{t} g^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})}(s) g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \\
&\le \eta(t) \left[\frac{1}{\eta(t)} \int_{t_1}^{t} g(s)^{(p-1+\delta_0)} g^{1-\delta_0}(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\le \left[\int_{t_1}^{t} g(s)^p \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\le c \left[\int_{t_1}^{t} -g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \right]^{\frac{\delta_0}{p-1+\delta_0}} \\
&\le c \left[-E'(t)\right]^{\frac{\delta_0}{p-1+\delta_0}} .
\end{aligned}
$$

Then, particularly for $\delta_0 = \frac{1}{2}$, we find that (3.10) becomes

$$F'(t) \le -mE(t) + c\left[-E'(t)\right]^{\frac{1}{2p-1}} .$$

Now, we multiply by $E^\alpha(t)$, with $\alpha = 2p-2$, to get, using (3.1),

$$(FE^\alpha)'(t) \le F'(t)E^\alpha(t) \le -mE^{1+\alpha}(t) + cE^\alpha(t)\left[-E'(t)\right]^{\frac{1}{1+\alpha}} .$$

Then, Young's inequality, with $q = 1+\alpha$ and $q' = \frac{1+\alpha}{\alpha}$, gives

$$(FE^\alpha)'(t) \le -mE^{1+\alpha}(t) + \varepsilon E^{1+\alpha}(t) + C_\varepsilon(-E'(t)).$$

Consequently, picking $\varepsilon < m$, we obtain

$$F_0'(t) \le -m' E^{1+\alpha}(t),$$

where $F_0 = FE^\alpha + C_\varepsilon E \sim E$. Hence, we have, for some $a_0 > 0$,

$$F_0'(t) \le -a_0 F_0^{1+\alpha}(t)$$

from which we easily deduce that

$$E(t) \le \frac{a}{(a't + a'')^{\frac{1}{2p-2}}}. \tag{3.12}$$

By recalling that $p < \frac{3}{2}$ and using (3.12), we find that $\int_0^{+\infty} E(s)ds < +\infty$. Hence, by noting that

$$\int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dx ds \le c \int_0^t E(s) ds,$$

(3.10) gives

$$
\begin{aligned}
F'(t) &\le -mE(t) + c(g^{p \cdot \frac{1}{p}} \circ \nabla u)(t) \le -mE(t) + c\left[(g^p \circ \nabla u)(t)\right]^{\frac{1}{p}} \\
&\le -mE(t) + c\left[(-g' \circ \nabla u)(t)\right]^{\frac{1}{p}} \le -mE(t) + c\left[-E'(t)\right]^{\frac{1}{p}}.
\end{aligned}
$$

Therefore, repeating the above steps, with $\alpha = p-1$, we arrive at

$$E(t) \le \frac{a}{(a't + a'')^{\frac{1}{p-1}}} = cG^{-1}(c't + c'').$$

(II) The general case. We define $I(t)$ by

$$I(t) := \int_{t_1}^{t} \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds,$$

where H_0 is such that (2.4) is satisfied. As in (3.11), we find that $I(t)$ satisfies, for all $t \geq t_1$,

$$I(t) < 1. \tag{3.13}$$

We also assume, without loss of generality that $I(t) \geq \beta > 0$, for all $t \geq t_1$; otherwise (3.10) yields an exponential decay. In addition, we define $\xi(t)$ by

$$\xi(t) := -\int_{t_1}^{t} g'(s) \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds$$

and infer from (A) and the properties of H_0 and D that

$$\frac{g(s)}{H_0^{-1}(-g'(s))} \leq \frac{g(s)}{H_0^{-1}(H(g(s)))} = \frac{g(s)}{D^{-1}(g(s))} \leq k_0$$

for some positive constant k_0. Then, using (2.2) and choosing t_1 even larger (if needed), one can easily see that $\xi(t)$ satisfies, for all $t \geq t_1$,

$$\begin{aligned} \xi(t) &\leq -k_0 \int_{t_1}^{t} g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &\leq -cE(0) \int_{t_1}^{t} g'(s) \leq cg(t_1)E(0) \\ &< \min\{r, H(r), H_0(r)\}. \end{aligned} \tag{3.14}$$

Since H_0 is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\theta x) \leq \theta H_0(x)$$

provided $0 \leq \theta \leq 1$ and $x \in (0, r]$. The use of this fact, the hypothesis (A), (2.6), (3.13), (3.14), and Jensen's inequality leads to

$$\begin{aligned} \xi(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &\geq \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g'(s))] \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \\ &\geq H_0 \left(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g'(s)) \frac{g(s)}{H_0^{-1}(-g'(s))} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \right) \\ &= H_0 \left(\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \right). \end{aligned}$$

This implies that

$$\int_{t_1}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \leq H_0^{-1}(\xi(t)),$$

and (3.10) becomes

$$F'(t) \leq -mE(t) + cH_0^{-1}(\xi(t)), \quad \forall t \geq t_1. \tag{3.15}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, using (3.15), and the fact that $E' \le 0$, $H_0' > 0, H_0'' > 0$ on $(0, r]$, we find that the functional F_1, defined by

$$F_1(t) := H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) F(t) + c_0 E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_1(t) \le E(t) \le \alpha_2 F_1(t) \tag{3.16}$$

and

$$\begin{aligned} F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H_0''\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) F(t) + H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) F'(t) + c_0 E'(t) \\ &\le -mE(t) H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + cH_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) H_0^{-1}(\xi(t)) + c_0 E'(t). \end{aligned} \tag{3.17}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [2, pp. 61–64]). Then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad \text{if } s \in (0, H_0'(r)] \tag{3.18}$$

and H_0^* satisfies the following Young's inequality

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in (0, H_0'(r)], B \in (0, r]. \tag{3.19}$$

With $A = H_0'\big(\varepsilon_0 \frac{E(t)}{E(0)}\big)$ and $B = H_0^{-1}(\xi(t))$, using (2.2), (3.14) and (3.17)–(3.19), we arrive at

$$\begin{aligned} F_1'(t) &\le -mE(t) H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + cH_1^*\left(H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)\right) + c\xi(t) + c_0 E'(t) \\ &\le -mE(t) H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + c\varepsilon_0 \frac{E(t)}{E(0)} H_0'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) - cE'(t) + c_0 E'(t). \end{aligned}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain, for all $t \ge t_1$,

$$F_1'(t) \le -k\left(\frac{E(t)}{E(0)}\right) H_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -kH_2\Big(\frac{E(t)}{E(0)}\Big), \tag{3.20}$$

where $H_2(t) = tH_0'(\varepsilon_0 t)$.

Since $H_2'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, then using the strict convexity of H_0 on $(0, r]$, we find that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \varepsilon \frac{\alpha_1 F_1(t)}{E(0)}, \quad 0 < \varepsilon < 1,$$

taking in account (3.16) and (3.20), we have

$$R(t) \sim E(t) \tag{3.21}$$

and for some $k_0 > 0$,

$$R'(t) \le -\varepsilon k_0 H_2(R(t)), \quad \forall t \ge t_1.$$

Then, a simple integration and a suitable choice of ε yield, for some $k_1, k_2 > 0$,

$$R(t) \le H_1^{-1}(k_1 t + k_2), \quad \forall t \ge t_1, \tag{3.22}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)} ds$.

Here, we have used, based on the properties of H_2, the fact that H_1 is strictly decreasing function on $(0,1]$ and $\lim_{t\to 0} H_1(t) = +\infty$. A combination of (3.21) and (3.22), the estimate (2.3) is established.

Moreover, if $\int_0^1 H_1(t)dt < +\infty$, then

$$\int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds \le c \int_0^t E(s)ds < +\infty.$$

Therefore, we can repeat the same procedures with

$$I(t) := \int_{t_1}^t \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds$$

and

$$\xi(t) := -\int_{t_1}^t g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \, dxds,$$

to establish (2.5).

4 Appendix

Let $0 < q < 1$ and consider the relaxation function

$$g(t) = a\exp(-t^q),$$

where $0 < a < 1$ is chosen so that g satisfies (2.1). Here, we show how to apply Theorem 2.2 to this specific type of relaxation functions. First, one can show that $g'(t) = -H(g(t))$ where

$$H(t) = \frac{qt}{[\ln(\frac{a}{t})]^{\frac{1}{q}-1}}.$$

Since

$$H'(t) = \frac{(1-q) + q\ln\left(\frac{a}{t}\right)}{\left[\ln\left(\frac{a}{t}\right)\right]^{\frac{1}{q}}} \quad \text{and} \quad H''(t) = \frac{(1-q)\left[\ln\left(\frac{a}{t}\right) + \frac{1}{q}\right]}{\left[\ln\left(\frac{a}{t}\right)\right]^{\frac{1}{q}+1}},$$

then the function H satisfies the hypothesis (A) on the interval $(0, r]$ for any $0 < r < a$. Also, by taking $D(t) = t^\alpha$, (2.4) is satisfied for any $\alpha > 1$. Therefore, an explicit rate of decay can be obtained by Theorem 2.2. The function $H_0(t) = H(t^\alpha)$ has derivative

$$H_0'(t) = \frac{q\alpha t^{\alpha-1}\left[\frac{1}{q} - 1 + \ln\left(\frac{a}{t^\alpha}\right)\right]}{\left[\ln\left(\frac{a}{t^\alpha}\right)\right]^{\frac{1}{q}}}.$$

Therefore,

$$\begin{aligned} H_1(t) &= \int_t^1 \frac{\left[\ln\left(\frac{a}{(\varepsilon_0 s)^\alpha}\right)\right]^{\frac{1}{q}}}{q\alpha\varepsilon_0^{\alpha-1} s^\alpha \left[\frac{1}{q} - 1 + \ln\left(\frac{a}{(\varepsilon_0 s)^\alpha}\right)\right]} ds \quad \text{take } u = \ln\left(\frac{a}{(\varepsilon_0 s)^\alpha}\right) \\ &= \frac{1}{q\alpha^2 a^{1-\frac{1}{\alpha}}} \int_{\ln[a\varepsilon_0{}^{-\alpha}]}^{\ln[a(\varepsilon_0 t)^{-\alpha}]} \frac{u^{\frac{1}{q}} e^{(1-\frac{1}{\alpha})u}}{\frac{1}{q} - 1 + u} du. \end{aligned}$$

Using the fact that $(\frac{1}{q} - 1 + u) > (\frac{1}{q} - 1)$ and the function $f(u) = u^{\frac{1}{q}}$ is increasing on $(0, +\infty)$ and taking $\varepsilon_0 < a$, we have

$$H_1(t) \le \frac{[-\alpha \ln kt]^{\frac{1}{q}}}{\alpha^2 a^{1-\frac{1}{\alpha}}(1-q)} \int_{-\alpha\ln k}^{-\alpha\ln kt} e^{(1-\frac{1}{\alpha})u} du \quad \left(k = \frac{\varepsilon_0}{a^{\frac{1}{\alpha}}}\right)$$

$$= \frac{[-\alpha \ln kt]^{\frac{1}{q}} \left[t^{1-\alpha} - 1\right]}{\alpha(1-q)(\alpha-1){\varepsilon_0}^{\alpha-1}} = b\,[-\ln kt]^{\frac{1}{q}}\,[t^{1-\alpha} - 1],$$

where $b = \frac{\alpha^{\frac{1}{q}-1}}{(1-q)(\alpha-1)\varepsilon_0^{\alpha-1}}$. Next, we find that

$$\int_0^1 H_1(t)dt \le \int_0^1 b\,[-\ln kt]^{\frac{1}{q}} \left[t^{1-\alpha} - 1\right] dt \quad (\text{take } v = -\ln kt)$$
$$= \frac{b}{k} \int_{-\ln k}^{+\infty} v^{\frac{1}{q}} \left[k^{\alpha-1} \mathrm{e}^{(\alpha-2)v} - \mathrm{e}^{-v}\right] dv.$$

Then, it is easily seen that $\int_0^1 H_1(t)dt < +\infty$ if $(\alpha - 2) < 0$, and so we choose $1 < \alpha < 2$. Therefore, we can use (2.5) to deduce

$$E(t) \le k_3 G^{-1}(k_1 t + k_2),$$

where

$$G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds = \int_t^1 \frac{\left[\ln \frac{a}{\varepsilon_0 s}\right]^{\frac{1}{q}}}{s\left[1 - q + q \ln \frac{a}{\varepsilon_0 s}\right]} ds$$
$$= \int_{\ln \frac{a}{\varepsilon_0}}^{\ln \frac{a}{\varepsilon_0 t}} \frac{u^{\frac{1}{q}}}{1 - q + qu} du = \frac{1}{q} \int_{\ln \frac{a}{\varepsilon_0}}^{\ln \frac{a}{\varepsilon_0 t}} u^{\frac{1}{q}-1} \left[\frac{u}{\frac{1-q}{q} + u}\right] du$$
$$\le \frac{1}{q} \int_{\ln \frac{a}{\varepsilon_0}}^{\ln \frac{a}{\varepsilon_0 t}} u^{\frac{1}{q}-1} du = \left[\ln \frac{a}{\varepsilon_0 t}\right]^{\frac{1}{q}} - \left[\ln \frac{a}{\varepsilon_0}\right]^{\frac{1}{q}}$$
$$\le \left[\ln \frac{a}{\varepsilon_0 t}\right]^{\frac{1}{q}}.$$

Hence, $G^{-1}(t) \le \frac{a}{\varepsilon_0} \exp(-t^q)$ and the enegy decays at the same rate of g, that is,

$$E(t) \le c \exp(-\omega t^q).$$

Acknowledgements The author thanks the referees for their time and comments.

References

[1] Alabau-Boussouira, F., Cannarsa, P.: A general method for proving sharp energy decay rates for memory dissipative evolution equations. *C. R. Acad. Sci. Paris Ser. I*, **347**, 867–872 (2009)

[2] Arnold, V. I.: Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1989

[3] Burns, J. A., Liu, Y. Z., Zheng, S.: On the energy decay of a linear thermoelastic bar. *J. Math. Anal. Appl.*, **179**, 574–591 (1993)

[4] Dafermos, C. M.: On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. *Arch. Rational Mech. Anal.*, **29**, 241–271 (1968)

[5] Hansen, S. W.: Exponential energy decay in a linear thermoelastic rod. *J. Math. Anal. Appl.*, **167**, 429–442 (1992)

[6] Irmscher, T., Racke, R.: Sharp decay rates in parabolic and hyperbolic thermoelasticity. *IMA J. Appl. Math.*, **71**, 459–478 (2006)

[7] Jiang, S., Racke, R.: Evolution Equations in Thermoelasticity, π Monographs Surveys Pure Appl. Math. 112, Chapman & Hall/CRC, Boca Raton, 2000

[8] Jiang, S., Munoz Rivera, J. E., Racke, R.: Asymptotic stability and global existence in thermoelasticity with symmetry. *Quart. Appl. Math.*, **56**(2), 259–275 (1998)

[9] Lebeau, G., Zuazua, E.: Sur la décroissance non uniforme de l'énergie dans le système de la thermoelasticité linéaire. *C. R. Acad. Sci. Paris. Sér. I Math.*, **324**, 409–415 (1997)

[10] Liu, W. J.: Partial exact controllability and exponential stability of the higher-dimensional linear thermoelasticity. *ESAIM. Control. Optm. Calc. Var.*, **3**, 23–48 (1998)

[11] Liu, Z., Zheng, S.: Exponential stability of the semigroup associated with a thermoelastic system. *Quart. Appl. Math.* , **51**, 535–545 (1993)

[12] Liu, W. J., Zuazua, E.: Uniform stabilization of the higher dimensional system of thermoelasticity with a nonlinear boundary feedback. *Quart. Appl. Math.*, **59**(2), 269–314 (2001)

[13] Messaoudi, S. A., Al-Shehri, A.: General boundary stabilization of memory-type thermoelasticity. *J. Math. Phys.*, **51**, 103514 (2010)

[14] Munoz Rivera, J. E., Qin, Y.: Global existence and exponential stability in one dimensional nonlinear thermoelasticity with thermal memory. *Nonlinear Anal.*: *TMA*, **51**, 11–32 (2002)

[15] Munoz Rivera, J. E., Racke, R.: Magneto-thermo-elasticity — Large time behavior for linear systems. *Adv. Differential Equations*, **6**(3), 359–384 (2001)

[16] Mustafa, M. I.: On the control of n-dimensional thermoelastic system. *Appl. Anal.*, **92**(1), 104–114 (2014)

[17] Oliveira, J. C., Charao, R. C.: Stabilization of a locally damped thermoelastic system. *Comput. Appl. Math.*, **27**(3), 319–357 (2008)

[18] Pereira, D. C., Menzala, G. P.: Exponential stability in linear thermoelasticity: the inhomogeneous case. *Appl. Anal.*, **44**, 21–36 (1992)

[19] Qin, Y., Munoz Rivera, J. E.: Global existence and exponential stability of solutions to thermoelastic equations of hyperbolic type. *J. Elasticity*, **75**(2), 125–145 (2004)

[20] Zheng, S.: Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems, Pitman Monographs Surveys Pure Applied Mathematics, Vol. 76, Longman, Harlow, 1995