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Some Results on Space-Like Self-Shrinkers

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Abstract We study space-like self-shrinkers of dimension n in pseudo-Euclidean space $\mathbb{R}^{m+n}_m$ with index m. We derive drift Laplacian of the basic geometric quantities and obtain their volume estimates in pseudo-distance function. Finally, we prove rigidity results under minor growth conditions in terms of the mean curvature or the image of Gauss maps.

Keywords Space-like self-shrinker, pseudo-distance, volume growth, rigidity

MR(2010) Subject Classification 58E20, 53A10

1 Introduction

Let $\mathbb{R}^{m+n}_m$ be an $(m+n)$-dimensional pseudo-Euclidean space with the index m. The indefinite flat metric on $\mathbb{R}^{m+n}_m$ is defined by $ds^2 = \sum_{i=1}^n (dx^i)^2 - \sum_{\alpha=n+1}^{m+n}(dx^\alpha)^2$. In what follows we agree with the following range of indices

$$A,\ B,\ C, \ldots = 1, \ldots, m+n; \quad i,\ j,\ k, \ldots = 1, \ldots, n;$$
$$s, t = 1,\ \ldots, m; \quad \alpha,\ \beta, \ldots = n+1, \ldots, m+n.$$

Let $F : M \to \mathbb{R}^{m+n}_m$ be a space-like n-dimensional submanifold in $\mathbb{R}^{m+n}_m$ with the second fundamental form B defined by $B_{XY} \overset{\text{def.}}{=} (\bar\nabla_X Y)^N$ for $X, Y \in \Gamma(TM)$. We denote $(\cdots)^T$ and $(\cdots)^N$ for the orthogonal projections into the tangent bundle TM and the normal bundle NM, respectively. For $\nu \in \Gamma(NM)$ we define the shape operator $A^\nu : TM \to TM$ by $A^\nu(V) = -(\bar\nabla_V \nu)^T$. Taking the trace of B gives the mean curvature vector H of M in $\mathbb{R}^{m+n}_m$ and $H \overset{\text{def.}}{=} \operatorname{trace}(B) = B_{e_ie_i}$, where $\{e_i\}$ is a local orthonormal frame field of M. Here and in the sequel we use the summation convention. The mean curvature vector is time-like, and a cross-section of the normal bundle.

We now consider a one-parameter family $F_t = F(\cdot, t)$ of immersions $F_t : M \to \mathbb{R}^{m+n}_m$ with

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the corresponding images $M_t = F_t(M)$ such that

$$\begin{aligned} &\frac{d}{dt}F(x,t) = H(x,t), \quad x \in M, \\ &F(x,0) = F(x) \end{aligned} \tag{1.1}$$

are satisfied, where $H(x,t)$ is the mean curvature vector of M_t at $F(x,t)$. There are many interesting results on mean curvature flow on space-like hypersurfaces in certain Lorentzian manifolds [10–13]. For higher codimension we refer to the previous work of the second author [19].

A special but important class of solutions to (1.1) are self-similar shrinking solutions, whose profiles, space-like self-shrinkers, satisfy a system of quasi-linear elliptic PDE of the second order

$$H = -\frac{X^N}{2}. \tag{1.2}$$

Besides the Lagrangian space-like self-shrinkers [2, 9, 14], there is an interesting paper on curves in the Minkowski plane [15]. The present paper is devoted to general situation on space-like self-shrinker.

For a space-like n-submanifold M in $\mathbb{R}^{m+n}_m$, we have the Gauss map $\gamma : M \to \mathbf{G}^m_{n,m}$. The target manifold is a pseudo-Grassmann manifold, dual space of the Grassmann manifold $\mathbf{G}_{n,m}$. In the next section, we will describe its geometric properties, which will be used in the paper.

Choose a Lorentzian frame field $\{e_i, e_\alpha\}$ in $\mathbb{R}^{m+n}_m$ with space-like $\{e_i\} \in TM$ and time-like $\{e_\alpha\} \in NM$ along the space-like submanifold $F : M \to \mathbb{R}^{m+n}_m$. Define coordinate functions

$$x^i = \langle F, e_i \rangle, \quad y^\alpha = -\langle F, e_\alpha \rangle.$$

We then have

$$|F|^2 = X^2 - Y^2,$$

where $X = \sqrt{\sum_{i=1}^n (x^i)^2}, Y = \sqrt{\sum_{\alpha=n+1}^{m+n} (y^\alpha)^2}$. We call $|F|^2$ the pseudo-distance function from the origin $0 \in M$.

We always put the origin on M in the paper. We see that $|F|^2$ is invariant under the Lorentzian action up to the choice of the origin in $\mathbb{R}^{m+n}_m$. Set $z = |F|^2$. It has been proved that z is proper provided M is closed with the Euclidean topology (see [4] for $m = 1$ and [16] for any codimension m).

Following Colding et al. [6], we can also introduce the drift Laplacian,

$$\mathcal{L} = \Delta - \frac{1}{2}\langle F, \nabla(\cdot) \rangle = \mathrm{e}^{\frac{z}{4}} \mathrm{div}(\mathrm{e}^{-\frac{z}{4}} \nabla(\cdot)). \tag{1.3}$$

It can be showed that $\mathcal{L}$ is self-adjoint with respect to the weighted volume element $\mathrm{e}^{-\frac{z}{4}} d\mu$, where $d\mu$ is the volume element of M with respect to the induced metric from the ambient space $\mathbb{R}^{m+n}_m$. In the present paper we carry out integrations with respect to this measure. We denote $\rho = \mathrm{e}^{-\frac{z}{4}}$ and the volume form $d\mu$ might be omitted in the integrations for notational simplicity.

For a space-like submanifold in $\mathbb{R}^{m+n}_m$, there are several geometric quantities, the squared norm of the second fundamental form $|B|^2$, the squared norm of the mean curvature $|H|^2$ and the w-function, which is related to the image of the Gauss map. In Section 3, we will calculate drift Laplacian $\mathcal{L}$ of those quantities, see Proposition 3.1.

Corresponding to the weighted measure and drift Laplacian there is so-called the Baker–Emery Ricci tensor. It is noted that in [3] $\mathrm{Ric}_f \geq \frac{z}{4}$ with $f = \frac{z}{4}$. Using the comparison technique, the weighted volume of the geodesic ball can be estimated from above in terms of the distance function [18].

For a space-like n-submanifold M in $\mathbb{R}^{m+n}_m$, there are several global conditions: closed one with Euclidean topology; entire graph; complete with induced Riemannian metric. A complete space-like one has to be entire graph, but the converse claim is not always the case. Closed one with Euclidean topology is complete under the parallel mean curvature assumption (see [4] for codimension one and [16] for higher codimension).

In our case of closed one with Euclidean topology, the pseudo-distance function z is always proper. It is natural to consider the volume growth in z. For the proper self-shrinkers in Euclidean space Ding–Xin [8] gave the volume estimates. It has been generalized in [5] for more general situation. But, the present case does not satisfy the conditions in Theorem 1.1 in [5]. However, the idea in [8] is still applicable for space-like self-shrinkers. In Section 4, we will give volume estimates for space-like self-shrinkers, in a similar manner as in [8], see Theorem 4.3.

Finally, using integral method we can obtain rigidity results as follows.

Theorem 1.1 *Let M be a space-like self-shrinker of dimension n in $\mathbb{R}^{n+m}_m$, which is closed with respect to the Euclidean topology. If there is a constant $\alpha < \frac{1}{8}$, such that $|H|^2 \leq \mathrm{e}^{\alpha z}$, then M is an affine n-plane.*

Theorem 1.2 *Let M be a complete space-like self-shrinker of dimension n in $\mathbb{R}^{n+m}_m$. If there is a constant $\alpha < \frac{1}{2}$, such that $\ln w \leq \mathrm{e}^{\alpha d^2(p,x)}$ for certain $p \in M$, where $d(p,\cdot)$ is the distance function from p, then M is affine n-plane.*

Remark 1.3 In the special situation, for the Lagrangian space-like self-shrinkers, the rigidity results hold without the growth condition (see [9]). Let $\mathbb{R}^{2n}_n$ be Euclidean space with null coordinates $(x, y) = (x_1, \ldots, x_n; y_1, \ldots, y_n)$, which means that the indefinite metric is defined by $ds^2 = \sum_i dx_i dy_i$. If $M = \{(x, Du(x))\big|\ x \in \mathbb{R}^n\}$ is a space-like submanifold in $\mathbb{R}^{2n}_n$, then u is convex and the induced metric on M is given by $ds^2 = \sum_{i,j} u_{ij} dx_i dx_j$. M is a space-like Lagrangian submanifold in $\mathbb{R}^{2n}_n$. It is worthy to point out that the potential function u is proper if M is an entire gradient graph, as the following consideration. On $\mathbb{R}^n$ set $\rho' = |x| = \sqrt{\sum x_i^2}$. At any direction $\theta \in S^{n-1}$,

$$u_i = u_{\rho'} \frac{\partial \rho'}{\partial x_i} = \frac{x_i}{\rho'} u_{\rho'}$$

and the pseudo-distance

$$z = x_i u_i = \rho' u_{\rho'},$$

which is positive when the origin is on M, since it is space-like. It implies that u is increasing in ρ'. Moreover,

$$z_{\rho'} = u_{\rho'} + \rho' u_{\rho'\rho'} > 0,$$

which means that z is also increasing in ρ'. Hence,

$$u(\rho') - u(\epsilon) = \int_\epsilon^{\rho'} u_{\rho'} d\rho' = \int_\epsilon^{\rho'} \frac{z}{\rho'} d\rho' \geq z(\epsilon) \int_\epsilon^{\rho'} \frac{1}{\rho'} d\rho' \geq z(\epsilon) \int_\epsilon^{\rho'} \frac{1}{\rho'} d\rho' \to \infty$$

as $\rho' \to \infty$.

Remark 1.4 Rigidity problem for space-like extremal submanifolds was raised by Calabi [1], and solved by Cheng–Yau [4] for codimension 1. Later, Jost–Xin generalized the results to higher codimension [16]. The rigidity problem for space-like submanifolds with parallel mean curvature was studied in [20, 22] and [16] (see also in Chap. 8 of [21]).

2 Geometry of $G^m_{n,m}$

In $\mathbb{R}^{n+m}_m$ all space-like n-subspaces form the pseudo-Grassmannian $G^m_{n,m}$. It is a specific Cartan–Hadamard manifold which is the noncompact dual space of the Grassmann manifold $G_{n,m}$.

Let P and $A \in G^m_{n,m}$ be two space-like n-plane in $\mathbb{R}^{m+n}_m$. The angles between P and A are defined by the critical values of angel θ between a nonzero vector x in P and its orthogonal projection x^* in A as x runs through P.

Assume that $e_1, \dots, e_n$ are orthonormal vectors which span the space-like P and $a_1, \dots, a_n$ for space-like A. For a nonzero vector in P,

$$x = \sum_i x_i e_i,$$

its orthonormal projections in A is

$$x^* = \sum_i x^*_i a_i.$$

Thus, for any $y \in A$, we have

$$\langle x - x^*, y\rangle = 0.$$

Set

$$W_{i\,j} = \langle e_i, a_j\rangle.$$

We then have

$$x^*_j = \sum_i W_{i\,j} x_i.$$

Since x is a vector in a space-like n-plane and its projection x^* in A is also a space-like vector, we then have a Minkowski plane R^2_1 spanned by x and x^*. Then angle θ between x and x^* is defined by

$$\cosh\theta = \frac{\langle x, x^*\rangle}{|x||x^*|}.$$

Let

$$W = (W_{i\,j}) = \begin{pmatrix} \langle e_1, a_1\rangle & \cdots & \langle e_n, a_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_n, a_1\rangle & \cdots & \langle e_n, a_n\rangle \end{pmatrix}.$$

Now define the w-function as

$$w = \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n\rangle = \det W.$$

$W^T W$ is symmetric, its eigenvalues are $\mu_1^2, \dots, \mu_n^2$, then there exist $e_1, \dots, e_n$ in P, such that

$$W^T W = \begin{pmatrix} \mu_1^2 & & 0 \\ & \ddots & \\ 0 & & \mu_n^2 \end{pmatrix},$$

in which $\mu_i \geq 1$ and $\mu_i = \cosh\theta_i$. Then

$$w = \prod_i \cosh\theta_i = \prod_i \frac{1}{\sqrt{1-\lambda_i^2}}, \quad \lambda_i = \tanh\theta_i. \tag{2.1}$$

The distance between P and A in the canonical Riemannian metric on $\mathbf{G}^m_{n,m}$ is (see [17] for example)

$$d(P,A) = \sqrt{\sum_i \theta_i^2}.$$

For the fixed $A \in G^m_{n,m}$, which is spanned by $\{a_i\}$, choose time-like $\{a_{n+s}\}$ such that $\{a_i, a_{n+s}\}$ form an orthonormal Lorentzian bases of R^{n+m}_m.

Set

$$\begin{aligned} e_i &= \cosh\theta_i a_i + \sinh\theta_i a_{n+i}, \\ e_{n+i} &= \sinh\theta_i a_i + \cosh\theta_i a_{n+i} \quad (\text{and } e_{n+\alpha} = a_{n+\alpha} \text{ if } m > n). \end{aligned}$$

Then $e_i \in T_pM, e_{n+i} \in N_pM$. In this case

$$\begin{aligned} w_{i\,\alpha} &= \langle e_1 \wedge \cdots \wedge e_{i-1} \wedge e_\alpha \wedge e_{i+1} \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\ &= \cosh\theta_1 \cosh\theta_{i-1} \sinh\theta_i \cosh\theta_{i+1} \cosh\theta_n = \lambda_i w \delta_{n+i\ \alpha}, \end{aligned}$$

which is obtained by replacing e_i by e_α in w. We also have $w_{i\alpha j\beta}$ by replacing e_j by e_β in $w_{i\,\alpha}$. We obtain

$$w_{i\alpha j\beta} = \begin{cases} \lambda_i\lambda_j w, & \alpha = n+i, \beta = n+j, \\ -\lambda_i\lambda_j w, & \alpha = n+j, \beta = n+i, \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

3 Drift Laplacian of Some Geometric Quantities

The second fundamental form B can be viewed as a cross-section of the vector bundle $\mathrm{Hom}(\odot^2 TM, NM)$ over M. A connection on $\mathrm{Hom}(\odot^2 TM, NM)$ can be induced from those of TM and NM naturally. There is a natural fiber metric on $\mathrm{Hom}(\odot^2 TM, NM)$ induced from the ambient space and it becomes a Riemannian vector bundle. There is the trace-Laplace operator ∇^2 acting on any Riemannian vector bundle.

In [19], we already calculate $\nabla^2 B$ for general space-like n-submanifolds in $\mathbb{R}^{m+n}_m$.

Set

$$B_{i\,j} = B_{e_i\,e_j} = h^\alpha_{i\,j} e_\alpha,\ S_{\alpha\,\beta} = h^\alpha_{i\,j} h^\beta_{i\,j}.$$

From Proposition 2.1 in [19], we have

$$\langle \nabla^2 B, B\rangle = \langle \nabla_i\nabla_j H, B_{i\,j}\rangle + \langle B_{i\,k}, H\rangle\langle B_{i\,l}, B_{k\,l}\rangle - |R^\perp|^2 - \sum_{\alpha,\beta} S^2_{\alpha\,\beta}, \tag{3.1}$$

where $R^\perp$ denotes the curvature of the normal bundle and

$$|R^\perp|^2 = -\langle R_{e_i\,e_j}\nu_\alpha, R_{e_i\,e_j}\nu_\alpha\rangle.$$

Then from the self-shrinker equation (1.2), we obtain

$$\nabla_i F^N = [\bar\nabla_i (F - \langle F, e_j\rangle e_j)]^N$$

$$= [e_i - \bar{\nabla}_i \langle F, e_j \rangle e_j - \langle F, e_j \rangle \bar{\nabla}_{e_i} e_j]^N$$
$$= -\langle F, e_j \rangle B_{i\,j},$$

and

$$\begin{aligned} \nabla_i \nabla_j F^N &= -\nabla_i [\langle F, e_k \rangle B_{k\,j}] \\ &= -\delta_i^k B_{k\,j} - \langle F^N, B_{k\,i} \rangle B_{k\,j} - \langle F, e_k \rangle \nabla_i B_{k\,j} \\ &= -B_{i\,j} - \langle F^N, B_{k\,i} \rangle B_{k\,j} - \langle F, e_k \rangle \nabla_k B_{i\,j} \\ &= -B_{i\,j} + \langle 2H, B_{k\,i} \rangle B_{k\,j} - \langle F, e_k \rangle \nabla_k B_{i\,j}. \end{aligned}$$

Set $P_{i\,j} = \langle B_{i\,j}, H \rangle$. Then

$$\nabla_i \nabla_j H = \frac{1}{2} B_{i\,j} - P_{k\,i} B_{k\,j} + \frac{1}{2} \langle F, e_k \rangle \nabla_k B_{i\,j}. \tag{3.2}$$

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned} \langle \nabla^2 B, B \rangle = &\left\langle \frac{1}{2} B_{i\,j}, B_{i\,j} \right\rangle - \langle H, B_{k\,i} \rangle \langle B_{k\,j}, B_{i\,j} \rangle + \frac{1}{2} \langle F, e_k \rangle \langle \nabla_k B_{i\,j}, B_{i\,j} \rangle \\ &+ \langle B_{i\,k}, H \rangle \langle B_{i\,l}, B_{k\,l} \rangle - |R^{\perp}|^2 - \sum_{\alpha,\beta} S_{\alpha\,\beta}^2. \end{aligned}$$

This also means that

$$\langle \nabla^2 B, B \rangle = \frac{1}{2} \langle B, B \rangle + \frac{1}{4} \langle F^T, \nabla \langle B, B \rangle \rangle - |R^{\perp}|^2 - \sum_{\alpha,\beta} S_{\alpha\,\beta}^2. \tag{3.3}$$

Note that $\Delta \langle B, B \rangle = 2 \langle \nabla^2 B, B \rangle + 2 \langle \nabla B, \nabla B \rangle$, so

$$\begin{aligned} \Delta \langle B, B \rangle = &\langle B, B \rangle + \frac{1}{2} \langle F^T, \nabla \langle B, B \rangle \rangle - 2|R^{\perp}|^2 - 2 \sum_{\alpha,\beta} S_{\alpha\,\beta}^2 \\ &+ 2 \langle \nabla B, \nabla B \rangle. \end{aligned} \tag{3.4}$$

We denote

$$\begin{aligned} &|B|^2 = -\langle B, B \rangle = \sum_{i,j,\alpha} h_{\alpha ij}^2, \ |\nabla B|^2 = -\langle \nabla B, \nabla B \rangle, \\ &|H|^2 = -\langle H, H \rangle, \ |\nabla H|^2 = -\langle \nabla H, \nabla H \rangle. \end{aligned}$$

Then

$$\Delta |B|^2 = |B|^2 + \frac{1}{2} \langle F^T, \nabla |B|^2 \rangle + 2|R^{\perp}|^2 + 2 \sum_{\alpha,\beta} S_{\alpha\,\beta}^2 + 2|\nabla B|^2. \tag{3.5}$$

From (3.2), we also obtain

$$\nabla^2 H = \frac{1}{2} H - P_{k\,i} B_{k\,j} + \frac{1}{2} \langle F, e_k \rangle \nabla_k H.$$

Since

$$\Delta |H|^2 = -\Delta \langle H, H \rangle = -2 \langle \nabla^2 H, H \rangle - 2 \langle \nabla H, \nabla H \rangle,$$

we obtain

$$\begin{aligned} \Delta |H|^2 &= -2 \left\langle \frac{1}{2} H - P_{k\,i} B_{k\,i} + \frac{1}{2} \langle F, e_k \rangle \nabla_k H, H \right\rangle - 2 \langle \nabla H, \nabla H \rangle \\ &= |H|^2 + 2|P|^2 + \frac{1}{2} \langle F^T, \nabla |H|^2 \rangle + 2|\nabla H|^2, \end{aligned} \tag{3.6}$$

where $|P|^2 = \sum_{i,j} P_{ij}^2$.

In the pseudo-Grassmann manifold $\mathbf{G}^m_{n,m}$, there are w-functions with respect to a fixed point $A \in \mathbf{G}^m_{n,m}$, as shown in Section 2. For the space-like n-submanifold M in $\mathbb{R}^{m+n}_m$ we define the Gauss map $\gamma : M \to \mathbf{G}^m_{n,m}$, which is obtained by parallel translation of T_pM for any $p \in M$ to the origin in $\mathbb{R}^{m+n}_m$. Then, we have functions $w \circ \gamma$ on M, which is still denoted by w for notational simplicity.

For any point $p \in M$ around p there is a local tangent frame field $\{e_i\}$, and which is normal at p. We also have a local orthonormal normal frame field $\{e_\alpha\}$, and which is normal at p. Define a w-function by

$$w = \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle,$$

where $\{a_i\}$ is a fixed orthonormal vectors which span a fixed space-like n-plane A. Denote

$$e_{i\,\alpha} = e_1 \wedge \cdots \wedge e_\alpha \wedge \cdots \wedge e_n,$$

which is got by substituting e_α for e_i in $e_1 \wedge \cdots \wedge e_n$ and $e_{i\alpha j\beta}$ is obtained by substituting e_β for e_j in $e_{i\,\alpha}$. Then

$$\begin{aligned}
\nabla_{e_j} w &= \sum_{i=1}^n \langle e_1 \wedge \cdots \wedge \bar{\nabla}_{e_j} e_i \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\
&= \sum_{i=1}^n \langle e_1 \wedge \cdots \wedge B_{i\,j} \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\
&= \sum_{i=1}^n h^\alpha_{i\,j} \langle e_1 \cdots \wedge e_\alpha \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\
&= \sum_{i=1}^n h^\alpha_{i\,j} \langle e_{i\,\alpha}, a_1 \wedge \cdots \wedge a_n \rangle. \qquad (3.7)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\nabla_{e_i} \nabla_{e_j} w &= \langle \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} (e_1 \wedge \cdots \wedge e_n), a_1 \wedge \cdots \wedge a_n \rangle \\
&= \sum_{k \neq l} \langle e_1 \wedge \cdots \wedge \bar{\nabla}_{e_j} e_k \wedge \cdots \wedge \bar{\nabla}_{e_i} e_l \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\
&\quad + \sum_k \langle e_1 \wedge \cdots \bar{\nabla}_{e_i} \bar{\nabla}_{e_j} e_k \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \\
&= \sum_{k \neq l} \langle e_1 \wedge \cdots \wedge B_{j\,k} \wedge \cdots \wedge B_{il} \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \qquad (3.8) \\
&\quad + \sum_k \langle e_1 \wedge \cdots \wedge (\bar{\nabla}_i \bar{\nabla}_j e_k)^T \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle \qquad (3.9) \\
&\quad + \sum_k \langle e_1 \wedge \cdots \wedge (\bar{\nabla}_i \bar{\nabla}_j e_k)^N \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle. \qquad (3.10)
\end{aligned}$$

Note that

$$\begin{aligned}
(3.8) &= \sum_{k \neq l} h^\alpha_{j\,k} h^\beta_{i\,l} \langle e_{\alpha k \beta l}, a_1 \wedge \cdots \wedge a_n \rangle, \\
(3.9) &= \langle \bar{\nabla}_i \bar{\nabla}_j e_k, e_k \rangle w = -\langle \bar{\nabla}_j e_k, \bar{\nabla}_i e_k \rangle w = -\langle B_{j\,k}, B_{i\,k} \rangle w = h^\alpha_{j\,k} h^\alpha_{i\,k} w, \\
(3.10) &= -\langle (\bar{\nabla}_i \bar{\nabla}_j e_k)^N, e_\alpha \rangle \langle e_{\alpha\,k}, a_1 \wedge \cdots \wedge a_n \rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\langle(\bar\nabla_i(B_{j\,k}+\nabla_{e_j}e_k))^N, e_\alpha\rangle\langle e_{\alpha\,k}, a_1\wedge\cdots\wedge a_n\rangle\\
&= -\langle\nabla_i B_{j\,k}, e_\alpha\rangle\langle e_{\alpha\,k}, a_1\wedge\cdots\wedge a_n\rangle = -\langle\nabla_k B_{i\,j}, e_\alpha\rangle\langle e_{\alpha\,k}, a_1\wedge\cdots\wedge a_n\rangle,
\end{aligned}
$$

where we use the Codazzi equation in the last step. Thus, we obtain

$$
\Delta w = \sum_{i,k\neq l} h^\alpha_{i\,k}h^\beta_{i\,l}\langle e^\alpha_{k\beta l}, a_1\wedge\cdots\wedge a_n\rangle + |B|^2 w - \langle\nabla_k H, e_\alpha\rangle\langle e_{\alpha k}, a_1\wedge\cdots\wedge a_n\rangle.
$$

Since

$$
\nabla_i F^N = -\langle F, e_j\rangle B_{i\,j},
$$

from (1.2), we obtain

$$
\nabla_i H = \frac{1}{2}\langle F, e_j\rangle B_{i\,j}, \quad \langle\nabla_i H, e_\alpha\rangle = -\frac{1}{2}\langle F, e_j\rangle h^\alpha_{i\,j}, \tag{3.11}
$$

so,

$$
\begin{aligned}
\Delta w &= |B|^2 w + \sum_{i,k\neq l} h^\alpha_{i\,k}h^\beta_{i\,l}\langle e_{\alpha k\beta l}, a_1\wedge\cdots\wedge a_n\rangle + \frac{1}{2}\langle F, e_i\rangle h^\alpha_{k\,i}\langle e_{\alpha k}, a_1\wedge\cdots\wedge a_n\rangle\\
&= |B|^2 w + \sum_{i,k\neq l} h^\alpha_{i\,k}h^\beta_{i\,l}\langle e_{\alpha k\beta l}, a_1\wedge\cdots\wedge a_n\rangle + \frac{1}{2}\langle F, \nabla w\rangle, \qquad (3.12)
\end{aligned}
$$

where (3.7) has been used in the last equality.

Proposition 3.1 *For a space-like self-shrinker M of dimension n in $\mathbb{R}^{m+n}_m$, we have*

$$
\mathcal{L}|B|^2 = |B|^2 + 2|R^\perp|^2 + 2\sum_{\alpha,\beta} S^2_{\alpha\,\beta} + 2|\nabla B|^2, \tag{3.13}
$$

$$
\mathcal{L}|H|^2 = |H|^2 + 2|P|^2 + 2|\nabla H|^2, \tag{3.14}
$$

$$
\mathcal{L}(\ln w) \ge \frac{|B|^2}{w^2}. \tag{3.15}
$$

Proof From (1.3), (3.5), (3.6), we can obtain (3.13) and (3.14) easily.

From (1.3), (3.12), we have

$$
\begin{aligned}
\mathcal{L}w &= |B|^2 w + \sum_{i,k\neq l} h^\alpha_{i\,k}h^\beta_{i\,l}\langle e_{\alpha k\beta l}, a_1\wedge\cdots\wedge a_n\rangle = |B|^2 w + \sum_{i,k\neq l} h^\alpha_{i\,k}h^\beta_{i\,l} w_{\alpha k\beta l}\\
&= |B|^2 w + \sum_{i,k\neq l}\lambda_k\lambda_l(h^{n+k}_{i\,k}h^{n+l}_{i\,l} - h^{n+l}_{i\,k}h^{n+k}_{i\,l})w. \qquad (3.16)
\end{aligned}
$$

Furthermore, since

$$
\mathcal{L}(\ln w) = \frac{1}{w}\mathcal{L}w - \frac{|\nabla w|^2}{w^2},
$$

we obtain

$$
\mathcal{L}(\ln w) = |B|^2 + \sum_{i,k\neq l}\lambda_k\lambda_l(h^{n+k}_{i\,k}h^{n+l}_{i\,l} - h^{n+l}_{i\,k}h^{n+k}_{i\,l}) - \frac{|\nabla w|^2}{w^2}.
$$

From (3.7), we obtain

$$
\begin{aligned}
|\nabla w|^2 &= \sum_{j=1}^n |\nabla_{e_j} w|^2 = \sum_{j=1}^n\Big(\sum_{i=1}^n\sum_\alpha h^\alpha_{ij} w_{i\,\alpha}\Big)^2\\
&= \sum_{j=1}^n\Big(\sum_{i=1}^n h^{n+i}_{ij}\lambda_i w\Big)^2 = \sum_{i,j,k=1}^n \lambda_i\lambda_k w^2 h^{n+i}_{i\,j}h^{n+k}_{k\,j}.
\end{aligned}
$$

In the case of $m \geq n$, we rewrite (otherwise, we treat the situation similarly)

$$|B|^2 = \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \sum_{i,j} (h^{n+i}_{ij})^2 + \sum_j \sum_{k<i} (h^{n+k}_{ij})^2 + \sum_j \sum_{i<k} (h^{n+k}_{ij})^2.$$

So, we obtain

$$\begin{aligned}
\mathcal{L}(\ln w) &= |B|^2 + \sum_{i,j,k\neq i} \lambda_i\lambda_k (h^{n+i}_{i\,j} h^{n+k}_{j\,k} - h^{n+k}_{i\,j} h^{n+i}_{j\,k}) - \sum_{i,j,k=1}^{n} \lambda_i\lambda_k h^{n+i}_{ij} h^{n+k}_{j\,k} \\
&= |B|^2 + \sum_{i,j,k\neq i} \lambda_i\lambda_k h^{n+i}_{i\,j} h^{n+k}_{j\,k} - \sum_{i,j,k\neq i} \lambda_i\lambda_k h^{n+k}_{i\,j} h^{n+i}_{j\,k} - \sum_{i,j,k=1}^{n} \lambda_i\lambda_k h^{n+i}_{ij} h^{n+k}_{j\,k} \\
&= \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \sum_{i,j} (h^{n+i}_{i\,j})^2 + \sum_j \sum_{k<i} (h^{n+k}_{ij})^2 + \sum_j \sum_{i<k} (h^{n+k}_{ij})^2 \\
&\quad - \sum_{i,j} \lambda_i^2 (h^{n+i}_{ij})^2 - \sum_{ij,k\neq i} \lambda_i\lambda_k h^{n+k}_{ij} h^{n+i}_{jk} \\
&= \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \sum_{i,j} (1-\lambda_i^2)(h^{n+i}_{ij})^2 + \sum_j \sum_{k<i} (h^{n+k}_{ij})^2 + \sum_j \sum_{i<k} (h^{n+k}_{ij})^2 \\
&\quad - 2\sum_j \sum_{k<i} \lambda_k\lambda_i h^{n+i}_{jk} h^{n+k}_{ij} \\
&\geq \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \sum_{i,j} (1-\lambda_i^2)(h^{n+i}_{ij})^2 + \sum_j \sum_{k<i} (1-\lambda_i^2)(h^{n+k}_{ij})^2 \\
&\quad + \sum_j \sum_{i<k} (1-\lambda_i^2)(h^{n+k}_{ij})^2 \\
&= \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \sum_{i,j,k} (1-\lambda_i^2)(h^{n+k}_{ij})^2 \\
&\geq \sum_{j,k,\alpha>n} (h^{n+\alpha}_{jk})^2 + \prod_i (1-\lambda_i^2) \sum_{i,j,k} (h^{n+k}_{ij})^2. \qquad (3.17)
\end{aligned}$$

Noting (2.1), the inequality (3.15) has been proved. □

Remark 3.2 For a space-like graph $M = (x, f(x))$ with $f : \mathbb{R}^n \to \mathbb{R}^m$ its induced metric is $ds^2 = (\delta_{ij} - f^\alpha_i f^\alpha_j) dx^i dx^j$. Set $g = \det(\delta_{ij} - f^\alpha_i f^\alpha_j)$. Then $w = \frac{1}{\sqrt{g}}$.

Remark 3.3 (3.15) is a generalization of a formula (5.8) for space-like graphical self-shrinkers in [7] to more general situation.

4 Volume Growth

To draw our results we intend to integrate those differential inequalities obtained in the last section. We need to know the volume growth in the pseudo-distance function z on the space-like submanifolds. In [16] the following property has been proved.

Proposition 4.1 ([16, Proposition 3.1]) *Let M be a space-like n-submanifold in $\mathbb{R}^{m+n}_m$. If M is closed with respect to the Euclidean topology, then when $0 \in M$, $z = \langle F, F\rangle$ is a proper function on M.*

we also need a lemma from [8]:

Lemma 4.2 *If $f(r)$ is a monotonic increasing nonnegative function on $[0, +\infty)$ satisfying $f(r) \leq C_1 r^n f(\frac{r}{2})$ on $[C_2, +\infty)$ for some positive constant n, C_1, C_2, here $C_2 > 1$, then $f(r) \leq$*

$C_3 e^{2n(\log r)^2}$ *on* $[C_2, +\infty)$ *for some positive constant* C_3 *depending only on* $n, C_1, C_2, f(C_2)$.

Using similar method as in [8], we obtain the following volume growth estimates.

Theorem 4.3 *Let* $z = \langle F, F\rangle$ *be the pseudo-distance of* R^{n+m}_m, *where* $F \in R^{n+m}_m$ *is the position vector with respect to the origin* $0 \in M$. *Let* M *be an* n*-dimensional space-like self-shrinker of* R^{n+m}_m. *Assume that* M *is closed with respect to the Euclidean topology. Then for any* $\alpha > 0$, $\int_M e^{-\alpha z} d\mu$ *is finite, in particular* M *has finite weighted volume.*

Proof We have

$$\begin{aligned} z_i &\overset{\text{def.}}{=} e_i(z) = 2\langle F, e_i\rangle, \\ z_{ij} &\overset{\text{def.}}{=} \text{Hess}(z)(e_i, e_j) = 2(\delta_{ij} - y^\alpha h^\alpha_{ij}), \\ \Delta z &= 2n - 2y^\alpha H^\alpha = 2n + Y^2, \end{aligned} \tag{4.1}$$

where the self-shrinker equation (1.2) has been used in third equality. For our self-shrinker M^n in $\mathbb{R}^{n+m}_m$, we define a functional F_t on any set $\Omega \subset M$ by

$$F_t(\Omega) = \frac{1}{(4\pi t)^{n/2}} \int_\Omega e^{-\frac{z}{4t}} d\mu, \quad \text{for } t > 0.$$

Set $B_r = \{p \in \mathbb{R}^{m+n}_m, z(p) < r^2\}$ and $D_r = B_r \cap M$. We differential $F_t(D_r)$ with respect to t,

$$F_t'(D_r) = (4\pi)^{-\frac{n}{2}} t^{-(\frac{n}{2}+1)} \int_{D_r} \Big(-\frac{n}{2} + \frac{z}{4t}\Big) e^{-\frac{z}{4t}} d\mu.$$

Noting (4.1),

$$\begin{aligned} -e^{\frac{z}{4t}} \text{div}(e^{-\frac{z}{4t}} \nabla z) &= -\Delta z + \frac{1}{4t} \nabla z \cdot \nabla z \\ &= -2n - Y^2 + \frac{X^2}{t} \\ &\geq \frac{z}{t} - 2n \quad (\text{when } 0 < t \leq 1). \end{aligned} \tag{4.2}$$

Since

$$\nabla z = 2F^T$$

and the unit normal vector to ∂D_r is $\frac{F^T}{X}$, then

$$\begin{aligned} F_t'(D_r) &\leq \pi^{-\frac{n}{2}} (4t)^{-(\frac{n}{2}+1)} \int_{D_r} -\text{div}(e^{-\frac{z}{4t}} \nabla z) d\mu \\ &= \pi^{-\frac{n}{2}} (4t)^{-(\frac{n}{2}+1)} \int_{\partial D_r} -2X e^{-\frac{z}{4t}} \leq 0. \end{aligned} \tag{4.3}$$

We integrate $F_t'(D_r)$ over t from $\frac{1}{r}$ to 1, $r \geq 1$, and get

$$\int_{D_r} e^{-\frac{z}{4}} d\mu \leq r^{\frac{n}{2}} \int_{D_r} e^{-\frac{zr}{4}} d\mu.$$

Note that

$$\int_{D_r} e^{-\frac{z}{4}} d\mu \geq e^{-\frac{r^2}{4}} \int_{D_r} 1 d\mu$$

and

$$\begin{aligned}\int_{D_r} \mathrm{e}^{-\frac{zr}{4}}\,d\mu &= \int_{D_r\setminus D_{\frac{r}{2}}} \mathrm{e}^{-\frac{zr}{4}}\,d\mu + \int_{D_{\frac{r}{2}}} \mathrm{e}^{-\frac{zr}{4}}\,d\mu\\ &\le \mathrm{e}^{-\frac{r^3}{16}}\int_{D_r} 1d\mu + \int_{D_{\frac{r}{2}}} 1d\mu.\end{aligned}$$

Set $V(r) = \int_{D_r} 1d\mu$. Then,

$$(\mathrm{e}^{-\frac{r^2}{4}} - \mathrm{e}^{-\frac{r^3}{16}} r^{\frac{n}{2}})V(r) \le r^{\frac{n}{2}} V\Big(\frac{r}{2}\Big).$$

Let $g(r) = \mathrm{e}^{-\frac{r^2}{4}} - \mathrm{e}^{-\frac{r^3}{16}} r^{\frac{n}{2}}$. $g(r) > 0$ when r sufficiently large (say $r \ge 8n$). Since

$$\begin{aligned}g'(r) &= -\frac{r}{2}\mathrm{e}^{-\frac{r^2}{4}} - \frac{n}{2}r^{\frac{n}{2}-1}\mathrm{e}^{-\frac{r^3}{16}} + \frac{3r^2}{16}r^{\frac{n}{2}}\mathrm{e}^{-\frac{r^3}{16}}\\ &> \Big(-\frac{r}{2} - \frac{n}{2}r^{\frac{n}{2}-1} + \frac{3}{16}r^{\frac{n}{2}+2}\Big)\mathrm{e}^{-\frac{r^3}{16}} > 0,\end{aligned}$$

$g(r)$ is increasing in r and $g^{-1}(r)$ is decreasing in r. Therefore,

$$g^{-1}(r) \le \frac{1}{\mathrm{e}^{-16n^2} - \mathrm{e}^{-32n^3}(8n)^{\frac{n}{2}}} = C_1.$$

We then have

$$V(r) \le C_1 r^n V\Big(\frac{r}{2}\Big) \quad \text{for } r \text{ sufficiently large (say, } r \ge 8n).$$

By Lemma 4.2, we have

$$V(r) \le C_4 \mathrm{e}^{2n(\log r)^2} \quad \text{for } r \ge 8n,$$

here C_4 is a constant depending only on $n, V(8n)$. Hence, for any $\alpha > 0$,

$$\begin{aligned}\int_M \mathrm{e}^{-\alpha z}d\mu &= \sum_{j=0}^{\infty}\int_{D_{8n(j+1)}\setminus D_{8nj}} \mathrm{e}^{-\alpha z}d\mu \le \sum_{j=0}^{\infty} \mathrm{e}^{-\alpha(8nj)^2}V(8n(j+1))\\ &\le C_4\sum_{j=0}^{\infty}\mathrm{e}^{-\alpha(8nj)^2}\mathrm{e}^{2n(\log(8n)+\log(j+1))^2} \le C_5,\end{aligned}$$

where C_5 is a constant depending only on $n, V(8n)$. So we obtain our estimates. Certainly, M has weighted finite volume. □

Corollary 4.4 *Any space-like self-shrinker M of dimension n in $\mathbb{R}^{m+n}_m$ with closed Euclidean topology has finite fundamental group.*

From the Gauss equation, we have

$$\mathrm{Ric}(e_i, e_i) = \langle H, B_{ii}\rangle - \sum_j \langle B_{ij}, B_{ij}\rangle,$$

and

$$\mathrm{Hess}(f)(e_i, e_i) = \frac{1}{4}\mathrm{Hess}(z)(e_i, e_i) = \frac{1}{2}\delta_{ij} + \frac{1}{2}\langle F, B_{ij}\rangle = \frac{1}{2}\delta_{ij} - \langle H, B_{ij}\rangle.$$

It follows that

$$\mathrm{Ric}_f(e_i, e_i) = \mathrm{Ric}(e_i, e_i) + \mathrm{Hess}(f)(e_i, e_i) \ge \frac{1}{2}.$$

Set $B_R(p) \subset M$, a geodesic ball of radius R and centered at $p \in M$. From Theorem 3.1 in [18], we know that for any r there are constant A, B and C such that

$$\int_{B_R(p)} \rho \leq A + B \int_r^R \mathrm{e}^{-\frac{1}{2}t^2 + Ct} dt. \tag{4.4}$$

5 Rigidity Results

Now, we are in a position to prove rigidity results mentioned in the introduction.

Theorem 5.1 *Let M be a space-like self-shrinker of dimension n in R_m^{n+m}, which is closed with respect to the Euclidean topology. If there is a constant $\alpha < \frac{1}{8}$, such that $|H|^2 \leq \mathrm{e}^{\alpha z}$, then M is an affine n-plane.*

Proof Let η be a smooth function with compact support in M. Then by (3.14), we obtain

$$\begin{aligned}\int_M \left(\frac{1}{2}|H|^2 + |P|^2 + |\nabla H|^2\right)\eta^2 \rho &= \frac{1}{2}\int_M (\mathcal{L}|H|^2)\eta^2\rho = \frac{1}{2}\int_M \operatorname{div}(\rho\nabla|H|^2)\eta^2 \\ &= -\int_M \eta\rho\nabla|H|^2 \cdot \nabla\eta \\ &= 2\int_M \eta\rho\langle\nabla_i H, H\rangle \cdot \nabla_i \eta \\ &\leq \int_M |\nabla H|^2\eta^2\rho + \int_M |H|^2|\nabla\eta|^2\rho. \end{aligned} \tag{5.1}$$

We then have

$$\int_M \left(\frac{1}{2}|H|^2 + |P|^2\right)\eta^2\rho \leq \int_M |H|^2|\nabla\eta|^2\rho. \tag{5.2}$$

Let $\eta = \phi(\frac{|F|}{r})$ for any $r > 0$, where ϕ is a nonnegative function on $[0, +\infty)$ satisfying

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{if } x \in [2, +\infty), \end{cases}$$

and $|\phi'| \leq C$ for some absolute constant. Since $\nabla z = 2F^T$,

$$\nabla\eta = \frac{1}{r}\phi'\nabla\sqrt{z} = \frac{1}{r}\phi'\frac{F^T}{\sqrt{z}}.$$

By (1.2), we have

$$|\nabla\eta|^2 \leq \frac{C^2}{r^2}\frac{|F^T|^2}{z} = \frac{1}{r^2 z}C^2(z + 4|H|^2).$$

It follows that (5.2) becomes

$$\int_{D_r} \left(\frac{1}{2}|H|^2 + |P|^2\right)\rho \leq \frac{C^2}{r^2}\int_{D_{2r}\setminus D_r} |H|^2\left(1 + \frac{4|H|^2}{z}\right)\rho. \tag{5.3}$$

By Theorem 4.3, then under the condition on $|H|$, we obtain that the right-hand side of (5.3) approaches to zero as $r \to +\infty$. This implies that $H \equiv 0$.

According to Theorem 3.3 in [16], we see that M is complete with respect to the induced metric from $\mathbb{R}_m^{m+n}$. In a geodesic ball $B_a(x)$ of radius a and centered at $x \in M$, we can make gradient estimates of $|B|^2$ in terms of the mean curvature. From (2.9) in [16], we have

$$|B|^2 \leq k\frac{2m(n-4)a^2}{(a^2 - r^2)^2}.$$

Since M is complete, we can fix x and let a go to infinity. Hence, $|B|^2 = 0$ at any $x \in M$ and M is an n-plane. $\square$

Theorem 5.2 *Let M be a complete space-like self-shrinker of dimension n in R^{n+m}_m. If there is a constant $\alpha < \frac{1}{2}$, such that $\ln w \leq \mathrm{e}^{\alpha d^2(p,x)}$ for certain $p \in M$, where $d(p,\cdot)$ is the distance function from p, then M is affine n-plane.*

Proof (3.15) tells us

$$\mathcal{L}(\ln w) \geq \frac{|B|^2}{w^2} \geq 0.$$

As an application to (4.4) (Theorem 3.1 in [18]), Corollary 4.2 in [18] tells us that $\ln w$ is constant. This forces $|B|^2 \equiv 0$. $\square$

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