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Cohomology of the Schrödinger Algebra $\mathcal{S}(1)$

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Abstract We explicitly compute the first and second cohomology groups of the Schrödinger algebra $\mathcal{S}(1)$ with coefficients in the trivial module and the finite-dimensional irreducible modules. We also show that the first and second cohomology groups of $\mathcal{S}(1)$ with coefficients in the universal enveloping algebras $U(\mathcal{S}(1))$ (under the adjoint action) are infinite dimensional.

Keywords Schrödinger algebra, first cohomology groups, second cohomology groups

MR(2010) Subject Classification 17B56, 17B10

1 Introduction

Cohomology is an important tool in mathematics. Its range of applications contains algebra and topology as well as the theory of smooth manifolds or of holomorphic functions. The cohomology theory of Lie algebras has its origins in the work of Cartan, but the foundation of the theory, as an independent topic of research, is due to Chevalley and Eilenberg [5], Koszul [10] and Hochschild and Serre [9]. A unifying treatment of the cohomology theory of groups, associative algebras, and Lie algebras has been given by Cartan and Eilenberg [4]. As is well-known, several classical results in Lie algebra theory have a cohomological interpretation. If $\mathfrak{g}$ is a Lie algebra, the structure of the extensions of $\mathfrak{g}$-modules (and hence the question of semi-simplicity of $\mathfrak{g}$-modules) is described by the 1-cohomology of $\mathfrak{g}$, and the structure of the Lie algebra extensions (and hence the Levi, Malcev theorem) is related to the 2-cohomology [5].

Non-semisimple Lie algebras and groups play a very important role in the physics applications, recall, e.g., the Poincaré algebra and groups (for this and other examples, cf., e.g., [2]). The Schrödinger group is the symmetry group of the free particle Schrödinger equation and

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its Lie algebra is the so-called Schrödinger algebra. The Schrödinger algebra is non-semisimple and plays an important role in mathematical physics and its applications (see, e.g., [1–3, 11] and the recent review [6] containing many relevant references).

In this paper we compute the first and second cohomology groups for the Schrödinger algebra $\mathcal{S}(1)$ with coefficients in the trivial module and finite-dimensional irreducible modules. These cohomology groups are of crucial importance to the understanding of extensions of modules, and also extensions of the Lie algebras themselves. We also determine that the dimensions of the first and second cohomology groups for $\mathcal{S}(1)$ with coefficients in the universal enveloping algebra are infinite.

Let us briefly comment on the actual computation of the cohomology groups. The computation is carried at an elementary level. Detailed knowledge on the structure of finite-dimensional irreducible modules is used extensively, which in fact constitutes the essential input of the calculations. To determine the dimensions of the first and second cohomology groups for $\mathcal{S}(1)$ with coefficients in the universal enveloping algebra, we explore the first and second cohomology groups with coefficients in a 3-dimensional indecomposable $\mathcal{S}$-module.

The organization of the paper is as follows. Section 2 provides some necessary background material on $\mathcal{S}(1)$. Sections 3 and 4 contain the computations of the first and second cohomology groups of $\mathcal{S}(1)$ with coefficients in the trivial module and the finite-dimensional irreducible modules respectively. Section 5 is devoted to determining the dimensions of the first and second cohomology groups of $\mathcal{S}(1)$ with coefficients in the universal enveloping algebra.

2 Preliminaries

2.1 Schrödinger Algebra $\mathcal{S}(1)$

We shall work over the field $\mathbb{C}$ of complex numbers throughout the paper. The *Schrödinger algebra* $\mathcal{S} := \mathcal{S}(1)$ is the Lie algebra with basis $\{f, q, h, z, p, e\}$ and relations

$$\begin{aligned}
&[h, e] = 2e, \quad &&[h, f] = -2f, \quad &&[e, f] = h,\\
&[h, p] = p, \quad &&[h, q] = -q, \quad &&[p, q] = z,\\
&[e, q] = p, \quad &&[p, f] = -q, \quad &&[f, q] = 0,\\
&[e, p] = 0, \quad &&[z, \mathcal{S}] = 0.
\end{aligned}$$

The Schrödinger algebra can be viewed as a semidirect product

$$\mathcal{S} = \mathcal{H} \rtimes \mathfrak{sl}_2$$

of two subalgebras: a Heisenberg subalgebra $\mathcal{H} = \operatorname{span}\{p, q, z\}$ and $\mathfrak{sl}_2 = \operatorname{span}\{e, h, f\}$. Observe that $\mathcal{S} = \bigoplus_{i=-2}^{2} \mathcal{S}_i$ is $\mathbb{Z}$-graded, where $\mathcal{S}_{-2} = \mathbb{C}f$, $\mathcal{S}_{-1} = \mathbb{C}q$, $\mathcal{S}_0 = \mathbb{C}h + \mathbb{C}z$, $\mathcal{S}_1 = \mathbb{C}p$, $\mathcal{S}_2 = \mathbb{C}e$. $\mathcal{S}$ also has a triangular decomposition

$$\mathcal{S} = \mathcal{S}^- \oplus \mathcal{S}_0 \oplus \mathcal{S}^+,$$

where $\mathcal{S}^- = \operatorname{span}\{f, q\}$, $\mathcal{S}_0 = \operatorname{span}\{h, z\}$, $\mathcal{S}^+ = \operatorname{span}\{p, e\}$. Denote by $U(\mathcal{S})$ the universal enveloping algebra of $\mathcal{S}$.

2.2 Lie Algebra Cohomology

In this subsection we explain some basic concepts of Lie algebra cohomology. The material can be found in many sources, say, [5]. For $n \geq 1$ and an $\mathcal{S}$-module V, let the *space* $C^n(\mathcal{S}, V)$ *of* n*-cochains* be the vector space of all n-linear maps $\varphi : \mathcal{S} \times \cdots \times \mathcal{S} \to V$ satisfying the *skew symmetry condition*

$$\varphi(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -\varphi(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

for $1 \leq i \leq n-1$. Set $C^0(\mathcal{S}, V) = V$. We define the *differential operator* $\delta : C^n(\mathcal{S}, V) \to C^{n+1}(\mathcal{S}, V)$ by

$$\begin{aligned}(\delta\varphi)(x_0, \ldots, x_n) &= \sum_{i=0}^{n}(-1)^i x_i \cdot \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_n) \\ &\quad + \sum_{i<j}(-1)^j \varphi(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_n) \end{aligned} \tag{2.1}$$

for $\varphi \in C^n(\mathcal{S}, V)$ and $x_0, \ldots, x_n \in \mathcal{S}$, where the sign $\hat{}$ means that the element below it is omitted. It can be verified that $\delta^2 = 0$. Set

$$\begin{aligned} Z^n(\mathcal{S}, V) &= \{\varphi \in C^n(\mathcal{S}, V) \,|\, \delta\varphi = 0\}, \\ B^n(\mathcal{S}, V) &= \{\delta\varphi \,|\, \varphi \in C^{n-1}(\mathcal{S}, V)\}, \\ H^n(\mathcal{S}, V) &= Z^n(\mathcal{S}, V)/B^n(\mathcal{S}, V). \end{aligned}$$

The elements in $Z^n(\mathcal{S}, V)$ are called n*-cocycles*, while those in $B^n(\mathcal{S}, V)$ are n*-coboundaries*. The space $H^n(\mathcal{S}, V)$ is called the n*-th Lie algebra cohomology group* of $\mathcal{S}$ with coefficients in the module V. Two elements of $Z^n(\mathcal{S}, V)$ are said to be *cohomologous* if their images in $H^n(\mathcal{S}, V)$ are equal.

For $x \in \mathcal{S}$, we define maps

$$i_x : C^n(\mathcal{S}, V) \to C^{n-1}(\mathcal{S}, V), \quad \theta_x : C^n(\mathcal{S}, V) \to C^n(\mathcal{S}, V)$$

respectively by

$$(i_x\varphi)(x_1, \ldots, x_{n-1}) = \varphi(x, x_1, \ldots, x_{n-1}), \tag{2.2}$$

$$(\theta_x\varphi)(x_1, \ldots, x_n) = x \cdot \varphi(x_1, \ldots, x_p) - \sum_{i=1}^{n} \varphi(x_1, \ldots, x_{i-1}, [x, x_i], x_{i+1}, \ldots, x_n). \tag{2.3}$$

One can verify that

$$\delta i_x + i_x\delta = \theta_x \quad \text{and} \quad \delta\theta_x = \theta_x\delta. \tag{2.4}$$

Note that $\theta : x \mapsto \theta_x$ defines an $\mathcal{S}$-module structure on $C^n(\mathcal{S}, V)$ such that $Z^n(\mathcal{S}, V)$, $B^n(\mathcal{S}, V)$ are submodules by (2.4). One can also see that $\mathcal{S}$ acts on $H^n(\mathcal{S}, V)$ trivially by (2.4). If V is finite-dimensional, we can decompose

$$Z^n(\mathcal{S}, V) = Z_0^n(\mathcal{S}, V) \oplus B^n(\mathcal{S}, V) \tag{2.5}$$

as a direct sum of $\mathfrak{sl}_2$-submodules. Then $Z_0^n(\mathcal{S}, V)$ is isomorphic to $H^n(\mathcal{S}, V)$. For any $\mathcal{S}$-module W, we denote

$$W^{\mathfrak{sl}_2} = \{v \in W \,|\, \mathfrak{sl}_2 \cdot v = 0\}. \tag{2.6}$$

The elements in $W^{\mathfrak{sl}_2}$ are said to be $\mathfrak{sl}_2$-*invariant*.

Since $Z^n(\mathcal{S}, V)$ is finite dimensional, $Z^n(\mathcal{S}, V)$ as $\mathfrak{sl}_2$-module is completely reducible. From this one can easily prove the following results.

Proposition 2.1 *Let V be any finite-dimensional $\mathcal{S}$-module.*

(1) *For any cocycle $\varphi \in Z^n(\mathcal{S}, V)$ there exists an $\mathfrak{sl}_2$-invariant cocycle $\varphi' \in Z^n(\mathcal{S}, V)$ which is cohomologous to φ.*

(2) *Any $\mathfrak{sl}_2$-invariant coboundary $b \in B^n(\mathcal{S}, V)$ is equal to $\delta^{n-1}g$ with an $\mathfrak{sl}_2$-invariant cochain $g \in C^{n-1}(\mathcal{S}, V)$.*

The following two propositions will also be used later.

Proposition 2.2 *Let U, V, W be three $\mathcal{S}$-modules such that*

$$0 \to U \xrightarrow{j} V \xrightarrow{g} W \to 0$$

is a short exact sequence, where j, g are $\mathcal{S}$-module homomorphisms. Then there exists a long exact sequence

$$\cdots \to H^n(\mathcal{S}, U) \xrightarrow{j^n} H^n(\mathcal{S}, V) \xrightarrow{g^n} H^n(\mathcal{S}, W) \xrightarrow{d^*} H^{n+1}(\mathcal{S}, U) \to \cdots,$$

where the maps j^n, g^n can be defined easily from j, g, and d^* is the connecting homomorphism.

Proposition 2.3 *Suppose that V is the direct sum of a family $(V_i)_{i\in I}$ of submodules, i.e., $V = \bigoplus_{i\in I} V_i$. Then as vector spaces*

$$\bigoplus_{i\in I} H^n(\mathcal{S}, V_i) \cong H^n(\mathcal{S}, V).$$

3 First and Second Cohomology Groups with Coefficients in Trivial Module

In this section we shall determine $H^1(\mathcal{S}, \mathbb{C})$ and $H^2(\mathcal{S}, \mathbb{C})$.

Consider a 1-cocyle $\varphi \in Z^1(\mathcal{S}, \mathbb{C})$. For $x_0, x_1 \in \mathcal{S}$, we have

$$0 = (\delta\varphi)(x_0, x_1) = x_0 \cdot \varphi(x_1) - x_1 \cdot \varphi(x_0) - \varphi([x_0, x_1]) = \varphi([x_0, x_1]).$$

Since $[\mathcal{S}, \mathcal{S}] = \mathcal{S}$, one can easily deduce that $Z^1(\mathcal{S}, \mathbb{C}) = 0$. Thus

$$H^1(\mathcal{S}, \mathbb{C}) = 0.$$

Now we compute $H^2(\mathcal{S}, \mathbb{C})$. Choose a 2-cocycle $\varphi \in Z^2(\mathcal{S}, \mathbb{C})$. By Proposition 2.1, we only need to consider $\mathfrak{sl}_2$-invariant cocycles. Thus we may assume that φ is $\mathfrak{sl}_2$-invariant. Define a linear map $\xi : \mathcal{S} \to \mathbb{C}$ by

$$\xi(z) = \varphi(p, q), \quad \xi(e) = \xi(f) = \xi(h) = \xi(p) = \xi(q) = 0.$$

One can easily check that ξ and $\delta\xi$ are $\mathfrak{sl}_2$-invariant. Replacing φ by $\varphi + (\delta\xi)$ if necessary, we may assume that

$$\varphi(p, q) = 0.$$

The $\mathfrak{sl}_2$-invariance and the cocycle condition imply that

$$-\varphi([x_0, x_1], x_2) + \varphi([x_0, x_2], x_1) + \varphi(x_0, [x_1, x_2]) = 0, \tag{3.1}$$

$$\varphi([e, x], y) + \varphi(x, [e, y]) = 0, \tag{3.2}$$

$$\varphi([f, x], y) + \varphi(x, [f, y]) = 0, \tag{3.3}$$

$$\varphi([h,x],y)+\varphi(x,[h,y])=0 \tag{3.4}$$

for $x_0, x_1, x_2, x, y \in \mathcal{S}$. Putting $(x,y)=(e,f),(h,f)$ in (3.2), we have

$$\varphi(e,h)=0, \quad \varphi(e,f)=0.$$

Putting $(x,y)=(e,f)$ in (3.3), we have

$$\varphi(h,f)=0.$$

Putting $(x,y)=(f,z),(q,z),(h,p)$ in (3.2), we obtain

$$\varphi(h,z)=0, \quad \varphi(p,z)=0, \quad \varphi(e,p)=0.$$

Similarly, one can obtain

$$\varphi(q,z)=\varphi(f,q)=0.$$

Putting $(x,y)=(h,p)$ in (3.3) and $(x,y)=(f,q)$ in (3.2) respectively, we have

$$2\varphi(f,p)+\varphi(h,q)=0, \quad \varphi(h,q)+\varphi(f,p)=0.$$

This forces that

$$\varphi(f,p)=\varphi(h,q)=0.$$

Similarly, we have

$$\varphi(h,p)=\varphi(e,q)=0.$$

Putting $(x,y)=(e,z)$ in (3.4), we have $\varphi(e,z)=0$. Similarly, we have $\varphi(f,z)=0$.

Now we have proved that

$$H^2(\mathcal{S},\mathbb{C})=0.$$

In conclusion, we obtain

Theorem 3.1 $H^1(\mathcal{S},\mathbb{C})=0$, $H^2(\mathcal{S},\mathbb{C})=0$.

4 First and Second Cohomology Groups with Coefficients in Finite-Dimensional Irreducible Modules

Let V be a finite-dimensional irreducible $\mathcal{S}$-module. Then V must be an irreducible $\mathfrak{sl}_2$-module with

$$p\cdot V=q\cdot V=z\cdot V=0 \tag{4.1}$$

as shown in [7].

4.1 First Cohomology

Choose any $\mathfrak{sl}_2$-invariant 1-cocycle $\varphi\in Z^1(\mathcal{S},V)$. Since V is an irreducible $\mathfrak{sl}_2$-module, it is well known that $H^1(\mathfrak{sl}_2,V)=0$. Replacing φ by $\varphi-b$ for some 1-coboundary $b\in B^1(\mathcal{S},V)$ if necessary, we can assume that

$$\varphi(\mathfrak{sl}_2)=0.$$

The cocycle condition implies that

$$0=(\delta\varphi)(x,y)=x\cdot\varphi(y)-y\cdot\varphi(x)-\varphi([x,y]) \tag{4.2}$$

for $x, y \in \mathcal{S}$. Putting $(x, y) = (p, q)$ in (4.2) and using (4.1), we obtain

$$\varphi(z) = 0.$$

Putting $(x, y) = (e, p), (h, p), (f, p), (e, q), (h, q), (f, q)$ in (4.2) respectively, we have

$$e \cdot \varphi(p) = 0, \quad h \cdot \varphi(p) = \varphi(p), \quad f \cdot \varphi(p) = \varphi(q),$$
$$e \cdot \varphi(q) = \varphi(p), \quad h \cdot \varphi(q) = -\varphi(q), \quad f \cdot \varphi(q) = 0,$$

which implies that $\varphi(p), \varphi(q)$ span an irreducible $\mathcal{S}$-module if $\varphi(p) \neq 0,\ \varphi(q) \neq 0$. Thus if $\dim V \neq 2$, we have

$$\varphi(p) = \varphi(q) = 0 \quad \text{and} \quad H^1(\mathcal{S}, V) = 0.$$

Now suppose that V is 2-dimensional and has a basis $\{v,\ f \cdot v\}$ such that $h \cdot v = v$. Then one can check that the linear map $\phi : \mathcal{S} \to V$ defined by

$$\phi(p) = v, \quad \phi(q) = f \cdot v,$$
$$\phi(e) = \phi(f) = \phi(h) = \phi(z) = 0$$

is a 1-cocycle. Furthermore, one can check ϕ is non-trivial because any 1-coboundary maps p, q to zero by (4.1). Using the representation theory of $\mathfrak{sl}_2$, we easily see that the space

$$\{\varphi \in Z^1(\mathcal{S}, V) \,|\, \varphi(e) = \varphi(f) = \varphi(h) = \varphi(z) = 0\}$$

is 1-dimensional.

Theorem 4.1 $H^1(\mathcal{S}, V) \cong \begin{cases} \mathbb{C}, & \textit{if} \dim V = 2, \\ 0, & \textit{otherwise.} \end{cases}$

4.2 Second Cohomology

Choose any $\mathfrak{sl}_2$-invariant 2-cocycle $\varphi \in Z^2(\mathcal{S}, V)$. Since V is an irreducible $\mathfrak{sl}_2$-module, it is well known that $H^2(\mathfrak{sl}_2, V) = 0$. Replacing φ by $\varphi - b$ for some 2-coboundary $b \in B^2(\mathcal{S}, V)$ if necessary, we can assume that

$$\varphi(\mathfrak{sl}_2, \mathfrak{sl}_2) = 0.$$

The $\mathfrak{sl}_2$-invariance and the cocycle condition imply that

$$x_0 \cdot \varphi(x_1, x_2) - x_1 \cdot \varphi(x_0, x_2) + x_2 \cdot \varphi(x_0, x_1)$$
$$-\varphi([x_0, x_1], x_2) + \varphi([x_0, x_2], x_1) + \varphi(x_0, [x_1, x_2]) = 0, \tag{4.3}$$
$$e \cdot \varphi(x, y) - \varphi([e, x], y) - \varphi(x, [e, y]) = 0, \tag{4.4}$$
$$f \cdot \varphi(x, y) - \varphi([f, x], y) - \varphi(x, [f, y]) = 0, \tag{4.5}$$
$$h \cdot \varphi(x, y) - \varphi([h, x], y) - \varphi(x, [h, y]) = 0 \tag{4.6}$$

for $x_0, x_1, x_2, x, y \in \mathcal{S}$. Putting $(x, y) = (p, q)$ in (4.4), (4.5) and (4.6) respectively, we obtain

$$e \cdot \varphi(p, q) = 0, \quad f \cdot \varphi(p, q) = 0, \quad h \cdot \varphi(p, q) = 0,$$

which implies that

$$\varphi(p, q) = 0. \tag{4.7}$$

Putting $(x, y) = (f, p)$ in (4.4), we have

$$e \cdot \varphi(f, p) = \varphi(h, p). \tag{4.8}$$

Putting $(x_0, x_1, x_2) = (e, f, p)$ in (4.3) and using (4.8), we obtain

$$f \cdot \varphi(e, p) = \varphi(e, q). \tag{4.9}$$

Putting $(x, y) = (e, p)$ in (4.5), using (4.9) and (4.8), we obtain

$$\varphi(h, p) = 0, \quad e \cdot \varphi(f, p) = 0. \tag{4.10}$$

Similarly, one can obtain

$$\varphi(h, q) = 0, \quad f \cdot \varphi(e, q) = 0. \tag{4.11}$$

Putting $(x, y) = (f, p)$ in (4.6), we have

$$h \cdot \varphi(f, p) = -\varphi(f, p). \tag{4.12}$$

Since V is finite dimensional, (4.12) and (4.10) imply

$$\varphi(f, p) = 0. \tag{4.13}$$

Similarly, we obtain

$$\varphi(e, q) = 0. \tag{4.14}$$

Putting $(x_0, x_1, x_2) = (e, f, q)$ in (4.3) and $(x, y) = (f, q)$ in (4.6) respectively, using (4.11), (4.13) and (4.14), we have

$$e \cdot \varphi(f, q) = 0, \quad h \cdot \varphi(f, q) = -3\varphi(f, q),$$

which forces that

$$\varphi(f, q) = 0.$$

Similarly, we obtain

$$\varphi(e, p) = 0.$$

Putting $(x_0, x_1, x_2) = (e, p, q), (f, p, q), (h, p, q)$ in (4.3) respectively and using (4.7), we have

$$\varphi(e, z) = 0, \quad \varphi(f, z) = 0, \quad \varphi(h, z) = 0.$$

Putting $(x, y) = (p, z), (q, z)$ in (4.4), (4.5) and (4.6) respectively, we have

$$e \cdot \varphi(p, z) = 0, \quad f \cdot \varphi(p, z) = \varphi(q, z), \quad h \cdot \varphi(p, z) = \varphi(p, z),$$
$$e \cdot \varphi(q, z) = \varphi(p, z), \quad f \cdot \varphi(q, z) = 0, \quad h \cdot \varphi(q, z) = -\varphi(q, z),$$

which implies that $\varphi(p, z), \varphi(q, z)$ span an irreducible $\mathcal{S}$-module if $\varphi(p, z) \neq 0$, $\varphi(q, z) \neq 0$. Thus if $\dim V \neq 2$, we have

$$\varphi(p, z) = \varphi(q, z) = 0 \quad \text{and} \quad H^2(\mathcal{S}, V) = 0.$$

Now suppose that V is 2-dimensional and has a basis $\{v, f \cdot v\}$ such that $h \cdot v = v$. Then one can check that the skew symmetric linear map $\phi : \mathcal{S} \times \mathcal{S} \to V$ defined by

$$\phi(p, z) = v, \quad \phi(q, z) = f \cdot v,$$
$$\phi(\mathcal{S}, \mathfrak{sl}_2) = \phi(p, q) = 0$$

is a 2-cocycle. Furthermore, one can check ϕ is non-trivial because any 2-coboundary maps $(p, z), (q, z)$ to zero by (4.1). Using the representation theory of $\mathfrak{sl}_2$, we easily see that the space

$$\{\varphi \in Z^2(\mathcal{S}, V) \,|\, \varphi(\mathcal{S}, \mathfrak{sl}_2) = \varphi(p, q) = 0\}$$

is 1-dimensional.

Theorem 4.2 $H^2(\mathcal{S}, V) \cong \begin{cases} \mathbb{C}, & \textit{if} \dim V = 2, \\ 0, & \textit{otherwise.} \end{cases}$

5 First and Second Cohomology Groups with Coefficients in the Universal Enveloping Algebra

In this section we want to determine $\dim H^1(\mathcal{S}, U(\mathcal{S}))$ and $\dim H^2(\mathcal{S}, U(\mathcal{S}))$. For convenience, set $U := U(\mathcal{S})$. Note that U is an $\mathcal{S}$-module with respect to the adjoint action.

For $m \geq 1$ denote by $\mathscr{S}_m$ the symmetric group on $\{1, 2, \ldots, m\}$. Set

$$\mathfrak{S}^m = \text{span}\bigg\{ \frac{1}{m!} \sum_{\sigma \in \mathscr{S}_m} x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(m)}} \,\bigg|\, x_{i_1}, \ldots, x_{i_m} \in \{f, q, h, z, p, e\} \bigg\}.$$

It is easy to see that $\mathfrak{S}^m$ are finite-dimensional $\mathcal{S}$-modules with respect to the adjoint action. Set $\mathfrak{S}^0 = \mathbb{C}$ and $\mathfrak{S} = \bigoplus_{m=0}^{\infty} \mathfrak{S}^m$. It is not difficult to see that $U \cong \mathfrak{S}$ regarded as $\mathcal{S}$-modules. Thus from Proposition 2.3, we have

$$H^i(\mathcal{S}, U) \cong H^i(\mathcal{S}, \mathfrak{S}) = \bigoplus_{m=0}^{\infty} H^i(\mathcal{S}, \mathfrak{S}^m). \tag{5.1}$$

Lemma 5.1 *For $m \geq 1$,*

$$\dim H^1(\mathcal{S}, \mathfrak{S}^m) \geq 1, \quad \dim H^2(\mathcal{S}, \mathfrak{S}^m) \geq 1.$$

Proof Each $\mathfrak{S}^m$ $(m \geq 1)$ can be decomposed as a direct sum of finite-dimensional indecomposable submodules

$$\mathfrak{S}^m = \bigoplus_{j \in J} \mathfrak{S}^m_j, \qquad \text{where } J \text{ is a finite set.} \tag{5.2}$$

Observe that each $\mathfrak{S}^m$ $(m \geq 1)$ contains a 3-dimensional indecomposable $\mathcal{S}$-submodule V spanned by pz^{m-1}, qz^{m-1}, z^m. Set $U = \mathbb{C}\, z^m$, $W = \text{span}\{pz^{m-1}, qz^{m-1}\}$. Regarded as $\mathfrak{sl}_2$-module, $V \cong U \oplus W$. It is well known that

$$H^1(\mathfrak{sl}_2, U) = H^1(\mathfrak{sl}_2, W) = 0, \quad H^2(\mathfrak{sl}_2, U) = H^2(\mathfrak{sl}_2, W) = 0.$$

Thus from Proposition 2.3, $H^1(\mathfrak{sl}_2, V) = H^1(\mathfrak{sl}_2, V) = 0$.

Define $\varphi \in Z^1(\mathcal{S}, V)$, $\psi \in Z^2(\mathcal{S}, V)$, respectively, by

$$\begin{aligned} &\varphi(p) = pz^{m-1}, \quad \varphi(q) = qz^{m-1}, \quad \varphi(\mathfrak{sl}_2) = 0, \\ &\psi(p, z) = pz^{m-1}, \quad \psi(q, z) = qz^{m-1}, \quad \psi(\mathfrak{sl}_2, \mathcal{S}) = \psi(p, q) = 0. \end{aligned}$$

Using similar arguments in Sections 4.1 and 4.2, it is not difficult to see that

$$H^1(\mathcal{S}, V) = \mathbb{C}\varphi, \quad H^2(\mathcal{S}, V) = \mathbb{C}\psi.$$

Thus $\dim H^1(\mathcal{S}, V) = \dim H^2(\mathcal{S}, V) = 1$. Combining this with (5.2) and Proposition 2.3, we have

$$\dim H^1(\mathcal{S}, \mathfrak{S}^m) \geq 1, \quad \dim H^2(\mathcal{S}, \mathfrak{S}^m) \geq 1.$$

□

Remark 5.2 To prove $\dim H^1(\mathcal{S}, \mathfrak{S}^m) \geq 1$, one can also consider the short exact sequence

$$0 \to U \xrightarrow{i} V \xrightarrow{\pi} V/U \to 0,$$

where i, π are the canonical maps. By Proposition 2.2, there exists a long exact sequence

$$\begin{aligned}\cdots \to H^0(\mathcal{S}, V/U) &\xrightarrow{d^*} H^1(\mathcal{S}, U) \xrightarrow{i^1} H^1(\mathcal{S}, V) \xrightarrow{\pi^1} H^1(\mathcal{S}, V/U) \\ &\xrightarrow{d^*} H^2(\mathcal{S}, U) \xrightarrow{i^2} H^2(\mathcal{S}, V) \xrightarrow{\pi^2} H^2(\mathcal{S}, V/U) \to \cdots . \end{aligned} \tag{5.3}$$

Note that U is a 1-dimensional trivial $\mathcal{S}$-module and the quotient V/U is a 2-dimensional simple $\mathcal{S}$-module. From Theorems 3.1 and 4.1, we have

$$\dim H^1(\mathcal{S}, U) = \dim H^2(\mathcal{S}, U) = 0, \quad \dim H^1(\mathcal{S}, V/U) = 1.$$

Thus from (5.3), we have

$$\dim H^1(\mathcal{S}, V) = \dim H^1(\mathcal{S}, V/U) = 1.$$

It follows from Proposition 2.3 that

$$\dim H^1(\mathcal{S}, \mathfrak{S}^m) \geq 1.$$

The following result directly follows from Lemma 5.1 and (5.1).

Theorem 5.3

$$\dim H^1(\mathcal{S}, U(\mathcal{S})) = \infty, \quad \dim H^2(\mathcal{S}, U(\mathcal{S})) = \infty.$$

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