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The Exceptional Set for Sums of Unlike Powers of Primes

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Abstract It is established that all even positive integers up to N but at most $O(N^{15/16+\varepsilon})$ exceptions can be expressed in the form $p_1^2 + p_2^3 + p_3^4 + p_4^5$, where p_1, p_2, p_3 and p_4 are prime numbers.

Keywords Circle method, Waring–Goldbach problem, exceptional set

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1 Introduction

In 1951, Roth [9] proved that almost all positive integers n can be represented in the form

$$n = m_1^2 + m_2^3 + m_3^4 + m_4^5,$$

where m_1, m_2, m_3 and m_4 are positive integers. In 1953, Prachar [4] refined the above result by establishing that almost all positive even integers n can be represented in the form

$$n = p_1^2 + p_2^3 + p_3^4 + p_4^5, \tag{1.1}$$

where p_1, p_2, p_3 and p_4 are prime numbers. Let $E(N)$ denote the number of positive even integers n up to N, which cannot be expressed in the form (1.1). Prachar [4] proved

$$E(N) \ll N(\log N)^{-\frac{30}{47}+\varepsilon}, \tag{1.2}$$

where $\varepsilon > 0$ is an arbitrary positive number. Bauer [1] improved (1.2) to

$$E(N) \ll N^{1-\delta+\varepsilon} \quad \text{with } \delta = \frac{1}{2742}. \tag{1.3}$$

The estimate (1.3) was further improved by Ren and Tsang to $\delta = 1/66$ in [7], and to $\delta = 1/48$ in [8]. Recently, Bauer [2] proved that one can take $\delta = 47/1680$.

In this note, we establish the following result.

Theorem 1.1 *Let $E(N)$ be defined as above. Then we have*

$$E(N) \ll N^{1-\frac{1}{16}+\varepsilon} \tag{1.4}$$

for any $\varepsilon > 0$.

Note that $47/1680 < 1/35.7$, Theorem 1.1 improves upon the result of Bauer [2] by a factor more than 2. We establish Theorem 1.1 by the Hardy–Littlewood method.

As usual, we abbreviate $\mathrm{e}^{2\pi\mathrm{i}\alpha}$ to $e(\alpha)$. The letter p, with or without a subscript, always denotes a prime number. We use ε to denote a sufficiently small positive number, and the value

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of ε may change from statement to statement. Denote by $d(n)$ the number of divisors of n, and denote by $\phi(n)$ the Euler function. We use $\Lambda(n)$ to stand for the von Mangoldt function.

2 Preliminary

Suppose that N is a large positive integer. For $2 \le k \le 5$, we define

$$f_k(\alpha) = \sum_{P_k < p \le 2P_k} (\log p) e(p^k \alpha),$$

where $P_k = (N/5)^k$. Let

$$r(n) = \sum_{\substack{p_2^2+p_3^3+p_4^4+p_5^5=n \\ P_k<p_k\le 2P_k (2\le k\le 5)}} (\log p_2)(\log p_3)(\log p_4)(\log p_5).$$

By orthogonality, we have

$$r(n) = \int_0^1 \bigg(\prod_{k=2}^5 f_k(\alpha)\bigg) e(-n\alpha)d\alpha.$$

We apply the Hardy–Littlewood method to show that $r(n) > 0$ for all even numbers $n \le N$ but at most $O(N^{1-\frac{1}{16}+\varepsilon})$ exceptions. Let

$$L = \log N, \quad P = N^{1/10-\varepsilon}, \quad Q = NP^{-1}. \tag{2.1}$$

Denote by $\mathfrak{M}(q,a)$ the interval $[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} - \frac{1}{qQ}]$. We write $\mathfrak{M}$ for the union of $\mathfrak{M}(q,a)$ for $1 \le a \le q \le P$ and $(a,q) = 1$. Then we define the minor arcs $\mathfrak{m}$ as the complement of $\mathfrak{M}$ in $[N^{-1/2}, 1+N^{-1/2}]$.

In order to describe the contribution from the major arcs, we introduce some notations. Let

$$C_k(q,a) = \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\bigg(\frac{ar^k}{q}\bigg)$$

and

$$B(q;n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\bigg(-\frac{an}{q}\bigg) \prod_{k=2}^5 C_k(q,a).$$

The singular series is defined by

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \frac{B(q;n)}{\phi(q)^4}.$$

We define the singular integral as

$$\mathfrak{I}(n) = \int_{-\infty}^{+\infty} \prod_{j=2}^5 \Phi_j(\lambda) e(-n\lambda) d\lambda,$$

where

$$\Phi_k(\lambda) = \frac{1}{k} \int_{P_k^k}^{(2P_k)^k} u^{\frac{1}{k}-1} e(\lambda u) du.$$

We point out that for even integers $n \in [N, 2N]$, one has

$$1 \ll \mathfrak{S}(n) \ll 1 \tag{2.2}$$

and

$$N^{17/60} \ll \mathfrak{J}(n) \ll N^{17/60}. \tag{2.3}$$

Lemma 2.1 ([8, Lemma 2.1]) *Let* $S_k(\alpha) = \sum_{P_k < m \le 2P_k} \Lambda(m) e(m^k \alpha)$. *Suppose that* $N \le n \le 2N$ *and* n *is even. Then we have*

$$\int_{\mathfrak{M}} \left(\prod_{k=2}^{5} S_k(\alpha) \right) e(-n\alpha) d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O(N^{\frac{17}{60}} L^{-A}),$$

where A *is arbitrary large.*

Lemma 2.2 ([6, Theorem 1.1]) *Suppose that* α *is a real number, and that* $|\alpha - a/q| \le q^{-2}$ *with* $(a,q) = 1$. *Let* $\beta = \alpha - a/q$. *Then one has*

$$f_k(\alpha) \ll d(q)^{c_k} (\log N)^c \left(P_k^{1/2} \sqrt{q(1+N|\beta|)} + P_k^{4/5} + \frac{P_k}{\sqrt{q(1+N|\beta|)}} \right),$$

where $c_k = 1/2 + \log k / \log 2$ *and* c *is a constant.*

Lemma 2.3 ([3, Theorem 3]) *Suppose that* α *is a real number, and that there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *with*

$$(a,q) = 1, \quad 1 \le q \le N^{1/2} \quad \text{and} \quad |q\alpha - a| \le N^{-1/2}.$$

Then for $k \in \{2, 4\}$, *one has*

$$f_k(\alpha) \ll P_k^{1-\eta_k+\varepsilon} + \frac{q^{-\frac{1}{2}} P_k^{1+\varepsilon}}{(1+N|\alpha - a/q|)^{1/2}},$$

where $\eta_2 = 1/8$ *and* $\eta_4 = 1/24$.

Let $\mathfrak{R}$ denote the union of intervals $\mathfrak{R}(q,a)$ for $1 \le a \le q \le N^{1/6}$ and $(a,q) = 1$, where $\mathfrak{R}(q,a) = [\frac{a}{q} - \frac{1}{qN^{5/6}}, \frac{a}{q} + \frac{1}{qN^{5/6}}]$. We write $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{R}$ and $\mathfrak{m}_2 = \mathfrak{m} \backslash \mathfrak{R}$.

For $\alpha \in \mathfrak{m}_1$, Lemma 2.2 provides the following estimate.

Lemma 2.4 *Suppose that* $\alpha \in \mathfrak{m}_1$. *Then we have*

$$f_3(\alpha) \ll P_3^{1-\frac{1}{5}+\varepsilon}.$$

Proof For $\alpha \in \mathfrak{m}_1$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$(a,q) = 1, \quad 1 \le a \le q \le N^{1/6} \quad \text{and} \quad |q\alpha - a| \le N^{-5/6}.$$

Since $\alpha \notin \mathfrak{M}$, one has either $q > P$ or $|q\alpha - a| > Q^{-1}$. Then we apply Lemma 2.2 to conclude that $f_3(\alpha) \ll P_3^{4/5+\varepsilon}$ for $\alpha \in \mathfrak{m}_1$. □

Lemma 2.5 *Suppose that* $\alpha \in \mathfrak{m}_2$. *Then we have*

$$f_2(\alpha) \ll P_2^{1-\frac{1}{8}+\varepsilon} \quad \text{and} \quad f_4(\alpha) \ll P_4^{1-\frac{1}{24}+\varepsilon}.$$

Proof By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q) = 1, \quad 1 \le q \le N^{1/2} \quad \text{and} \quad |q\alpha - a| \le N^{-1/2}.$$

Since $\alpha \in \mathfrak{m} \backslash \mathfrak{R}$, we have either $q > N^{1/6}$ or $N|q\alpha - a| > N^{1/6}$. The conclusions follow from Lemma 2.3. □

Lemma 2.6 *Let* $g_k(\alpha) = \sum_{P_k<x\le 2P_k} e(m^k\alpha)$, *where* x *denotes an integer. Suppose that* $F \in \{g_3^4, g_4^4, g_3^2g_5^2, g_5^6\}$. *Then one has*

$$\int_0^1 |g_2^2(\alpha)F(\alpha)|d\alpha \ll F(0)^{1+\varepsilon}.$$

Proof The conclusion can be deduced by counting the number of solutions for the underlying diophantine equation. We only give the details for

$$\int_0^1 |g_2^2(\alpha)g_5^6(\alpha)|d\alpha \ll g_5(0)^{6+\varepsilon}.$$

The above integration is no more than the number of solutions for the equation

$$x_1^2 - x_2^2 = y_1^5 + y_2^5 + y_3^5 - y_4^5 - y_5^5 - y_6^5$$

with $P_2 < x_j \le 2P_2$ and $P_5 < y_k \le 2P_5$. If $x_1 \ne x_2$, the contribution is bounded by $P_5^{6+\varepsilon}$. If $x_1 = x_2$, the contribution is bounded by $P_2 \int_0^1 |g_5(\alpha)|^6 d\alpha$. An application of Hua's lemma (see [10, Lemma 2.5]) gives

$$\int_0^1 |g_5(\alpha)|^6 d\alpha \ll P_5^{7/2+\varepsilon}.$$

The conclusion is established. □

3 The Minor Arcs Estimate

We define the multiplicative function $w(q)$ by taking

$$w(p^{3u+v}) = \begin{cases} 3p^{-u-1/2}, & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le 3. \end{cases} \tag{3.1}$$

Lemma 3.1 *For* $\gamma \in \mathbb{R}$, *we define*

$$\mathcal{L}(\gamma) = \sum_{w(q)\ge P_3^{-1/4}} \sum_{\substack{1\le a\le q \\ (a,q)=1}} \int_{|\alpha-a/q|\le N} \frac{w(q)^2|\sum_{P_5<p\le 2P_5} e(p^k(\alpha+\gamma))|^2}{1+N|\alpha-a/q|} d\alpha.$$

One has uniformly for $\gamma \in \mathbb{R}$ *that*

$$\mathcal{L}(\gamma) \ll P_5^2 N^{-1+\varepsilon}. \tag{3.2}$$

Proof We have

$$\sum_{1\le a\le q}\Big|\sum_{P_5<p\le 2P_5} e\Big(p^k\frac{a}{q} + p^k(\beta+\gamma)\Big)\Big|^2 = q \sum_{\substack{P_5<p_1,p_2\le 2P_5 \\ p_1^k\equiv p_2^k \,(\mathrm{mod}\ q)}} e((p_1^k - p_2^k)(\beta+\gamma))$$
$$\ll q \sum_{P_5<p_1\le 2P_5} \sum_{\substack{P_5<p_2\le 2P_5 \\ p_2^k\equiv p_1^k \,(\mathrm{mod}\ q)}} 1.$$

One can deduce from (3.1) and $w(q) \ge P_3^{-1/4}$ that $(p,q) = 1$ for $P_5 < p \le 2P_5$. Then we have

$$\sum_{1\le a\le q}\Big|\sum_{P_5<p\le 2P_5} e\Big(p^k\frac{a}{q} + p^k(\beta+\gamma)\Big)\Big|^2 \ll q \sum_{P_5<p_1\le 2P_5} \sum_{\substack{1\le b<q \\ b^k\equiv p_1^k \,(\mathrm{mod}\ q)}} \Big(\frac{P_5}{q} + 1\Big)$$

$$\ll (P_5^2 + qP_5) \sum_{\substack{1 \le b < q \\ b^k \equiv 1 (\mathrm{mod}\ q)}} 1$$
$$\ll (P_5^2 + qP_5) q^{\varepsilon}.$$

We conclude from above that

$$\mathcal{L}(\gamma) \ll \sum_{w(q) \ge P_3^{-1/4}} w(q)^2 (P_5^2 + qP_5) N^{-1+\varepsilon}.$$

For any $q \in \mathbb{N}$, one has the unique decomposition $q = sr^3$ with s cube-free. Then by the definition of $w(q)$, we have $w(q) \ll s^{-1/2} r^{-1} P_3^{\varepsilon}$. Therefore,

$$\sum_{w(q) \ge P_3^{-1/4}} w(q)^2 q \ll P_3^{\varepsilon} \sum_{s^{-1/2} r^{-1} \gg P_3^{-1/4-\varepsilon}} r \ll P_3^{1/2+\varepsilon}.$$

On applying Lemma 2.1 in [11], we can obtain $\mathcal{L}(\gamma) \ll P_5^2 N^{-1+\varepsilon}$. □

Let

$$\mathcal{M}(q,a) = \left\{ \alpha : \ |\alpha - a/q| \le \frac{P_3^{3/4}}{qN} \right\} \quad \text{and} \quad \mathcal{M}(q) = \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(q,a).$$

We define $\mathcal{M}$ to be the union of $\mathcal{M}(q)$ with $w(q) \ge P_3^{-1/4}$. Then we define the function $\Psi(\alpha)$ on $\mathcal{M}$ by taking

$$\Psi(\alpha) = \frac{1}{1 + N|\alpha - a/q|} \quad \text{if } \alpha \in \mathcal{M}(q,a)$$

for a and q satisfying $1 \le a \le q$ and $w(q) \ge P_3^{-1/4}$. Let

$$\mathfrak{J} = \sup_{\beta \in [0,1)} \int_{\mathcal{M}} w(q)^2 \Psi(\alpha)^2 |f_5(\alpha + \beta)|^2 \alpha. \tag{3.3}$$

For $t \ge 1/2$, we define

$$\mathcal{I}(t) = \int_{\mathfrak{m}_2} |f_2(\alpha)^2 f_3(\alpha)^t f_4(\alpha)^2 f_5(\alpha)^2| d\alpha \tag{3.4}$$

and

$$J(t) = P_2^2 P_3^t P_4^2 P_5^2. \tag{3.5}$$

Lemma 3.2 *Let* $\mathcal{I}(t)$ *be defined in* (3.4). *Then we have*

$$\mathcal{I}(2) \ll P_3 \mathfrak{J}^{1/4} \left(\int_{\mathfrak{m}_2} |f_2(\alpha)^4 f_3(\alpha)^2 f_4(\alpha)^4 f_5(\alpha)^2| d\alpha \right)^{1/4} \mathcal{I}(1)^{1/2} + P_3^{1-1/8+\varepsilon} \mathcal{I}(1),$$

where $\mathfrak{J}$ *is given in* (3.3).

Proof In view of [11, Lemma 2.3], one has

$$\sum_{P_3 < x \le 2P_3} e(x^3 \alpha) \ll P_3 w(q) \Psi(\alpha)$$

if $\alpha \in \mathcal{M}(q,a)$ for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying $(a,q) = 1$ and $w(q) \ge P_3^{-1/4}$. Otherwise, one has

$$\sum_{P_3 < x \le 2P_3} e(x^3 \alpha) \ll P_3^{3/4+\varepsilon}.$$

Then the desired conclusion can be established by following the proof of Lemma 3.1 in [11]. □

Lemma 3.3 *Let $\mathcal{I}(t)$ be defined in* (3.4). *Then we have*

$$\mathcal{I}(2) \ll (P_3^4 P_5^2 N^{-1+\varepsilon})^{1/4} \bigg(\int_{\mathfrak{m}_2} |f_2(\alpha)^4 f_3(\alpha)^2 f_4(\alpha)^4 f_5(\alpha)^2| d\alpha \bigg)^{1/4} \mathcal{I}(1)^{1/2} + P_3^{1-1/8+\varepsilon} \mathcal{I}(1).$$

Proof This follows from Lemmas 3.1 and 3.2. □

Lemma 3.4 *Let $J(1/2)$ be given by* (3.5). *Then one has*

$$\int_0^1 |f_2(\alpha)^2 f_3(\alpha)^{1/2} f_4(\alpha)^2 f_5(\alpha)^2| d\alpha \ll J(1/2) N^{-1+\varepsilon}.$$

Proof By Hölder's inequality,

$$\int_0^1 |f_2^2 f_3^{1/2} f_4^2 f_5^2| d\alpha \ll \bigg(\int_0^1 |f_2^2 f_4^4| d\alpha \bigg)^{1/2} \bigg(\int_0^1 |f_2^2 f_3^2 f_5^2| d\alpha \bigg)^{1/4} \bigg(\int_0^1 |f_2^2 f_5^6| d\alpha \bigg)^{1/4}.$$

The desired estimate follows by applying Lemma 2.6. □

Lemma 3.5 *We have*

$$\mathcal{I}(2) \ll J(2) N^{-1-\frac{1}{16}+\varepsilon}.$$

Proof By Lemma 2.5, one has

$$\begin{aligned} \int_{\mathfrak{m}_2} |f_2(\alpha)^4 f_3(\alpha)^2 f_4(\alpha)^4 f_5(\alpha)^2| d\alpha &\ll \sup_{\alpha \in \mathfrak{m}_2} |f_2(\alpha)^2 f_4(\alpha)^2| \mathcal{I}(2) \\ &\ll P_3^{-7/16+\varepsilon} P_2^2 P_4^2 \mathcal{I}(2). \end{aligned}$$

Then we conclude from Lemma 3.3 that

$$\mathcal{I}(2) \ll (P_3^4 P_5^2 N^{-1+\varepsilon})^{1/4} \big(P_3^{-7/16+\varepsilon} P_2^2 P_4^2 \mathcal{I}(2) \big)^{1/4} \mathcal{I}(1)^{1/2} + P_3^{1-1/8+\varepsilon} \mathcal{I}(1).$$

By Hölder's inequality, $\mathcal{I}(1) \le \mathcal{I}(2)^{1/3} \mathcal{I}(1/2)^{2/3}$. Therefore,

$$\begin{aligned} \mathcal{I}(2) \ll\ & P_3^{-7/64} (P_3^4 P_5^2 N^{-1+\varepsilon} P_2^2 P_4^2)^{1/4} \mathcal{I}(2)^{5/12} \mathcal{I}(1/2)^{1/3} \\ & + P_3^{1-1/8+\varepsilon} \mathcal{I}(2)^{1/3} \mathcal{I}(1/2)^{2/3}, \end{aligned}$$

and this implies

$$\mathcal{I}(2) \ll P_3^{-3/16} (P_3^4 P_5^2 N^{-1+\varepsilon} P_2^2 P_4^2)^{3/7} \mathcal{I}(1/2)^{4/7} + P_3^{3/2-3/16+\varepsilon} \mathcal{I}(1/2).$$

The desired conclusion now follows by applying Lemma 3.4. □

4 Proof of Theorem 1.1

Proof of Theorem 1.1 By Bessel's inequality,

$$\sum_{N<n\le 2N} \bigg| \int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha) f_4(\alpha) f_5(\alpha) e(-n\alpha) d\alpha \bigg|^2 \le \int_{\mathfrak{m}} |f_2(\alpha) f_3(\alpha) f_4(\alpha) f_5(\alpha)|^2 d\alpha. \tag{4.1}$$

On applying Lemma 3.5, we have

$$\int_{\mathfrak{m}_2} |f_2(\alpha) f_3(\alpha) f_4(\alpha) f_5(\alpha)|^2 d\alpha \ll N^{17/30+1-1/16+\varepsilon}. \tag{4.2}$$

For the contribution from $\mathfrak{m}_1$, one has

$$\int_{\mathfrak{m}_1} |f_2(\alpha)f_3(\alpha)f_4(\alpha)f_5(\alpha)|^2 d\alpha$$
$$\ll \sup_{\alpha\in\mathfrak{m}_1} |f_3(\alpha)^{3/2}| \int_{\mathfrak{m}_1} |f_2(\alpha)^2 f_3(\alpha)^{1/2} f_4(\alpha)^2 f_5(\alpha)^2| d\alpha.$$

Then we can conclude from Lemmas 2.4 and 3.4 that

$$\int_{\mathfrak{m}_1} |f_2(\alpha)f_3(\alpha)f_4(\alpha)f_5(\alpha)|^2 d\alpha \ll N^{17/30+1-1/16+\varepsilon}. \tag{4.3}$$

Thus (4.2) and (4.3) yield

$$\int_{\mathfrak{m}} |f_2(\alpha)f_3(\alpha)f_4(\alpha)f_5(\alpha)|^2 d\alpha \ll N^{17/30+1-1/16+\varepsilon}. \tag{4.4}$$

Inserting (4.4) into (4.1), we arrive at

$$\sum_{N<n\le 2N} \left| \int_{\mathfrak{m}} f_2(\alpha)f_3(\alpha)f_4(\alpha)f_5(\alpha)e(-n\alpha)d\alpha \right|^2 \ll N^{17/30+1-1/16+\varepsilon}.$$

Thus for all even integers $n \in (N, 2N]$ but at most $O(N^{1-1/16+\varepsilon})$ exceptions, one has the estimate

$$\left| \int_{\mathfrak{m}} f_2(\alpha)f_3(\alpha)f_4(\alpha)f_5(\alpha)e(-n\alpha)d\alpha \right| \ll N^{17/60-\varepsilon}. \tag{4.5}$$

Next we show that

$$\sum_{N\le n\le 2N} \left| \int_{\mathfrak{M}} \left(\prod_{k=2}^{5} f_k(\alpha) - \prod_{k=2}^{5} S_k(\alpha) \right) e(-n\alpha)d\alpha \right|^2 \ll N^{17/30+1-2/15+\varepsilon}. \tag{4.6}$$

In view of Bessel's inequality, our task is to prove

$$\int_{\mathfrak{M}} \left| \prod_{k=2}^{5} f_k(\alpha) - \prod_{k=2}^{5} S_k(\alpha) \right|^2 d\alpha \ll N^{17/30+1-2/15+\varepsilon}. \tag{4.7}$$

Note that

$$f_2f_3f_4f_5 - S_2S_3S_4S_5 = (f_2 - S_2)f_3f_4f_5 + (f_3 - S_3)S_2f_4f_5$$
$$+ (f_4 - S_4)S_2S_3f_5 + (f_5 - S_5)S_2S_3S_4.$$

The estimate (4.7) follows from

$$\int_{\mathfrak{M}} |(f_5(\alpha) - S_5(\alpha))S_2(\alpha)S_3(\alpha)S_4(\alpha)|^2 d\alpha \ll N^{17/30+1-1/5+\varepsilon}, \tag{4.8}$$

$$\int_{\mathfrak{M}} |(f_4(\alpha) - S_4(\alpha))S_2(\alpha)S_3(\alpha)S_5(\alpha)|^2 d\alpha \ll N^{17/30+1-1/4+\varepsilon}, \tag{4.9}$$

and

$$\int_{\mathfrak{M}} |F(\alpha)|^2 d\alpha \ll N^{17/30+1-2/15+\varepsilon}, \tag{4.10}$$

where $F \in \mathcal{F} := \{(f_2 - S_2)f_3f_4f_5, (f_3 - S_3)S_2f_4f_5\}$.

For $2 \le k \le 5$, one has $f_k(\alpha) - S_k(\alpha) \ll P_k^{1/2+\varepsilon}$. By Lemma 2.6, we have

$$\int_{\mathfrak{M}} |(f_5(\alpha) - S_5(\alpha))S_2(\alpha)S_3(\alpha)S_4(\alpha)|^2 d\alpha$$

$$\ll N^{1/5+\varepsilon}\bigg(\int_0^1 |S_2(\alpha)^2S_3(\alpha)^4|d\alpha\bigg)^{1/2}\bigg(\int_0^1 |S_2(\alpha)^2S_4(\alpha)^4|d\alpha\bigg)^{1/2}$$
$$\ll N^{17/30+1-1/5+\varepsilon}.$$

This establishes (4.8). The estimate (4.9) can be established similarly. For $F \in \mathcal{F}$, one has

$$\int_{\mathfrak{M}} |F(\alpha)|^2 d\alpha \ll (P_2P_3P_4P_5)^2 P_3^{-1+\varepsilon} \int_{\mathfrak{M}} d\alpha \ll N^{17/30+1-2/15+\varepsilon}.$$

Thus (4.10) is established. Now by (4.6), for all even integers $n \in (N, 2N]$ but at most $O(N^{1-2/15+\varepsilon})$ exceptions, one has

$$\int_{\mathfrak{M}} \bigg(\prod_{k=2}^{5} f_k(\alpha)\bigg) e(-n\alpha)d\alpha = \int_{\mathfrak{M}} \bigg(\prod_{k=2}^{5} S_k(\alpha)\bigg) e(-n\alpha)d\alpha + O(N^{\frac{17}{60}-\varepsilon}).$$

Then by Lemma 2.1, for all even integers $n \in (N, 2N]$ but at most $O(N^{1-2/15+\varepsilon})$ exceptions, we have

$$\int_{\mathfrak{M}} \bigg(\prod_{k=2}^{5} f_k(\alpha)\bigg) e(-n\alpha)d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O(N^{\frac{17}{60}} L^{-A}). \tag{4.11}$$

The argument around (4.5) and (4.11) together with (2.2) and (2.3) imply that $r(n) \gg N^{17/60}$ for all even integers $n \in (N, 2N]$ but at most $O(N^{1-1/16+\varepsilon})$ exceptions. We now complete the proof of Theorem 1.1 by the dyadic argument.

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