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Regularity of the Inverse of a Homeomorphism with Finite Inner Distortion

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Abstract Let $f:\Omega\to f(\Omega)\subset\mathbb{R}^n$ be a $W^{1,1}$-homeomorphism with L^1-integrable inner distortion. We show that finiteness of $\min\{\mathrm{lip}_f(x),k_f(x)\}$, for every $x\in\Omega\backslash E$, implies that $f^{-1}\in W^{1,n}$ and has finite distortion, provided that the exceptional set E has σ-finite $\mathscr{H}^1$-measure. Moreover, f has finite distortion, differentiable a.e. and the Jacobian $J_f>0$ a.e.

Keywords Mapping of finite distortion, mappings of finite inner distortion, bi-Sobolev homeomorphism, Condition N on a.e. sphere, modulus of rectifiable surfaces

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1 Introduction

Recall that a non-constant continuous mapping $f:\Omega\to\mathbb{R}^n$ is said to be a mapping of finite distortion if f belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$, $J(\cdot,f)\in L^1_{\mathrm{loc}}(\Omega)$ and $J(\cdot,f)$ is strictly positive almost everywhere on the set where $|Df|\neq 0$. Here $|Df(\cdot)|$ and $J(\cdot,f)$ are the operator norm and the Jacobian determinant of $Df(\cdot)$, respectively. Associated with such a mapping the (outer) distortion $K(\cdot,f)$ is defined as

$$K(x,f)=\frac{|Df(x)|^n}{J(x,f)},\quad \text{if } J(x,f)>0,$$

and $K(x,f)=1$ otherwise. Similarly, a non-constant continuous mapping $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is said to have finite inner distortion if $J(\cdot,f)\geq 0$ a.e., $J(\cdot,f)\in L^1_{\mathrm{loc}}(\Omega)$, and $D^{\#}f(\cdot)$ vanishes a.e. in the zero set of $J(\cdot,f)$. Here $D^{\#}f(\cdot)$ represents the adjoint matrix of the differential $Df(\cdot)$. With such mappings we associate an inner distortion function $K_I:\Omega\to[1,\infty)$ defined as

$$K_I(x,f)=\frac{|D^{\#}f(x)|^n}{J(x,f)^{n-1}}$$

when $J(x,f)>0$, and $K(x,f)=1$ otherwise. The trivial inequality

$$K_I(x,f)\leq K(x,f)^n$$

implies that mappings of finite distortion are necessarily mappings of finite inner distortion. But the converse is not true in general. For instance, the mapping $f:\mathbb{R}^n\to\mathbb{R}^n$, $n\geq 3$,

$$f(x_1,\ldots,x_n)=(x_1,0,\ldots,0)$$

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is a mapping of finite inner distortion, but not a mapping of finite distortion. On the other hand, the reverse estimate

$$K(x,f) \le K_I(x,f)^{n-1} \quad \text{if } J(x,f) > 0$$

implies that the two classes of mappings coincide if $J(\cdot, f)$ is positive a.e. From now on, we sometimes write $J_f(\cdot)$ for the Jacobian determinant of $Df(\cdot)$.

An important and remarkable result of Hencl and Koskela [10] asserts that *if $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ is a homeomorphism with finite distortion, then f is a bi-Sobolev homeomorphism, i.e., $f \in W^{1,1}$ and $f^{-1} \in W^{1,1}$. Moreover, if the distortion is integrable, then $f^{-1} \in W^{1,2}$.* The converse is also true, see [11]. Such a conclusion is not valid in higher dimensions [12, 13].

A recent result of Csörnyei et al. [2] implies that homeomorphisms in $W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ with finite (outer) distortion are bi-Sobolev mappings. Moreover, the inverse mapping also has finite (outer) distortion. This has been extended to a wider class of mappings of finite inner distortion in [6].

The connection between general bi-Sobolev mappings and mappings of finite inner distortion are given by the following two results. The first result was proved in [13].

Let $f : \Omega \to \mathbb{R}^n$ be a bi-Sobolev mapping. If $J_f = 0$ a.e. on a measurable set A, then $|D^{\#} f(x)| = 0$ a.e. on A. If we moreover assume that $J_f \ge 0$, it follows that f has finite inner distortion.

In the opposite direction, we have the second result [6, 14, 18].

Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a homeomorphism with finite L^1-integrable inner distortion. Then f^{-1} is a $W^{1,n}(f(\Omega), \Omega)$ mapping of finite distortion. Moreover,

$$|Df^{-1}(y)|^n \le K_I(f^{-1}(y), f) J_{f^{-1}}(y) \quad \textit{a.e. in } \Omega,$$

$$\int_{f(\Omega)} |Df^{-1}(y)| dy = \int_\Omega |D^{\#} f(x)| dx, \tag{1.1}$$

and

$$\int_{f(\Omega)} |Df^{-1}(y)|^n dy = \int_\Omega K_I(x,f) dx. \tag{1.2}$$

In this paper, we prove an analogous result by relaxing the Sobolev regularity $W^{1,n-1}$ but additionally requiring finiteness of some pointwise constants.

Theorem 1.1 *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a homeomorphism of L^1-integrable inner distortion and let $E \subset \mathbb{R}^n$ be a set of σ-finite $\mathscr{H}^1$-measure. Suppose that for every $x \in \Omega \backslash E$, $\min\{\text{lip}_f(x), k_f(x)\} < \infty$. Then $f^{-1} \in W^{1,n}(f(\Omega), \mathbb{R}^n)$ and has finite distortion. Moreover,*

$$|Df^{-1}(y)|^n \le K_I(f^{-1}(y), f) J_{f^{-1}}(y) \quad \textit{a.e. in } \Omega$$

and (1.1), (1.2) *hold.*

Remark 1.2 It follows from Theorem 1.1 and Equation (1.1) that $D^{\#} f(\cdot) \in L^1(\Omega)$.

Recall that for a homeomorphism $f : \Omega \to \mathbb{R}^n$, the pointwise constants lip_f and k_f are defined by

$$\text{lip}_f(x) := \liminf_{r \to 0} \frac{L(x,r)}{r}, \quad k_f(x) := \liminf_{r \to 0} k_f(x,r),$$

where

$$L_f(x,r) = \sup_{x' \in B(x,r)\cap\Omega} |f(x') - f(x))|, \quad k_f(x,r) = \left(\frac{L_f(x,r)^n}{|f(B(x,r))|} \right)^{\frac{1}{n-1}}.$$

Note that in general the finiteness of lip_f does not imply Sobolev regularity for f and so Theorem 1.1 is not covered by the previous mentioned result. On the other hand, Sobolev regularity is implied by the integrability of lip_f. More precisely, $f \in W^{1,p}_{\mathrm{loc}}$ if we assume that $\mathrm{lip}_f \in L^p_{\mathrm{loc}}$, see [21, Corollary 1.4].

We will prove Theorem 1.1 in two steps. In the first step, we prove that under the analytic assumption that $\min\{\mathrm{lip}_f(x), k_f(x)\} < \infty$ for all $x \in \Omega\backslash E$, f satisfies condition N on a.e. sphere (see Section 2 below for definition). In this step, we follow closely the approach used by Williams [21], whereas the idea goes back to Balogh et al. [1]. In the second step, we prove that if $f \in W^{1,1}$ is a homeomorphism with L^1-integrable inner distortion such that f satisfies Condition N on a.e. sphere, then $f^{-1} \in W^{1,n}$ and has finite distortion. In this step, we follow the approach used by Csörnyei et al. [2].

This paper is organized as follows. Section 2 contains the basic definitions and some auxiliary results. In Section 3, we prove that f satisfies Condition N on a.e. spheres. The proof of Theorem 1.1 is given in Section 4. The last section, Section 5, contains a short discussion on related results in the non-homeomorphic case.

2 Preliminaries and Auxiliary Results

2.1 Condition N on m-Rectifiable Surfaces

In this paper, we call a Borel measurable set $S \subset \Omega$ an m-rectifiable surface, $m \geq 1$, if S is m-rectifiable with $0 < \mathscr{H}^m(S) < \infty$ and there exist a constant $\delta > 0$ and a constant $C(m) > 0$ such that $\mathscr{H}^m(B(x,r)\cap S) \geq C(m)r^m$ for every $x \in S$ and every $r < \delta$.

Recall that for a set $A \subset \mathbb{R}^n$ and $s > 0$, the s-Hausdorff measure of A is defined as follows. First, for each $\delta > 0$, we define the premeasure

$$\mathscr{H}^s_\delta(A) = \inf \sum_{B\in\mathcal{B}} (\operatorname{diam} B)^s,$$

where the infimum is taken over all covers $\mathcal{B}$ of A by countable balls of diameter at most δ. When δ gets smaller, the number of possible covers decreases, so the limit

$$\lim_{\delta\to 0} \mathscr{H}^s_\delta(A) = \mathscr{H}^s(A) \in [0,\infty]$$

exists and is called the s-Hausdorff measure of A.

A map $f : \Omega \to \mathbb{R}^n$ is said to satisfy Lusin's Condition N on an m-rectifiable surface S if for any set $E \subset S$ with $\mathscr{H}^m(E) = 0$, $\mathscr{H}^m(f(E)) = 0$. The case $m = n-1$ is of particular interest for us. We say that f satisfies Condition N on almost every sphere, if for any ball $B(x,r) \subset \Omega$, f satisfies Condition N on almost every $(n-1)$-dimensional sphere $S(x,t)$, $0 < t < r$ with respect to $\mathscr{H}^{n-1}$, i.e., for a.e. $0 < t < r$, $\mathscr{H}^{n-1}(f(A)) = 0$ whenever $A \subset S(x,t)$ and $\mathscr{H}^{n-1}(A) = 0$. Similarly, we say that f satisfies Condition N^{-1} on almost every sphere, if for any ball $B(y,r) \subset f(\Omega)$ and a.e. $0 < t < r$, it holds that $\mathscr{H}^{n-1}(f^{-1}(E)) = 0$ whenever $E \subset S(y,t)$ and $\mathscr{H}^{n-1}(E) = 0$.

If f is a homeomorphism, then it is easy to see that f satisfies Condition N^{-1} (or N) on a.e. sphere if and only if f^{-1} satisfies Condition N (or N^{-1}) on a.e. sphere.

2.2 Conformal Modulus for m-Rectifiable Surfaces

Let Λ be a family of m-rectifiable surfaces in $\mathbb{R}^n$, we set the q-modulus of Λ to be

$$\operatorname{Mod}_q \Lambda = \inf \left\{ \int_{\mathbb{R}^n} \rho(x)^q dx : \rho : \mathbb{R}^n \to [0, \infty] \text{ is admissible for } \Lambda \right\}.$$

Recall that a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is said to be admissible for a surface family Λ if $\int_S \rho(x) d\mathscr{H}^m(x) \geq 1$ for all $S \in \Lambda$. If $q = \frac{n}{m}$, the q-modulus is called the conformal modulus for Λ. We say that a property F holds for q-almost every m-rectifiable surfaces S if the family of m-rectifiable surfaces S such that the property F fails has zero q-modulus. By [5, Theorem 1], the map

$$\Lambda \mapsto \operatorname{Mod}_q(\Lambda)$$

defines an outer measure on the collection of all families of m-rectifiable surfaces in $\mathbb{R}^n$.

2.3 Approximate Differential and Chain Rule

Let $f : \Omega \to \mathbb{R}^n$ be a mapping. We say that f is approximately differentiable at x if there exists a linear mapping $L : \Omega \to \mathbb{R}^n$ such that

$$\text{ap-}\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0. \tag{2.1}$$

It is well known that such an L, if exists, is unique [4]. We will use the notation $Df(x)$ for L and call $Df(x)$ the approximate differential of f at x, since if f is differentiable at x, then the approximate differential coincides with the usual differential. We will denote by

$$\mathcal{D}_f = \{x \in \Omega : f \text{ is approximately differentiable at } x\}.$$

Recall that a Sobolev mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is approximately differentiable a.e. in Ω.

We need the following chain rule, which was originally proved in [6, Lemma 2.1].

Lemma 2.1 ([18, Lemma 3.6]) *Let $f : \Omega \to \Omega'$ be a homeomorphism such that f and f^{-1} are approximately differentiable a.e. Set $E = \{y \in \mathcal{D}_{f^{-1}} : |J_{f^{-1}}(y)| > 0\}$. Then there exists a Borel set $A \subset E$ with $|E \backslash A| = 0$ such that $f^{-1}(A) \subset \{x \in \mathcal{D}_f : |J_f(x)| > 0\}$, with the property that*

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1} \quad \text{for all } y \in A.$$

2.4 Discrete and Open Mapping

Let $f : \Omega \to \mathbb{R}^n$ be open and discrete, the latter meaning that $f^{-1}(y)$ cannot have accumulation points in Ω for any $y \in \mathbb{R}^n$. In particular, $N(y, f, A) < \infty$ whenever $A \subset\subset \Omega$. Recall that $N(y, f, \Omega)$ is defined as the number of preimages of y under f in Ω. We also put $N(f, \Omega) := \sup_{y \in \mathbb{R}^n} N(y, f, \Omega)$.

A domain $U \subset\subset \Omega$ is called a normal domain of f if $f(\partial U) = \partial f(U)$. The openness of f implies that $\partial f(U) \subset f(\partial U)$ holds for every domain U. An open, connected neighborhood $U \subset\subset \Omega$ of a point $x \in \Omega$ is called a normal neighborhood of x if $f(\partial U) = \partial f(U)$ and $U \cap f^{-1}(f(x)) = \{x\}$. Denote by $U(x, f, r)$ the x-component of $f^{-1}(B(f(x), r))$. For the following lemma, see [19, Chapter I, Lemma 4.9].

Lemma 2.2 ([19, Lemma 4.9]) *For each point $x \in \Omega$, there is $\sigma_x > 0$ such that $U(x, f, r)$ is a normal neighborhood of x whenever $0 < r \le \sigma_x$. Moreover,* $\operatorname{diam} U(x, f, r) \to 0$ *as* $r \to 0$.

Next, suppose that $x \in \Omega$ and $0 < r \le \sigma_x$, where σ_x is as in the above lemma. The local index $i(x, f)$ of f at x is defined as

$$i(x, f) = N(f, U(x, f, r)).$$

Note in particular that $N(f, U(x, f, r))$ is independent of $r \le \sigma_x$. Thus, $i(x, f) = 1$ if and only if $x \in \Omega \backslash B_f$, where B_f is the branch set of f.

The local index is in fact a topological concept. Recall that for a continuous mapping $f : \Omega \to \mathbb{R}^n$, we can define the local degree $\mu(y, f, U)$ for any domain $U \subset\subset \Omega$ and every $y \in \mathbb{R}^n \backslash f(\partial U)$. If U is a normal domain, we define $\mu(f, U) = \mu(y, f, U)$ for some $y \in f(U)$. This is well-defined because $\mu(y, f, U) = \mu(z, f, U)$ whenever $y, z \in f(U)$. We call f sense-preserving if $\mu(y, f, U) > 0$ whenever $U \subset\subset \Omega$ and $y \in f(U) \backslash f(\partial U)$.

With these concepts, the local index $i(\cdot, f)$ can be defined as follows. Let $x \in \Omega$ and U be a normal neighborhood of x, we define $i(x, f) = \mu(f, U)$. It is easy to show that $i(x, f)$ does not depend on the normal neighborhood of U. We refer the interested reader to [19, Section I 4] for more information.

2.5 Area Formula

We discuss the well-known area formula in this subsection. Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Then for every Borel set $E \subset\subset \Omega$ and for any nonnegative Borel function η on $\mathbb{R}^n$, we have

$$\int_E \eta(f(x))|J(x, f)|dx \le \int_{\mathbb{R}^n} \eta(y) N(y, f, E)dy. \tag{2.2}$$

The above equality holds if f satisfies the so-called Lusin Condition N. Recall that a mapping $f : \Omega \to \mathbb{R}^n$ is said to satisfy the Lusin Condition N if the implication "$|E| = 0 \Rightarrow |f(E)| = 0$" holds for all measurable sets $E \subset \Omega$. On the other hand, let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a continuous, discrete and open mapping. Then the area formula holds in $\mathcal{D}_f$, i.e.,

$$\int_{\mathcal{D}_f} \eta(f(x))|J(x, f)|dx = \int_{f(\mathcal{D}_f)} \eta(y) N(y, f, \mathcal{D}_f)dy \tag{2.3}$$

for every nonnegative Borel function η in $\mathbb{R}^n$; see for instance [18, Theorem 3.3].

We need also the following simple lemma from harmonic analysis, see for instance [15, Lemma 2.8].

Lemma 2.3 *Fix $1 \le p < \infty$. Let $B_1, B_2, \dots$ be balls in $\mathbb{R}^n$, $a_j \ge 0$ and $\lambda > 1$. Then*

$$\Big\| \sum a_j \chi_{\lambda B_j} \Big\|_p \le C(\lambda, p, n) \Big\| \sum a_j \chi_{B_j} \Big\|_p.$$

3 Condition N on Almost Every Sphere

In this section, we prove that under the analytic condition as in Theorem 1.1, f satisfies the Condition N on almost every sphere in Ω.

Theorem 3.1 *Let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a homeomorphism and let $E \subset \mathbb{R}^n$ be a set of σ-finite $\mathscr{H}^1$-measure. Suppose that for every $x \in \Omega \backslash E$, $\min\{\mathrm{lip}_f(x), k_f(x)\} < \infty$. Then on 1-almost every $(n-1)$-rectifiable surface S in Ω, f satisfies Condition N with respect to $\mathscr{H}^{n-1}$,* *and $f(S)$ has σ-finite $(n-1)$-measure. In particular, f satisfies Condition N on almost every sphere in Ω.*

In the case that lip_f is finite outside a set of σ-finite $\mathscr{H}^1$-measure, Theorem 3.1 holds for any continuous mapping f as well.

Theorem 3.2 *Let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a continuous map and let $E \subset \mathbb{R}^n$ be a set of σ-finite $\mathscr{H}^1$-measure. Suppose that for every $x \in \Omega \backslash E$, $\mathrm{lip}_f(x) < \infty$. Then on 1-almost every $(n-1)$-rectifiable surface S in Ω, f satisfies Condition N with respect to $\mathscr{H}^{n-1}$, and $f(S)$ has σ-finite $(n-1)$-measure. In particular, f satisfies Condition N on almost every sphere in Ω.*

We will follow the approach of Williams [21] and we need several auxiliary results to prove both results. The first one is a version of Fuglede's lemma for rectifiable surfaces.

Lemma 3.3 (Fuglede's lemma) *Let $\{g_i\}$ be a sequence of Borel functions that converges in $L^q(\Omega)$. Then there is a subsequence $\{g_{i_k}\}$ with the following property: if g is any Borel representative of the L^q-limit of $\{g_i\}$, then*

$$\lim_{k\to\infty} \int_S |g_{i_k} - g| d\mathscr{H}^m = 0$$

for q-almost every m-rectifiable surface S in Ω.

Proof Let g be an arbitrary Borel representative of the L^p-limit of $\{g_i\}$. Choose a subsequence $\{g_{i_k}\}$ of $\{g_i\}$ so that

$$\int_\Omega |g_{i_k} - g|^q dx \le 2^{-kq-k}.$$

Note that this subsequence is independent of the particular representative g. Define

$$\rho_k = |g_{i_k} - g|.$$

Let Λ be the family of m-rectifiable surfaces S in Ω for which the statement

$$\lim_{k\to\infty} \int_S \rho_k d\mathscr{H}^m = 0$$

fails to hold, and let Λ_k be the family of all m-rectifiable surfaces S in Ω for which

$$\int_S \rho_k d\mathscr{H}^m > 2^{-k}.$$

Then

$$\Lambda \subset \bigcup_{k=j}^{\infty} \Lambda_k$$

for each $j \ge 1$. On the other hand, $2^k\rho_k$ is admissible for Λ_k for each k, so that

$$\mathrm{Mod}_q(\Lambda_k) \le 2^{kq} \int_\Omega \rho_k^q dx < 2^{-k}.$$

Consequently, we have

$$\mathrm{Mod}_q(\Lambda) \le \sum_{k=j}^{\infty} \mathrm{Mod}_q(\Lambda_k) \le 2^{-j+1}. \qquad \square$$

The next result is an adjustment of [1, Lemma 3.5], see also [22, Proposition 3.2].

Proposition 3.4 *Let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a continuous map. Let $E \subset \Omega$ have σ-finite $\mathscr{H}^{n-p}$-measure for some $1 \le p \le n$. Then*

$$\mathrm{Mod}_q(\{S \in \Lambda : \mathscr{H}^m(f(S \cap E)) > 0\}) = 0,$$

where $q = \frac{p}{m}$ and Λ is the family of m-rectifiable surfaces in Ω.

Proof In light of the sub-additivity of modulus, it suffices to consider the case $\mathscr{H}^{n-p} < \infty$. Fix $\varepsilon > 0$ and set

$$\Lambda_\varepsilon = \{S \in \Lambda : \mathscr{H}^m(f(S \cap E)) > \varepsilon^m\}.$$

We need to show that Λ_ε has zero modulus; the claim will then follow, again by sub-additivity of modulus. We may assume that $E \subset \frac{1}{2}B_0$ for some ball B_0. Since $\mathbb{R}^n$ is proper, the closed ball $\overline{B}_0$ is compact, and since f is continuous, its restriction to $\overline{B}_0$ is uniformly continuous. Hence, for each $k \in \mathbb{N}$, there exists a $\delta_k > 0$ such that, for all $x, x' \in \overline{B}_0$,

$$|x - x| < \delta_k \Rightarrow |f(x) - f(x')| < \frac{\varepsilon}{2^{k+3}}. \tag{3.1}$$

We may assume that $\{\delta_k\}$ is a sequence of positive numbers decreasing to zero. Fix $\tilde{\varepsilon} > 0$. Using the definition of Hausdorff measure and applying the $5r$-covering theorem, we find a sequence of balls $\{B_i^k\}_i$ such that $E \subset \bigcup_i 5B_i^k$, $B_i^k \cap B_j^k = \emptyset$ when $i \ne j$, $\mathrm{diam}(B_i^k) < \frac{\delta_k}{5}$, $2B_i^k \subset B_0$ and such that

$$\sum_i (\mathrm{diam}(B_i^k))^{n-p} < \mathscr{H}^{n-p}(E) + \tilde{\varepsilon}. \tag{3.2}$$

Consider the sequence $\{\rho_k\}_k$ of Borel functions, defined by

$$\rho_k(x) = \frac{1}{2^k} \sum_i \frac{1}{\mathrm{diam}(B_i^k)^m} \chi_{12B_i^k}(x).$$

By Lemma 2.3 we see that

$$\begin{aligned}\int_\Omega \rho_k^q d\mu &= \frac{1}{2^{kq}} \int_\Omega \Big(\sum_i \frac{\chi_{12B_i^k}}{\mathrm{diam}(B_i^k)^m} \Big)^q dx \\ &\le \frac{C}{2^{kq}} \sum_i \int_\Omega \frac{\chi_{B_i^k}}{(\mathrm{diam}(B_i^k))^{mq}} dx \\ &\le \frac{C}{2^{kq}} \sum_i \frac{|B_i^k|}{\mathrm{diam}(B_i^k)^{mq}}.\end{aligned}$$

By (3.2), we finally conclude that

$$\int_\Omega \rho_k^q dx \le \frac{C}{2^{kq}} \sum_i \mathrm{diam}(B_i^k)^{n-mq} \le \frac{C}{2^{kq}} (\mathscr{H}^{n-p}(E) + \tilde{\varepsilon}). \tag{3.3}$$

Set

$$\tilde{\rho}_k(x) = \sum_{j=k}^\infty \rho_j(x).$$

Then

$$\Big(\int_\Omega \tilde{\rho}_k^q dx \Big)^{\frac{1}{q}} = \Big(\int_\Omega \Big(\sum_{j=k}^\infty \rho_j \Big)^q dx \Big)^{\frac{1}{q}} \le \sum_{j=k}^\infty \Big(\int_\Omega \rho_j^q dx \Big)^{\frac{1}{q}}$$

$$\leq \sum_{j=k}^{\infty} \left(\frac{C}{2^{jq}} (\mathscr{H}^{n-p}(E) + \tilde{\varepsilon}) \right)^{\frac{1}{q}}$$

$$\leq \frac{C}{2^k} (\mathscr{H}^{n-p}(E) + \tilde{\varepsilon})^{\frac{1}{q}}. \tag{3.4}$$

Next, we show that the functions $4\tilde{\rho}_k$ are admissible for the modulus. For any $S \in \Lambda_\varepsilon$, $\mathscr{H}^m(f(S \cap E)) > \varepsilon^m$. This implies that, for sufficiently large integers k, there exist points $y_1, \dots, y_{2^{(k-2)}} \in f(S \cap E)$ such that

$$|y_i - y_j| > \frac{\varepsilon}{2^{k+2}} \quad \text{for } i \neq j. \tag{3.5}$$

By inequalities (3.1) and (3.5), at least 2^{k-2} sets $f(5B_i^k)$ will be needed to cover $f(S \cap E)$. Hence, there are at least 2^{k-2} balls $5B_i^k$ that hit S. It follows that for any $j \in \mathbb{N}$, we may find $k \geq j$ such that S is not entirely contained in any ball $12B_i^k$ and there exist 2^{k-2} points y_m such that (3.5) holds. Therefore,

$$\int_S \tilde{\rho}_j d\mathscr{H}^m \geq \int_S \rho_k d\mathscr{H}^m = \frac{1}{2^k} \sum_i \int_S \frac{\chi_{12B_i^k}}{\operatorname{diam}(B_i^k)^m} d\mathscr{H}^m$$

$$\geq \frac{1}{2^k} \sum_{5B_i^k \cap S \neq \emptyset} \frac{\operatorname{diam}(B_i^k)^m}{\operatorname{diam}(B_i^k)^m} \geq \frac{1}{2^k} \cdot 2^{k-2} \geq \frac{1}{4},$$

where we have used the estimated number of balls intersecting S and the fact that S is not entirely contained in any ball $12B_i^k$. The claim follows by letting $k \to \infty$ in (3.3). □

Finally, we need the following result, whose proof is a modification of [21, Proposition 3.1]. Recall that for a fixed Borel set $A \subset \mathbb{R}^n$ and an m-rectifiable surface S, f is said to satisfy Condition N^A with respect S if for all $O \subset S$ with $\mathscr{H}^m(O) = 0$, we have $\mathscr{H}^m(f(O \cap A)) = 0$.

Proposition 3.5 *Let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a continuous map. Let $A \subset \Omega$ be a Borel set, and assume that there is a number $M < \infty$ such that one of the following two conditions is satisfied*:

i) *For every $x \in A$, $\operatorname{lip}_f(x) \leq M$;*

ii) *f is a homeomorphism and for every $x \in A$, $k_f(x) \leq M$.*

Then on q-almost every m-rectifiable surface S in Ω, $f(S \cap A)$ has finite m-measure, and f satisfies Condition N^A with respect to S. Above $q = \frac{n}{m}$.

Proof We first consider the case $\operatorname{lip}_f(x) \leq M$ on A. Let $\varepsilon > 0$. Fix an m-rectifiable surface S and a subset $E \subset S$. By the Vitali covering lemma, we may cover E with a sequence of balls $B_i = B(x_i, r_i)$, $x_i \in E$ and $r_i < \frac{\varepsilon}{2}$, such that $\sum_i \left(\operatorname{diam} B_i \right)^m < \mathscr{H}^m(E) + \varepsilon$, such that $\frac{1}{5}B_i$ are pairwise disjoint, and such that

$$\operatorname{diam}(f(B_i)) \leq 2L_f(x_i, r_i) \leq 2Mr_i.$$

Then

$$\mathscr{H}^m_\varepsilon(f(E)) \leq \sum_{i=1}^{\infty} \operatorname{diam}(f(B_i))^m \leq C_0 M^m \sum_{i=1}^{\infty} \mathscr{H}^m(B_i)$$

$$\leq C_0 M^m \sum_{i=1}^{\infty} \mathscr{H}^m \left(\frac{1}{5} B_i \right) \leq C_0 M^m (\mathscr{H}^m(E) + \varepsilon).$$

Passing to limit as ε approaches 0 gives the desired inequality

$$\mathscr{H}^m(f(E)) \leq C_0 M^m \mathscr{H}^m(E).$$

Next, we assume that $k_f(x) \leq M$ for all $x \in A$. Since our considerations are entirely local, we may assume that Ω and $f(\Omega)$ are bounded. Fix $\varepsilon > 0$. For each $x \in A$, there is by definition a radii $r_x < \varepsilon$ such that $k_f(x, r_x) \leq 2k_f(x) \leq 2M$ and $B(x, 2r_x) \subset \Omega$.

By the Besicovitch covering theorem, there is a constant C_n depending only on n, and a sequence of points $x_i \in A$ such that the balls $B_i = B(x_i, r_{x_i})$ satisfy the following properties:

i) $\sum_{i=1}^{\infty} \chi_{B_i} \leq C_n$;

ii) $\bigcup_{i=1}^{\infty} B_i \supset A$.

To ease notation, let $r_i = r_{x_i}$ and $L_i = L_f(x_i, r_i)$. We now define $\rho_\varepsilon : \Omega \to \mathbb{R}$ by

$$\rho_\varepsilon = 2 \sum_{i=1}^{\infty} \left(\frac{L_i}{r_i} \right)^m \chi_{2B_i}.$$

Then for every m-rectifiable surface S in Ω such that $\mathscr{H}^m(S) > \varepsilon^m$, we may conclude that for each i such that $B_i \cap S \neq \emptyset$, S joins B_i with $X \backslash 2B_i$. Since S is rectifiable and r_i is sufficiently small, $\int_S \chi_{2B_i} d\mathscr{H}^m \geq C(m) r_i^m$. Therefore,

$$\int_S \rho_\varepsilon d\mathscr{H}^m \geq \sum_{S \cap B_i \neq \emptyset} 2C(m) L_i^m \geq C(m) \mathscr{H}_\varepsilon^m(f(S \cap A)).$$

Moreover, for $q = \frac{n}{m}$, by Lemma 2.3,

$$\begin{aligned} \int_\Omega \rho_\varepsilon^q dx &\leq C \int_\Omega \left(\sum_{i=1}^{\infty} \left(\frac{L_i}{r_i} \right)^m \chi_{2B_i} \right)^q dx \leq C C_n^q \int_\Omega \sum_{i=1}^{\infty} \left(\frac{L_i}{r_i} \right)^{qm} \chi_{B_i} dx \\ &= C \sum_{i=1}^{\infty} \left(\frac{L_i}{r_i} \right)^{qm} |B_i| = C \sum_{i=1}^{\infty} L_i^n \leq C \sum_{i=1}^{\infty} M^{n-1} |f(B_i)| \\ &\leq C M^{n-1} |f(\Omega)|. \end{aligned}$$

Because the preceding estimate is independent of ε, a standard application of reflexivity, Mazur's lemma and Fuglede's Lemma 3.3, gives a subsequence of convex combinations of the functions ρ_ε, converging in $L^q(\Omega)$ to some function ρ, such that for q-almost every m-rectifiable surface S, $\int_S \rho d\mathscr{H}^m < \infty$, and

$$\int_S \rho d\mathscr{H}^m \geq C(m) \mathscr{H}^m(f(S \cap A)), \tag{3.6}$$

so that $f(S \cap A)$ has finite m-measure. It follows immediately from the basic properties of modulus [5, p. 177 (c)] that on q-almost every m-rectifiable surface $S' \subset S$, whence for every $O \subset S$,

$$\int_O \rho d\mathscr{H}^m \geq C(m) \mathscr{H}^m(f(O \cap A)),$$

from which the condition N^A immediately follows. □

Proof of Theorems 3.1 *and* 3.2 It follows from Proposition 3.4 that for 1-almost every $(n-1)$-rectifiable surface S in Ω, $\mathscr{H}^{n-1}(f(S \cap E)) = 0$. The functions $\frac{L_f(x,r)}{r}$ are continuous in both x and r, and $k_f(x, r)$ is upper semi-continuous in x and continuous from the left in

r. Thus $\mathrm{lip}_f = \liminf_{\mathbb{Q}\ni r\to 0} \frac{L_f(x,r)}{r}$ is measurable, and likewise for k_f. We may thus apply Proposition 3.5 repeatedly, with $A = \mathrm{lip}_f^{-1}([0,M])$, and with $A = k_f^{-1}([0,M])$, and for each $M \in \mathbb{N}$. Both theorems follow immediately from the countable sub-additivity of modulus and countable sub-additivity of measures. □

4 Proof of Main Results

In this section, we are going to prove our main results. With the aid of Theorem 3.1, Theorem 1.1 is a direct consequence of Theorem 4.1.

Theorem 4.1 *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a homeomorphism with L^1-integrable inner distortion. If f satisfies Condition N on almost every sphere, then $f^{-1} \in W^{1,n}(f(\Omega), \mathbb{R}^n)$ and has finite distortion. Moreover,*

$$|Df^{-1}(y)|^n \le K_I(f^{-1}(y), f) J_{f^{-1}}(y) \quad \text{a.e. in } \Omega$$

and (1.1), (1.2) *hold.*

We need the following two auxiliary results to prove Theorem 4.1.

Theorem 4.2 *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a bi-Sobolev homeomorphism such that f satisfies the condition N^{-1} on almost every sphere. Then $|Df(\cdot)|$ vanishes a.e. in the zero set of $J(\cdot, f)$. In particular, f is a mapping of finite distortion if additionally $J(\cdot, f) \ge 0$ a.e. in Ω.*

Proof This is essentially contained in [2, Proof of Theorem 4.5], see also [11, Theorem 5.8]. We briefly indicate the ideas here.

Let $\mathbb{H}_i$ be the i-th coordinate hyperplane, i.e.,

$$\mathbb{H}_i = \{x \in \mathbb{R}^n : x_i = 0\}$$

and denote by π_i the orthogonal projection to $\mathbb{H}_i$, so that

$$\pi_i(x) = x - x_i \mathbf{e_i}, \quad x \in \mathbb{R}^n.$$

We denote the projection to j-th coordinate by $\pi^j(x) = x_j$. Then the assumption that f satisfies condition N^{-1} on a.e. sphere implies that f satisfies condition N^{-1} on a.e. hyperplane, whence for each measurable set $E \subset f(\Omega)$,

$$\int_E |D^{\#}(\pi_i \circ f^{-1})| = \int_{\pi_i(\mathbb{R}^n)} \mathscr{H}^1(E \cap (\pi_i \circ f^{-1})^{-1}(x)) dx. \tag{4.1}$$

Suppose that f is not a mapping of finite distortion. Then we can find a set $\tilde{A} \subset \Omega$ such that $|\tilde{A}| > 0$ and $J(\cdot, f) = 0$ on $\tilde{A}$ but $|Df(\cdot)| > 0$ on $\tilde{A}$. The Sobolev regularity for f allows us to assume that f is absolutely continuous on all lines parallel to coordinate axes that intersect $\tilde{A}$ and that f has classical derivatives at every point of $\tilde{A}$.

The area formula (2.3) implies that we may find a Borel set $A \subset \tilde{A}$ such that $|A| > 0$ and $|f(A)| = 0$. Note that there is $i \in \{1, 2, \ldots, n\}$ such that the subset of A where $\frac{\partial f(x)}{\partial x_i} \ne 0$ has positive measure. We may thus assume that $\frac{\partial f(x)}{\partial x_i} \ne 0$ for all $y \in A$. Letting $E = f(A)$ and using (4.1) we obtain

$$\int_{\mathbb{H}_i} \mathscr{H}^1(\pi^j(\{y \in E : \pi_i \circ f^{-1}(y) = z\})) dz = 0$$

for each $j \in \{1, 2, \dots, n\}$. By Fubini theorem,

$$\int_{\mathbb{H}_i} \mathscr{H}^1(A \cap \pi_i^{-1}(z))dz = |A| > 0.$$

Therefore, there exists $z \in \mathbb{H}_i$ with

$$\mathscr{H}^1(\pi^j(E \cap f(\pi_i^{-1}(z)))) = \mathscr{H}^1(\pi^j(\{y \in E : \pi_i \circ f^{-1}(y) = z\})) = 0,$$

and

$$\mathscr{H}^1(A \cap \pi_i^{-1}(z)) > 0.$$

Note that

$$0 < \int_{A\cap\pi_i^{-1}(z)} \left|\frac{\partial f}{\partial x_i}(x)\right| d\mathscr{H}^1(x)$$

and thus we can find j such that for $h = \pi^j \circ f$ we have

$$0 < \int_{A\cap\pi_i^{-1}(z)} \left|\frac{\partial h}{\partial x_i}(x)\right| d\mathscr{H}^1(x)$$

A contradiction follows easily by applying the one-dimensional area formula to the absolutely continuous mapping

$$t \mapsto h(z + t\mathbf{e}_i).$$ □

Theorem 4.3 *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a homeomorphism with L^1-integrable inner distortion. If f satisfies Condition N on almost every sphere, then f has finite* (*outer*) *distortion and f is differentiable almost everywhere.*

Proof This is essentially contained in [20, Proof of Theorem 5.1] and one only needs to notice that the assumption $f \in W^{1,n-1}$ there was only used to deduce the fact that f satisfies Condition N on almost every sphere. □

Proof of Theorem 4.1 We first prove that $f^{-1} \in W^{1,n}$. Since the proof is now well-known and has been used in [2, 11], we only sketch it here. Following [2, 11], we use the notation $\pi_r(x) = |x|$ for radial projection and $\pi_S(x) = \frac{x}{|x|}$ for projection to the unit sphere. Set $h = \pi_S \circ f$. Then the following variant of coarea formula holds:

$$\int_{\partial B(0,1)} \mathscr{H}^1(\pi_r(\{x \in E : h(x) = z\}))d\mathscr{H}^{n-1}(z) \le \int_E |D^\# h(x)|dx \tag{4.2}$$

for all measurable $E \subset \Omega$. The proof of (4.2) is based on Condition N on sphere and Fubini theorem, see, e.g., [2, Proof of Lemma 4.2].

The next step is to deduce from (4.2) the $(1, 1)$-Poincaré inequality for f^{-1}. More precisely, one follows the proof of Theorem 4.3 in [2] to deduce the following estimate:

$$\int_B |f^{-1}(y) - c|dy \le Cr_0 \int_{f^{-1}(B)} |D^\# f(x)|dx$$

for each ball $B \subset f(\Omega)$, where

$$c = \fint_B f^{-1}(y)dy.$$

Then we claim that there is a function $g \in L^n$ such that

$$\int_{f^{-1}(B)} |D^\# f| \le \int_B g. \tag{4.3}$$

This understood, with [8, Theorem 9.4.2], gives us the desired Sobolev regularity for the inverse. To establish (4.3), define a function $g : f(\Omega) \to \mathbb{R}$ by setting

$$g(f(x)) = \begin{cases} \dfrac{|D^{\#}f(x)|}{J_f(x)}, & \text{if } x \in \mathcal{D}_f \text{ and } J_f(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since by Theorem 5.1, f is a mapping of finite distortion, we have

$$|D^{\#}f(x)| = g(f(x))J_f(x) \quad \text{a.e. in } \Omega.$$

Now by (2.3),

$$\int_{f(\Omega)} g(y)^n dy \le \int_{\Omega \cap \mathcal{D}_f} g(f(x))^n J_f(x) dx \le \int_{\Omega} K_I(x, f) dx$$

and

$$\int_{f^{-1}(B)} |D^{\#}f(x)| dx \le \int_{f^{-1}(B) \cap \mathcal{D}_f} g(f(x)) J_f(x) dx \le \int_B g(y) dy.$$

The proof of the desired regularity is complete.

Since f is bi-Sobolev, Theorem 4.2 allows us to conclude that f^{-1} has finite distortion. In the following, we only prove (1.2). Regarding the other statements in Theorem 4.1, simply proceed the proofs given in [6, Proof of Theorem 2.3].

In order to prove (1.2), let A be the Borel set provided by Lemma 2.1. The area formula implies that

$$\begin{aligned} \int_{f(\Omega)} |Df^{-1}(y)|^n dy &= \int_A |Df^{-1}(y)|^n dy = \int_A \frac{|D^{\#}f(f^{-1}(y))|^n}{(J_f(f^{-1}(y)))^n} dy \\ &\le \int_A K_I(f^{-1}(y), f) J_{f^{-1}}(y) dy \le \int_{\Omega} K_I(x, f) dx. \end{aligned}$$

It remains to show the opposite inequality in (1.2). For this purpose, let

$$Z = \{x \in \Omega : J_f(x) = 0\}.$$

Then the area formula (2.3) implies that

$$0 = \int_{\mathcal{D}_f} \chi_{f(Z)}(f(x)) J_f(x) dx = \int_{f(\mathcal{D}_f)} \chi_{f(Z)}(y) dy.$$

Hence,

$$|f(\mathcal{D}_f \cap Z)| = |f(\mathcal{D}_f) \cap f(Z)| = 0.$$

Since f^{-1} satisfies Condition N (see for instance [17]),

$$|\mathcal{D}_f \cap Z| = 0,$$

i.e., $J_f > 0$ a.e. in Ω. Thus, using the area formula and chain rule again, we obtain

$$\begin{aligned} \int_{\Omega} K_I(x, f) dx &= \int_{\mathcal{D}_f} K_I(x, f) dx = \int_{\mathcal{D}_f \setminus Z} \frac{|D^{\#}f(x)|^n}{(J_f(x))^n} J_f(x) dx \\ &= \int_B \frac{|D^{\#}f(x)|^n}{(J_f(x))^n} J_f(x) dx = \int_B |Df^{-1}(f(x))|^n J_f(x) dx \end{aligned}$$

$$\le \int_{\Omega'} |Df^{-1}(y)|^n dy,$$

where B is the Borel set determined by Lemma 2.1. This completes the proof of (1.2). □

Remark 4.4 Let f be as in Theorem 4.1. We know from the proof of Theorem 4.1 that $J_f > 0$ a.e. in Ω. Note that the a.e. positiveness of the Jacobian is a highly non-trivial fact, see [3, 9, 16] for more in this direction.

As a corollary of the proof of Theorem 4.1, we formulate the following simple result.

Corollary 4.5 *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ be a homeomorphism such that $J_f \ge 0$ a.e. in Ω. Then f satisfies Condition N^{-1} if and only if $J(\cdot, f) > 0$ a.e.*

Proof The proof of Condition N^{-1} implies $J(\cdot, f) > 0$ a.e. is already demonstrated in the proof of Theorem 4.1 and hence we only need to show the reverse direction. To this end, let $E \subset \Omega$ with $|f(E)| = 0$ and let $A \subset \mathbb{R}^n$ be a Borel set of measure zero that contains $f(E)$. Then E is contained in the Borel set $E' = f^{-1}(A)$. Let h be the characteristic function of A. By the area formula,

$$\int_{E'} J_f(x)dx = \int_{\Omega} h(f(x))J_f(x)dx \le \int_{\mathbb{R}^n} h(y)dy = 0.$$

Since $J_f > 0$ a.e., it follows that $|E| = 0$. □

5 Weaken the Topological Assumption

Very recently, Tengvall [20] proved that *if $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a continuous, discrete and open mapping of finite L^1_{loc}-integrable inner distortion, then f is differentiable a.e. and has finite distortion.* In particular, he verified that Lusion Condition N on a.e. sphere holds for continuous, discrete and open Sobolev mappings $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

As already seen in the previous results, Condition N on a.e. sphere plays a crucial role in proving the Sobolev regularity of the inverse of a homeomorphism $f \in W^{1,1}$. On the other hand, in geometric function theory, associated with a continuous, discrete and open mapping f, there is a "generalized inverse" map g_U defined for each normal domain U of f as follows (see, e.g., [19]).

$$g_U(y) = \frac{1}{m} \sum_{x \in f^{-1}(y) \cap U} i(x, f)x, \tag{5.1}$$

where $m = \mu(f, U)$ is the local degree on the normal domain U and $i(x, f)$ is the topological index of f at x. Observe that, in the homeomorphic case, the generalized inverse map g_U is just the inverse map of f.

In [7], it was proved that for a continuous, discrete and open mapping $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ of finite L^1_{loc}-integrable inner distortion, if $|f(B_f)| = 0$, then the generalized inverse map $g_U : V \to \mathbb{R}^n$ belongs to $W^{1,n}(V, \mathbb{R}^n)$.

Theorem 5.1 *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a continuous, discrete and open mapping of finite L^1_{loc}-integrable inner distortion. If f satisfies Condition N on a.e. sphere, then f is differentiable a.e., $|B_f| = 0$, and has finite distortion. Moreover, if $|f(B_f)| = 0$, then the generalized inverse $g_U : V \to \mathbb{R}^n$ belongs to $W^{1,n}(V, \mathbb{R}^n)$ for each normal domain $U \subset\subset \Omega$ such that $f(U) = V$.*

Proof The proof of the fact that f is differentiable a.e. and that $|B_f| = 0$ is basically contained in [20]. One can check that the regularity assumption $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ in [20] was only used to deduce that f satisfies Condition N on a.e. sphere. Similarly, the proof of the latter part of Theorem 5.1 can be found in [7]. □

Combining Theorem 5.1 with Theorem 4.1, we obtain the following corollary.

Corollary 5.2 *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a continuous, discrete and open mapping of finite L^1_{loc}-integrable inner distortion. If f satisfies Condition N on a.e. sphere, then for almost every $x \in \Omega$ there is an open neighborhood $U_x \subset\subset \Omega$ of x such that the restriction map $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism. For the inverse map we have $(f|_{U_x})^{-1} \in W^{1,n}_{\mathrm{loc}}(f(U_x), \mathbb{R}^n)$ and has finite distortion. Moreover, f and $(f|_{U_x})^{-1}$ are differentiable almost everywhere.*

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