Acta Mathematica Sinica, English Series
Sep., 2013, Vol. 29, No. 9, pp. 1655–1678
Published online: April 12, 2013
DOI: 10.1007/s10114-013-2284-z
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A General Vectorial Ekeland's Variational Principle with a P-distance

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Abstract In this paper, by using p-distances on uniform spaces, we establish a general vectorial Ekeland variational principle (in short EVP), where the objective function is defined on a uniform space and taking values in a pre-ordered real linear space and the perturbation involves a p-distance and a monotone function of the objective function. Since p-distances are very extensive, such a form of the perturbation in deed contains many different forms of perturbations appeared in the previous versions of EVP. Besides, we only require the objective function has a very weak property, as a substitute for lower semi-continuity, and only require the domain space (which is a uniform space) has a very weak type of completeness, i.e., completeness with respect to a certain p-distance. Such very weak type of completeness even includes local completeness when the uniform space is a locally convex topological vector space. From the general vectorial EVP, we deduce a general vectorial Caristi's fixed point theorem and a general vectorial Takahashi's nonconvex minimization theorem. Moreover, we show that the above three theorems are equivalent to each other. We see that the above general vectorial EVP includes many particular versions of EVP, which extend and complement the related known results.

Keywords Vectorial Ekeland's variational principle, vectorial Caristi's fixed point theorem, vectorial Takahashi's minimization theorem, p-distance, Gerstewitz's function

MR(2010) Subject Classification 58E30, 49J40, 46N10, 46A03

1 Introduction

Since Ekeland (see [13]) obtained a nonconvex minimization theorem for a lower semi-continuous function on a complete metric space in 1972, it has received a great deal of attention and has been applied to numerous problems in various fields of nonlinear analysis; see, e.g. [3, 7, 14, 15, 19, 48]. It is well known that Ekeland's variational principle (in short EVP) is equivalent to Caristi's fixed point theorem [6], to Takahashi's nonconvex minimization theorem [48, 49],

Received May 8, 2012, accepted November 16, 2012
Supported by National Natural Science Foundation of China (Grant No. 10871141)

to the drop theorem [11, 12] and to the petal theorem [38]. A number of generalizations in various different directions of these results have been investigated by several authors; see, e.g. [2–5, 7–10, 16–23, 25–28, 32–38, 41–53] and references therein. In particular, Kada et al. [28] introduced the notion of w-distances on metric spaces and improved EVP, Caristi's fixed point theorem and Takahashi's nonconvex minimization theorem. Moreover, Lin and Du [32] introduced the notion of τ-functions, which generalizes the notion of w-distances, and established a more general version of EVP, where the perturbation involves a τ-function and a nondecreasing function of the objective function value. Inspired by the above results, we introduced [46] the notions of p-distances and q-distances on uniform spaces and obtained a generalized EVP in uniform spaces, which not only extends the above mentioned versions of EVP but also includes almost all known extensions of EVP in uniform spaces and in topological vector spaces (for example, the results in [16, 21, 26, 35, 42–44]). All the above generalizations of EVP are concerned with scalar-valued functions. Recently, generalizing EVP to vector-valued functions (which is called a vectorial EVP) has aroused one's interest; see, e.g. [2, 4, 7, 9, 17, 19, 20, 22, 23, 25, 27, 36, 45, 50] and the references therein. In this paper, by using p-distances (or q-distances) on uniform spaces, we shall establish a general vectorial EVP, where the objective function is defined on a uniform space and taking values in a pre-ordered real linear space and the perturbation involves a p-distance (or a q-distance) and a nondecreasing function of the objective function. Since p-distances (or q-distances) are very extensive (see [46]), such a form of the perturbation indeed contains many different forms of perturbations appeared in the previous versions of EVP. Besides, we only require the objective function has a very weak property, as a substitute for lower semi-continuity, and only require the domain space (which is a uniform space) has a very weak type of completeness (i.e., completeness with respect to a p-distance or a q-distance). For example, when the uniform space is a locally convex space, the very weak type of completeness includes local completeness. As we know, local completeness is strictly weaker than sequential completeness and it seems to be the weakest type of completeness in the extent of locally convex spaces (for details, see [39, 40]). From the general vectorial EVP, we deduce a general vectorial Caristi's fixed point theorem and a general vectorial Takahashi's nonconvex minimization theorem. Moreover, we show that the above three theorems are equivalent to each other. We shall see that our general vectorial EVP includes many particular versions of EVP, which extend and complement the related known results. And it has many applications in vector optimization. Since a scalar-valued function can be regarded as a particular type of vector-valued function, our vectorial EVP is also an extension of the corresponding scalar-valued version of EVP in [46].

2 Convex Cones in a Real Linear Space and Gerstewitz's Function

A useful approach for solving a vector problem is to reduce it to a scalar problem. Gerstewitz's function is often used as the basis of the scalarization. In the framework of topological vector spaces, Gerstewitz's function generated by a closed convex (solid) cone and its properties have been investigated thoroughly, for example, see [7, 19, 20, 31] and the references therein. In order to obtain a more general version of vectorial Ekeland's variational principle, we need to consider Gerstewitz's function generated by a general convex cone in a real linear space and re-examine its properties in the more general setting. The main ideas of the section originate from [7, 19, 20, 22, 23, 31] and the results exhibited here also have their independent interest.

In the following, we always assume that Y is a real linear space. Let $A \subset Y$ be nonempty. The vector closure of A is defined as follows (refer to [1]):

$$\mathrm{vcl}(A) = \{y \in Y : \exists\, v \in Y, \exists\, \lambda_n \geq 0, \lambda_n \to 0 \text{ such that } y + \lambda_n v \in A, \forall\, n \in \mathbb{N}\}.$$

Moreover, for any given $v_0 \in Y$, we define the v_0-vector closure of A as follows:

$$\mathrm{vcl}_{v_0}(A) = \{y \in Y : \exists\, \lambda_n \geq 0, \lambda_n \to 0 \text{ such that } y + \lambda_n v_0 \in A, \forall\, n \in \mathbb{N}\}.$$

Obviously,

$$A \subset \mathrm{vcl}_{v_0}(A) \subset \bigcup_{v \in Y} \mathrm{vcl}_v(A) = \mathrm{vcl}(A).$$

All the above inclusions are proper. Moreover, if Y is a topological vector space and $\mathrm{cl}(A)$ denotes the closure of A, then $\mathrm{vcl}(A) \subset \mathrm{cl}(A)$ and the inclusion is also proper. A nonempty set $K \subset Y$ is called a convex cone if $K + K \subset K$ and $\lambda K \subset K$ for any $\lambda > 0$. We shall see that there exists a convex cone K containing the zero element 0 in Y and $k_0, k_1 \in K \backslash -K$ such that $K \neq \mathrm{vcl}_{k_1}(K)$ and $\mathrm{vcl}_{k_0}(K) \neq \mathrm{vcl}(K)$.

Example 2.1 Let $Y = \mathbb{R}^3$ and let

$$K = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 \geq 0, \xi_2 > 0, \xi_3 > 0\} \cup \{(0, \xi_2, 0) \in \mathbb{R}^3 : \xi_2 \geq 0\}.$$

Obviously, K is a convex cone containing 0. Let $k_0 = (0, 1, 0)$ and $k_1 = (0, 1, 1)$. Then $k_0, k_1 \in K \backslash -K$. Put $y = (1, 1, 0)$. Then $y \notin K$. For any sequence (λ_n) with $\lambda_n > 0$ and $\lambda_n \to 0$, we have

$$y + \lambda_n k_1 = (1, 1, 0) + (0, \lambda_n, \lambda_n) = (1, 1 + \lambda_n, \lambda_n) \in K,$$

and

$$y + \lambda_n k_0 = (1, 1, 0) + (0, \lambda_n, 0) = (1, 1 + \lambda_n, 0) \notin K.$$

Thus, we conclude that

$$y \in \mathrm{vcl}_{k_1}(K) \subset \mathrm{vcl}(K) \quad \text{and} \quad y \notin \mathrm{vcl}_{k_0}(K).$$

Hence

$$K \subset (\neq)\mathrm{vcl}_{k_1}(K) \quad \text{and} \quad \mathrm{vcl}_{k_0}(K) \subset (\neq)\mathrm{vcl}(K).$$

A subset A of Y is called vectorially closed if $A = \mathrm{vcl}(A)$; and called v_0-vectorially closed (briefly, v_0-closed) if $A = \mathrm{vcl}_{v_0}(A)$. In general, a set $A \subset Y$ need not be v_0-closed; and a v_0-closed set need not be vectorially closed. We shall see that there exists a convex cone K containing 0 and $k_0, k_1 \in K \backslash -K$ such that K is k_0-closed but it is not k_1-closed and hence it is not vectorially closed.

Example 2.2 Let $Y = \mathbb{R}^3$ and let

$$K = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 \geq 0, \xi_2 \geq 0, \xi_3 > 0\} \cup \{(0, \xi_2, 0) : \xi_2 \geq 0\}.$$

Then K is a convex cone containing 0. Let $k_0 = (0, 1, 0)$ and $k_1 = (0, 0, 1)$. Then $k_0, k_1 \in K \backslash -K$. Let $y = (\xi_1, \xi_2, \xi_3) \in \mathrm{vcl}_{k_0}(K)$. Then there exists a sequence (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that

$$y + \lambda_n k_0 \in K, \quad \text{i.e.,} \quad (\xi_1, \xi_2, \xi_3) + (0, \lambda_n, 0) = (\xi_1, \xi_2 + \lambda_n, \xi_3) \in K.$$

From this,

$$\xi_1 \geq 0,\, \xi_2 + \lambda_n \geq 0,\, \xi_3 > 0 \quad \text{or} \quad \xi_1 = 0,\, \xi_2 + \lambda_n \geq 0,\, \xi_3 = 0.$$

This implies that

$$\xi_1 \geq 0,\, \xi_2 \geq 0,\, \xi_3 > 0 \quad \text{or} \quad \xi_1 = 0,\, \xi_2 \geq 0,\, \xi_3 = 0.$$

Clearly, $y = (\xi_1, \xi_2, \xi_3) \in K$. Thus, $K = \mathrm{vcl}_{k_0}(K)$ and K is k_0-closed. However, we shall see that K is not k_1-closed. In fact, put $y = (1, 1, 0)$. Then $y \notin K$. For any given sequence (λ_n) with $\lambda_n > 0$ and $\lambda_n \to 0$,

$$y + \lambda_n k_1 = (1, 1, 0) + (0, 0, \lambda_n) = (1, 1, \lambda_n) \in K.$$

Thus, $y \in \mathrm{vcl}_{k_1}(K) \subset \mathrm{vcl}(K)$. From this, we know that K is not k_1-closed. Certainly, it is not vectorially closed.

As shown in Example 2.1, in general $\mathrm{vcl}_{k_0}(K) \subset \mathrm{vcl}(K)$ and $\mathrm{vcl}_{k_0}(K) \neq \mathrm{vcl}(K)$. However, if $k_0 \in \mathrm{cor}(K)$, where $\mathrm{cor}(K)$ denotes the algebraic interior of K (see [24, p. 7]), then we have the following.

Proposition 2.3 *Let K be a convex cone and $k_0 \in \mathrm{cor}(K)$. Then* $\mathrm{vcl}_{k_0}(K) = \mathrm{vcl}(K)$.

Proof Obviously, $\mathrm{vcl}_{k_0}(K) \subset \mathrm{vcl}(K)$. We only need to prove that $\mathrm{vcl}(K) \subset \mathrm{vcl}_{k_0}(K)$. Let $y \in \mathrm{vcl}(K)$. Then there exist $v \in Y$ and a sequence (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that

$$y + \lambda_n v \in K. \tag{2.1}$$

Since $k_0 \in \mathrm{cor}(K)$, there exists $\alpha > 0$ such that $k_0 - \alpha v \in K$. Put $k' = k_0 - \alpha v$. Then $k' \in K$. From this,

$$v = \frac{k_0 - k'}{\alpha}. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$y + \lambda_n \frac{k_0 - k'}{\alpha} \in K.$$

Thus,

$$y + \frac{\lambda_n}{\alpha} k_0 \in \frac{\lambda_n}{\alpha} k' + K \subset K, \quad \text{where} \quad \frac{\lambda_n}{\alpha} \to 0.$$

Hence, $y \in \mathrm{vcl}_{k_0}(K)$ and $\mathrm{vcl}(K) \subset \mathrm{vcl}_{k_0}(K)$. □

Proposition 2.4 *Let $K \subset Y$ be a convex cone and $k_0 \in K \backslash -K$. Then*

(i) $\mathrm{vcl}_{k_0}(K)$ *is a convex cone containing $K \cup \{0\}$, but not containing $-k_0$;*

(ii) *for any sequence (ϵ_n) with $\epsilon_n > 0$ and $\epsilon_n \to 0$,*

$$\mathrm{vcl}_{k_0}(K) = \bigcap_{\epsilon > 0} (K - \epsilon k_0) = \bigcap_{n=1}^{\infty} (K - \epsilon_n k_0);$$

(iii) $\mathrm{vcl}_{k_0}(\mathrm{vcl}_{k_0}(K)) = \mathrm{vcl}_{k_0}(K)$.

Proof (i) Obviously, $K \subset \mathrm{vcl}_{k_0}(K)$. For any sequence (λ_n) with $\lambda_n > 0$ and $\lambda_n \to 0$, $0 + \lambda_n k_0 = \lambda_n k_0 \in K$. Thus $0 \in \mathrm{vcl}_{k_0}(K)$ and hence $K \cup \{0\} \subset \mathrm{vcl}_{k_0}(K)$. Let $y, y' \in \mathrm{vcl}_{k_0}(K)$. Then there exist sequences (λ_n) and (λ'_n) with $\lambda_n \geq 0$, $\lambda'_n \geq 0$, $\lambda_n \to 0$ and $\lambda'_n \to 0$ such that $y + \lambda_n k_0 \in K$ and $y' + \lambda'_n k_0 \in K$. Thus, $y + y' + (\lambda_n + \lambda'_n) k_0 \in K + K \subset K$, where $\lambda_n + \lambda'_n \geq 0$ and $\lambda_n + \lambda'_n \to 0$. Hence $y + y' \in \mathrm{vcl}_{k_0}(K)$. Also, for any $\lambda > 0$, $\lambda y + \lambda \lambda_n k_0 \in \lambda K \subset K$, where $\lambda \lambda_n \geq 0$ and

$\lambda\lambda_n \to 0$. Thus, $\lambda y \in \mathrm{vcl}_{k_0}(K)$. Hence $\mathrm{vcl}_{k_0}(K)$ is a convex cone. Assume that $-k_0 \in \mathrm{vcl}_{k_0}(K)$. Then there exists (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that $(-1+\lambda_n)k_0 = -k_0 + \lambda_n k_0 \in K$. Obviously, there exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} < 1$. Thus $-1+\lambda_{n_0} < 0$, which leads to that $k_0 \in -K$, contradicting the assumption that $k_0 \in K\backslash -K$.

(ii) First we remark that, for any $\lambda' > \lambda \geq 0$, $K - \lambda k_0 \subset K - \lambda' k_0$. Let $y \in \mathrm{vcl}_{k_0}(K)$. Then there exists (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that $y + \lambda_n k_0 \in K$. For any given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} < \epsilon$. Thus, $y + \lambda_{n_0} k_0 \in K$ implies that $y \in K - \lambda_{n_0} k_0 \subset K - \epsilon k_0$. This shows that $y \in \bigcap_{\epsilon>0}(K - \epsilon k_0)$. Also, $\bigcap_{\epsilon>0}(K-\epsilon k_0) \subset \bigcap_{n=1}^{\infty}(K - \epsilon_n k_0)$ is obvious. Hence we have

$$\mathrm{vcl}_{k_0}(K) \subset \bigcap_{\epsilon>0}(K - \epsilon k_0) \subset \bigcap_{n=1}^{\infty}(K - \epsilon_n k_0).$$

On the other hand, for any $y \in \cap_{n=1}^{\infty}(K - \epsilon_n k_0)$, we have $y + \epsilon_n k_0 \in K$, where $\epsilon_n > 0$ and $\epsilon_n \to 0$. Hence, $y \in \mathrm{vcl}_{k_0}(K)$ and we have

$$\bigcap_{n=1}^{\infty}(K - \epsilon_n k_0) \subset \mathrm{vcl}_{k_0}(K).$$

(iii) By (i), we know that $\mathrm{vcl}_{k_0}(K)$ is a convex cone containing $K \cup \{0\}$ and $k_0 \in \mathrm{vcl}_{k_0}(K)\backslash -\mathrm{vcl}_{k_0}(K)$. We only need to prove that $\mathrm{vcl}_{k_0}(\mathrm{vcl}_{k_0}(K)) \subset \mathrm{vcl}_{k_0}(K)$. In fact, by (ii), we have

$$\begin{aligned}
\mathrm{vcl}_{k_0}(\mathrm{vcl}_{k_0}(K)) &= \bigcap_{\epsilon'>0}(\mathrm{vcl}_{k_0}(K) - \epsilon' k_0)\\
&= \bigcap_{\epsilon'>0}\left(\bigcap_{\epsilon>0}(K - \epsilon k_0) - \epsilon' k_0\right)\\
&= \bigcap_{\epsilon'>0}\bigcap_{\epsilon>0}(K - (\epsilon + \epsilon')k_0)\\
&= \bigcap_{\epsilon''>0}(K - \epsilon'' k_0)\\
&= \mathrm{vcl}_{k_0}(K).
\end{aligned}$$

□

Proposition 2.5 *Let* $K \subset Y$ *be a convex cone and* $k_0 \in K\backslash -K$. *Put* $\mathrm{vint}_{k_0}(K) = K + (0,\infty)k_0$. *Then we have the following*:

(i) $\mathrm{vint}_{k_0}(K)$ *is a convex cone contained in* K;

(ii) $\mathrm{vint}_{k_0}(\mathrm{vint}_{k_0}(K)) = \mathrm{vint}_{k_0}(K)$;

(iii) $\mathrm{vcl}_{k_0}(\mathrm{vint}_{k_0}(K)) = \mathrm{vcl}_{k_0}(K)$;

(iv) *if* $k_0 \in \mathrm{cor}(K)$, *then* $\mathrm{vint}_{k_0}(K) = \mathrm{cor}(K)$.

Proof It is easy to see that (i) and (ii) hold.

(iii) Obviously, $\mathrm{vcl}_{k_0}(\mathrm{vint}_{k_0}(K)) \subset \mathrm{vcl}_{k_0}(K)$. So we only need to show that $\mathrm{vcl}_{k_0}(\mathrm{vint}_{k_0}(K)) \supset \mathrm{vcl}_{k_0}(K)$. In fact,

$$\begin{aligned}
\mathrm{vcl}_{k_0}(\mathrm{vint}_{k_0}(K)) &= \bigcap_{\epsilon>0}[(K + (0,\infty)k_0) - \epsilon k_0]\\
&\supset \bigcap_{\epsilon>0}\left(K + \frac{\epsilon}{2}k_0 - \epsilon k_0\right)\\
&= \bigcap_{\epsilon>0}\left(K - \frac{\epsilon}{2}k_0\right) = \mathrm{vcl}_{k_0}(K).
\end{aligned}$$

(iv) Let $k_1 \in \mathrm{cor}(K)$. Then there exists $\alpha > 0$ such that $k_1 - \alpha k_0 \in K$. Thus, $k_1 \in \alpha k_0 + K \subset (0,\infty)k_0 + K = \mathrm{vint}_{k_0}(K)$. Hence $\mathrm{cor}(K) \subset \mathrm{vint}_{k_0}(K)$.

Conversely, let $k_1 \in \mathrm{vint}_{k_0}(K) = (0,\infty)k_0 + K$. Then there exist $k' \in K$ and $\alpha > 0$ such that $k_1 = \alpha k_0 + k'$. Since $k_0 \in \mathrm{cor}(K)$, for any $v \in Y$, there exists $\epsilon > 0$ such that

$$k_0 + \frac{\epsilon}{\alpha} v \in K.$$

Thus,

$$k_1 + \epsilon v = \alpha k_0 + k' + \epsilon v = k' + \alpha\left(k_0 + \frac{\epsilon}{\alpha} v\right) \in K + \alpha K \subset K.$$

This means that $k_1 \in \mathrm{cor}(K)$ and $\mathrm{vint}_{k_0}(K) \subset \mathrm{cor}(K)$. □

Let $K \subset Y$ be a convex cone containing 0 and $k_0 \in K \backslash -K$. Here, we need not assume that $\mathrm{cor}(K) \neq \emptyset$. Let $y \in Y$ and $t \in \mathbb{R}$ be such that $y \in tk_0 - K$. Then for any $t' > t$, we have

$$y \in t'k_0 - (t' - t)k_0 - K \subset t'k_0 - K.$$

Thus, we may define a function $\xi_{k_0} : Y \to \mathbb{R} \cup \{\pm\infty\}$ as follows

$$\xi_{k_0}(y) = \inf\{t \in \mathbb{R} : y \in tk_0 - K\}.$$

Such a function ξ_{k_0} is called Gerstewitz's function generated by K and k_0. When Y is a topological vector space and K is a closed convex (solid) cone, the function ξ_{k_0} and its properties have been studied in [7, 19, 20, 31] and the references therein. Now, in the setting of real linear spaces, we investigate properties of the function ξ_{k_0}.

Lemma 2.6 *There exists $u \in Y$ such that $\xi_{k_0}(u) = -\infty$ if and only if $k_0 \in -\mathrm{vcl}(K)$.*

Proof Assume that there exists $u \in Y$ such that $\xi_{k_0}(u) = -\infty$. Then for any $n \in \mathbb{N}$, $u \in -nk_0 - K$. Thus, $-k_0 - u/n \in K$ for all n. Hence $k_0 \in -\mathrm{vcl}(K)$.

Conversely, assume that $k_0 \in -\mathrm{vcl}(K)$, i.e., $-k_0 \in \mathrm{vcl}(K)$. Then there exist $v \in Y$ and a sequence (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that $-k_0 + \lambda_n v \in K$. Since $-k_0 \notin K$, we have $\lambda_n > 0$ for every n. Thus,

$$-v \in -\frac{1}{\lambda_n} k_0 - K, \quad \text{where } -\frac{1}{\lambda_n} \to -\infty.$$

Put $u = -v$. Then $\xi_{k_0}(u) = -\infty$. □

Remark 2.7 From Lemma 2.6, we know that if $k_0 \in K \backslash -\mathrm{vcl}(K)$, then $\xi_{k_0}(y) > -\infty$ for every $y \in Y$. If we only assume that $k_0 \in K \backslash -K$, then it may occur that $\xi_{k_0}(y) = -\infty$ for some $y \in Y$.

The following lemma says that ξ_{k_0} is sub-additive and positively homogeneous. Its proof is routine and omitted.

Lemma 2.8 (i) *For any $y_1, y_2 \in Y$, $\xi_{k_0}(y_1 + y_2) \leq \xi_{k_0}(y_1) + \xi_{k_0}(y_2)$ except that the right side becomes $\infty - \infty$.*

(ii) *For any $y \in Y$ and $\alpha \geq 0$, $\xi_{k_0}(\alpha y) = \alpha \xi_{k_0}(y)$.*

Lemma 2.9 (Compare it with [7, Proposition 1.43]) *Let $y \in Y$ and $r \in \mathbb{R}$. Then we have*

(i) $\xi_{k_0}(y) < r \;\Leftrightarrow\; y \in rk_0 - \mathrm{vint}_{k_0}(K)$.

(ii) $\xi_{k_0}(y) \leq r \;\Leftrightarrow\; y \in rk_0 - \mathrm{vcl}_{k_0}(K)$.

(iii) $\xi_{k_0}(y) = r \;\Leftrightarrow\; y \in rk_0 - (\mathrm{vcl}_{k_0}(K) \backslash \mathrm{vint}_{k_0}(K))$.

(iv) $\xi_{k_0}(y) \geq r \;\Leftrightarrow\; y \notin rk_0 - \mathrm{vint}_{k_0}(K)$.

(v) $\xi_{k_0}(y) > r \;\Leftrightarrow\; y \notin rk_0 - \mathrm{vcl}_{k_0}(K)$.

Here, $\mathrm{vint}_{k_0}(K) = (0,\infty)k_0 + K$ *and* $\mathrm{vcl}_{k_0}(K) = \bigcap_{\epsilon>0}(K - \epsilon k_0)$.

Proof (i) Let $-\infty < \xi_{k_0}(y) < r$. Then there exists $\epsilon > 0$ such that $\xi_{k_0}(y) + \epsilon < r$. From this,

$$\begin{aligned} y &\in (\xi_{k_0}(y) + \epsilon)k_0 - K \\ &= rk_0 - (r - \xi_{k_0}(y) - \epsilon)k_0 - K \\ &\subset rk_0 - (0,\infty)k_0 - K \\ &= rk_0 - \mathrm{vint}_{k_0}(K). \end{aligned}$$

Let $\xi_{k_0}(y) = -\infty$. Then for any $\lambda > 0$,

$$y \in -\lambda k_0 - K = rk_0 - (\lambda + r)k_0 - K.$$

Take $\lambda > 0$ such that $\lambda + r > 0$. Then

$$\begin{aligned} y &\in rk_0 - (\lambda + r)k_0 - K \\ &\subset rk_0 - (0,\infty)k_0 - K \\ &= rk_0 - \mathrm{vint}_{k_0}(K). \end{aligned}$$

Conversely, let $y \in rk_0 - \mathrm{vint}_{k_0}(K) = rk_0 - ((0,\infty)k_0 + K)$. Then there exists $\delta \in (0,\infty)$ such that $y \in rk_0 - \delta k_0 - K = (r-\delta)k_0 - K$. From this, we conclude that $\xi_{k_0}(y) \leq r - \delta < r$.

(ii) Let $\xi_{k_0}(y) \leq r$. Take a sequence (ϵ_n) such that $\epsilon_1 > \epsilon_2 > \cdots > 0$ and $\epsilon_n \to 0$. Then $y \in (r+\epsilon_n)k_0 - K$. From this, $rk_0 - y + \epsilon_n k_0 \in K$ for all n. This means that $rk_0 - y \in \mathrm{vcl}_{k_0}(K)$ and $y - rk_0 \in -\mathrm{vcl}_{k_0}(K)$. That is, $y \in rk_0 - \mathrm{vcl}_{k_0}(K)$.

Conversely, let $y \in rk_0 - \mathrm{vcl}_{k_0}(K)$. Then there exists a sequence (ϵ_n) with $\epsilon_n \geq 0$ and $\epsilon_n \to 0$ such that $rk_0 - y + \epsilon_n k_0 \in K$, i.e., $y \in (r+\epsilon_n)k_0 - K$. From this, $\xi_{k_0}(y) \leq r + \epsilon_n$ for every n. Letting $n \to \infty$, we have $\xi_{k_0}(y) \leq r$.

(iii) It follows from (i) and (ii).

Obviously, (i) is equivalent to (iv); (ii) is equivalent to (v). □

Lemma 2.10 *For any* $y \in Y$ *and any* $\lambda \in \mathbb{R}$, $\xi_{k_0}(y + \lambda k_0) = \xi_{k_0}(y) + \lambda$.

Proof We show the result according to the following three different cases.

Case 1 Let $\xi_{k_0}(y) = \infty$. Then $y \notin (-\infty,\infty)k_0 - K$. Thus, $y + \lambda k_0 \notin (-\infty,\infty)k_0 + \lambda k_0 - K = (-\infty,\infty)k_0 - K$. This means that $\xi_{k_0}(y+\lambda k_0) = \infty$ and hence $\xi_{k_0}(y+\lambda k_0) = \xi_{k_0}(y) + \lambda$ holds.

Case 2 Let $\xi_{k_0}(y) = -\infty$. Then

$$y \in rk_0 - K \quad \text{for every } r \in \mathbb{R}. \tag{2.3}$$

Assume that $\xi_{k_0}(y + \lambda k_0) > -\infty$. Then there exists $t \in \mathbb{R}$ such that $y + \lambda k_0 \notin tk_0 - K$, i.e., $y \notin (t-\lambda)k_0 - K$, which contradicts (2.3). Thus, $\xi_{k_0}(y+\lambda k_0) = -\infty$ and hence $\xi_{k_0}(y+\lambda k_0) = \xi_{k_0}(y) + \lambda$ holds too.

Case 3 Let $\xi_{k_0}(y)$ satisfy $-\infty < \xi_{k_0}(y) < \infty$. For any $\epsilon > 0$, $y \in (\xi_{k_0}(y) + \epsilon)k_0 - K$. Thus, $y + \lambda k_0 \in (\xi_{k_0}(y) + \lambda + \epsilon)k_0 - K$. Hence

$$\xi_{k_0}(y + \lambda k_0) \leq \xi_{k_0}(y) + \lambda + \epsilon \quad \text{and} \quad \xi_{k_0}(y + \lambda k_0) \leq \xi_{k_0}(y) + \lambda. \tag{2.4}$$

On the other hand, by the assumption that $-\infty < \xi_{k_0}(y) < \infty$, we can conclude that $-\infty < \xi_{k_0}(y + \lambda k_0) < \infty$. Thus, for any $\epsilon > 0$, $y + \lambda k_0 \in (\xi_{k_0}(y + \lambda k_0) + \epsilon)k_0 - K$. From this, $y \in (\xi_{k_0}(y + \lambda k_0) - \lambda + \epsilon)k_0 - K$. Hence

$$\xi_{k_0}(y) \leq \xi_{k_0}(y + \lambda k_0) - \lambda + \epsilon \quad \text{and} \quad \xi_{k_0}(y) + \lambda \leq \xi_{k_0}(y + \lambda k_0).$$

That is,

$$\xi_{k_0}(y) + \lambda \leq \xi_{k_0}(y + \lambda k_0). \tag{2.5}$$

Combining (2.4) and (2.5), we have $\xi_{k_0}(y + \lambda k_0) = \xi_{k_0}(y) + \lambda$. □

According to the cases that ξ_{k_0} takes $-\infty$, ∞ and finite real-values, we can easily verify the monotony of the function ξ_{k_0}.

Lemma 2.11 *If $y_1 - y_2 \in -K$, then $\xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$.*

3 P-distances and Q-distances in Uniform Spaces

In this section, we always assume that $(X, \mathcal{U})$ is a separated uniform space and the uniformity $\mathcal{U}$ defines the topology on X (see [29, 30]). The following p-distances and q-distances are generalizations of w-distances (see [28]) and τ-functions (see [32]).

Definition 3.1 ([46, Definition 2.3]) *Let X be a separated uniform space. An extended real-valued function $p : X \times X \to [0, +\infty]$ is called a p-distance on X if the following conditions are satisfied*:

(p1) *for any $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$;*

(p2) *every sequence $\{y_n\} \subset X$ with $p(y_n, y_m) \to 0$ $(m > n \to \infty)$ is a Cauchy sequence and in the case $p(y_n, y) \to 0$ is equivalent to $y_n \to y$ in X;*

(p3) *for $x, y, z \in X, p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

If the condition (p2) *is replaced by the following weaker condition*:

(q2) *every sequence $\{y_n\} \subset X$ with $p(y_n, y_m) \to 0$ $(m > n \to \infty)$ is a Cauchy sequence and in the case $p(y_n, y) \to 0$ implies that $y_n \to y$ in X,*
then p is called a q-distance on X.

Here, $p(y_n, y_m) \to 0$ $(m > n \to \infty)$ means that for any $\epsilon > 0$, there exists $n_0 \in N$ such that $p(y_n, y_m) < \epsilon$ for all $m > n \geq n_0$.

In the following, for brevity, we always denote "with respect to" by "w.r.t.".

Definition 3.2 ([46, Definition 3.1]) *A uniform space $(X, \mathcal{U})$ is said to be sequentially complete w.r.t. a p-distance* (*resp. a q-distance*) *p, if for any sequence $\{x_n\}$ in X with $p(x_n, x_m) \to 0$ $(m > n \to \infty)$, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ in X* (*resp. $p(x_n, \bar{x}) \to 0$*). *A nonempty subset S of X is said to be sequentially closed w.r.t. a p-distance* (*resp. a q-distance*) *p if for any sequence $\{x_n\} \subset S$ and $\bar{x} \in X$ satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ and $x_n \to \bar{x}$* (*resp. $p(x_n, \bar{x}) \to 0$*) *in X, we have $\bar{x} \in S$.*

Let Y be a real linear space and $K \subset Y$ be a convex cone containing 0 (in general, we always assume that $K \neq \{0\}$ and $K \neq Y$). The convex cone K can specify a pre-order " $\leq_K$ " as follows:

$$\text{for any } y_1, y_2 \in Y, \text{ define } y_1 \leq_K y_2 \text{ if and only if } y_2 - y_1 \in K.$$

We extend Y by an additional element ∞ such that $y \leq_K \infty$, $y + \infty = \infty$ and $t\,\infty = \infty$ for any $y \in Y$ and any $t > 0$. Denote $Y \cup \{\infty\}$ by $Y^\bullet$. A function $f : X \to Y^\bullet$ is said to be proper if $\mathrm{dom} f := \{x \in X : f(x) \in Y\} \neq \emptyset$.

Definition 3.3 *A proper function $f : X \to Y^\bullet$ is said to be sequentially lower K-monotone (if no confusion, "K" can be omitted) if for a sequence $\{x_n\} \subset X$ satisfying $x_n \to \bar{x}$ in X and $f(x_{n+1}) \leq_K f(x_n)$ for each $n \in \mathbb{N}$, we have $f(\bar{x}) \leq_K f(x_n)$ for every $n \in \mathbb{N}$ (see* [20, 22, 36]*; in* [20] *it is said that f satisfies the condition* (H4)*). A proper function $f : X \to Y^\bullet$ is said to be sequentially lower K-monotone w.r.t. a p-distance (resp. a q-distance) p if for a sequence $\{x_n\} \subset X$ satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$, $x_n \to \bar{x}$ in X (resp. $p(x_n, \bar{x}) \to 0$) and $f(x_{n+1}) \leq_K f(x_n)$, we have $f(\bar{x}) \leq_K f(x_n)$ for every $n \in \mathbb{N}$.*

Definition 3.4 ([46, Definition 3.3]) *Let $(X, \mathcal{U})$ be a uniform space, p be a p-distance (resp. a q-distance) on X and $f : X \to Y^\bullet$ be a proper function. $(X, \mathcal{U})$ is said to be sequentially complete w.r.t. $(p, f\downarrow)$ if for a sequence $\{x_n\}$ in X satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ and $f(x_{n+1}) \leq_K f(x_n)$ for each $n \in \mathbb{N}$, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$ in X (resp. $p(x_n, \bar{x}) \to 0$). A nonempty subset S of X is said to be sequentially closed w.r.t. $(p, f\downarrow)$ if for a sequence $\{x_n\} \subset S$ satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$, $x_n \to \bar{x}$ in X (resp. $p(x_n, \bar{x}) \to 0$) and $f(x_{n+1}) \leq_K f(x_n)$, we have $\bar{x} \in S$.*

Obviously, we have the following implications:

$$\begin{aligned}\text{sequential completeness} &\Rightarrow \text{sequential completeness w.r.t. } p \\ &\Rightarrow \text{sequential completeness w.r.t. } (p, f\downarrow),\end{aligned}$$

where p is a p-distance or a q-distance on X and f is any proper function. However, none of the converses is true (see [46]). For sequential closedness, sequential closedness w.r.t. p and sequential closedness w.r.t. $(p, f\downarrow)$, we also have similar implications. We shall establish a general vectorial EVP under the weakest type of completeness assumption.

4 A General Vectorial Ekeland's Variational Principle and its Equivalents

In this section, we give a general vectorial Ekeland's variational principle, Caristi's fixed point theorem and Takahashi's nonconvex minimization theorem. Moreover, we show that the three theorems are equivalent to each other. For the definiteness, we only consider the case that the perturbation contains a p-distance and exhibit a number of results; the correspondent versions on a q-distance can be deduced similarly.

Theorem 4.1 (Vectorial Ekeland's variational principle) *Let $(X, \mathcal{U})$ be a separated uniform space and p be a p-distance on X. Let Y be a real linear space pre-ordered by a convex cone K containing 0 and $k_0 \in K \setminus -K$. Let $\varphi : (Y, \leq_K) \to (0, +\infty)$ be a nondecreasing function (i.e., $y_1 \leq_K y_2 \Rightarrow \varphi(y_1) \leq \varphi(y_2)$) and let $\varphi(\infty) := \sup\{\varphi(y) : y \in Y\}$. Let $f : X \to Y^\bullet$ be a proper function such that for any $x \in X$, the set*

$$S(x) := \left\{ x' \in X : f(x') + \frac{p(x, x')}{\varphi \circ f(x)} k_0 \leq_K f(x) \right\}$$

is sequentially closed w.r.t. $(p, f\downarrow)$ and $(X, \mathcal{U})$ is sequentially complete w.r.t. $(p, f\downarrow)$. Let $x_0 \in X, y_0 \in Y$ and $\epsilon > 0$ such that

$$f(x_0) \in y_0 + (-\infty, +\infty)k_0 - K, \tag{4.1}$$

and

$$f(X) \cap (y_0 - \epsilon k_0 - K) = \emptyset. \tag{4.2}$$

Then either

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \not\leq_K f(x_0), \quad \forall x \in X \backslash \{x_0\},$$

or there exists $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + \frac{p(x_0,\bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{p(\bar{x},x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}),\ \forall x \in X \backslash \{\bar{x}\}$.

Proof For any $u \in \mathrm{dom} f$ and $v \in S(u)$, clearly,

$$f(v) \leq_K f(u). \tag{4.3}$$

Moreover, we claim that

$$S(v) \subset S(u). \tag{4.4}$$

In fact, for any $x \in S(v)$, we have

$$\begin{aligned} f(x) + \frac{p(u, x)}{\varphi \circ f(u)} k_0 &\leq_K f(x) + \frac{p(u, v) + p(v, x)}{\varphi \circ f(u)} k_0 \\ &\leq_K f(x) + \frac{p(u, v)}{\varphi \circ f(u)} k_0 + \frac{p(v, x)}{\varphi \circ f(v)} k_0 \\ &\leq_K f(v) + \frac{p(u, v)}{\varphi \circ f(u)} k_0 \\ &\leq_K f(u). \end{aligned}$$

That is, $x \in S(u)$ and (4.4) holds.

Now we begin to prove the conclusion. Assume that the statement that

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \not\leq_K f(x_0), \quad \forall x \in X \backslash \{x_0\}$$

does not hold. That is, there exists $x' \in X \backslash \{x_0\}$ such that

$$f(x') + \frac{p(x_0, x')}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0).$$

Then $S(x_0) \supset S(x_0) \backslash \{x_0\} \neq \emptyset$. For any $x \in X$, define $g(x) := \xi_{k_0}(f(x) - y_0)$. By (4.2), $f(x) - y_0 \notin -\epsilon k_0 - K$, $\forall x \in X$. By Lemma 2.9 (iv), $g(x) = \xi_{k_0}(f(x) - y_0) \geq -\epsilon$, so $g : X \to (-\infty, +\infty]$ is bounded from below. For any $x \in S(x_0)$,

$$f(x) - y_0 + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0) - y_0.$$

By Lemmas 2.10 and 2.11, we have

$$g(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} = \xi_{k_0}\left(f(x) - y_0 + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \right) \leq \xi_{k_0}(f(x_0) - y_0) = g(x_0).$$

Here, by (4.1), we know that $g(x_0) < +\infty$. Thus, g is bounded from below and takes finite real-value on $S(x_0)$. Choose $x_1 \in S(x_0)$ such that

$$g(x_1) < \inf_{S(x_0)} g + \frac{1}{2},$$

where $\inf_{S(x_0)} g$ denotes $\inf\{g(x) : x \in S(x_0)\}$. If x_1 satisfies

$$f(x) + \frac{p(x_1, x)}{\varphi \circ f(x_1)} k_0 \not\leq_K f(x_1), \quad \forall x \in X \backslash \{x_1\},$$

then we may take $\bar{x} := x_1$ and the conclusion holds. If not, then $S(x_1) \supset S(x_1)\backslash\{x_1\} \neq \emptyset$. As above, choose $x_2 \in S(x_1)$ such that

$$g(x_2) < \inf_{S(x_1)} g + \frac{1}{2^2}.$$

We may repeat the above process. If for some step n, we have

$$f(x) + \frac{p(x_n, x)}{\varphi \circ f(x_n)} k_0 \not\leq_K f(x_n), \quad \forall x \in X \backslash \{x_n\}. \tag{4.5}$$

By (4.4), we have $x_n \in S(x_{n-1}) \subset \cdots \subset S(x_0)$. Combining this with (4.5), we know that $\bar{x} := x_n$ satisfies (i) and (ii), so the conclusion holds. If not, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in S(x_n)$ and

$$g(x_{n+1}) < \inf_{S(x_n)} g + \frac{1}{2^{n+1}}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{4.6}$$

Since $x_{n+1} \in S(x_n)$, by (4.3) and (4.4), we have

$$f(x_{n+1}) \leq_K f(x_n), \tag{4.7}$$

and

$$S(x_{n+1}) \subset S(x_n). \tag{4.8}$$

By (4.7) and Lemma 2.11, we have

$$g(x_{n+1}) = \xi_{k_0}(f(x_{n+1}) - y_0) \leq \xi_{k_0}(f(x_n) - y_0) = g(x_n).$$

Thus, $\{g(x_n)\}_{n\in\mathbb{N}}$ is a nonincreasing sequence, bounded from below. Hence there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{n\to\infty} g(x_n) = \alpha. \tag{4.9}$$

Clearly,

$$g(x_m) \geq \alpha, \quad \forall\, m \in \mathbb{N}. \tag{4.10}$$

When $m > n$, $x_m \in S(x_{m-1}) \subset S(x_n)$, i.e.,

$$f(x_m) + \frac{p(x_n, x_m)}{\varphi \circ f(x_n)} k_0 \leq_K f(x_n). \tag{4.11}$$

Hence

$$f(x_m) - y_0 + \frac{p(x_n, x_m)}{\varphi \circ f(x_n)} k_0 \leq_K f(x_n) - y_0,$$

and so

$$g(x_m) + \frac{p(x_n, x_m)}{\varphi \circ f(x_n)} \leq g(x_n).$$

From this and using (4.10) and (4.9), we have

$$\begin{aligned} p(x_n, x_m) &\leq \varphi \circ f(x_n)(g(x_n) - g(x_m)) \\ &\leq \varphi \circ f(x_0)(g(x_n) - \alpha) \to 0, \quad m > n \to \infty. \end{aligned} \tag{4.12}$$

Since $(X,\mathcal{U})$ is sequentially complete w.r.t. $(p, f\downarrow)$, by (4.7) and (4.12), we know that there exists $\bar{x} \in X$ such that

$$x_n \to \bar{x} \quad \text{in}\, X. \tag{4.13}$$

By the assumption, $S(x_n)$ is sequentially closed w.r.t. $(p, f\downarrow)$ for any n. Remarking that $\{x_m\}_{m>n} \subset S(x_n)$ and combining (4.7), (4.12) and (4.13), we have

$$\bar{x} \in S(x_n). \tag{4.14}$$

Particularly, $\bar{x} \in S(x_0)$, i.e., $\bar{x}$ satisfies (i). We assert that $\bar{x}$ also satisfies (ii), i.e.,

$$f(x) + \frac{p(\bar{x}, x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \quad \forall x \in X \backslash \{\bar{x}\}.$$

If not, there exists $x' \in X$, $x' \neq \bar{x}$ such that

$$f(x') + \frac{p(\bar{x}, x')}{\varphi \circ f(\bar{x})} k_0 \leq_K f(\bar{x}), \quad \text{i.e., } x' \in S(\bar{x}). \tag{4.15}$$

Combining (4.15) and (4.14), we have

$$x' \in S(x_n) \quad \text{for every } n. \tag{4.16}$$

That is,

$$f(x') + \frac{p(x_n, x')}{\varphi \circ f(x_n)} k_0 \leq_K f(x_n).$$

From this, we have

$$f(x') - y_0 + \frac{p(x_n, x')}{\varphi \circ f(x_n)} k_0 \leq_K f(x_n) - y_0,$$

and so

$$g(x') + \frac{p(x_n, x')}{\varphi \circ f(x_n)} \leq g(x_n). \tag{4.17}$$

By (4.6), (4.16) and (4.17), we have

$$\begin{aligned} p(x_n, x') &\leq \varphi \circ f(x_n)(g(x_n) - g(x')) \\ &\leq \varphi \circ f(x_0)\Big(g(x_n) - \inf_{S(x_{n-1})} g\Big) \\ &< \varphi \circ f(x_0) \cdot \frac{1}{2^n} \to 0, \quad n \to \infty. \end{aligned}$$

By (p2) in Definition 3.1,

$$x_n \to x' \quad \text{in } X. \tag{4.18}$$

Comparing (4.18) with (4.13), we have $\bar{x} = x'$, which contradicts the assumption that $x' \neq \bar{x}$. □

Remark 4.2 In Theorem 4.1, if the condition that for any $x \in X$, $S(x)$ is sequentially closed w.r.t. $(p, f\downarrow)$ is replaced by a stronger condition that $S(x)$ is sequentially closed (or sequentially closed w.r.t. p) or/and the condition that $(X,\mathcal{U})$ is sequentially complete w.r.t. $(p, f\downarrow)$ is replaced by a stronger condition that $(X,\mathcal{U})$ is sequentially complete (or sequentially complete w.r.t. p), then the result certainly holds. Obviously, Theorem 4.1 generalizes [2, Theorem 3.2], [20, Corollary 2], [33, Theorems 2.4–2.5 and Corollaries 2.7–2.9], and [34, Theorem 4.2].

Remark 4.3 Moreover, if we assume that $p(x_0, x_0) = 0$, then

$$x_0 \in S(x_0) := \left\{ x \in X : f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0) \right\}.$$

Thus, the result of Theorem 4.1 can be unilaterally rewritten as follows: there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{p(x_0,\bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{p(\bar{x},x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

Remark 4.4 (1) If $k_0 \in \mathrm{cor}(K)$, then for any $x_0 \in \mathrm{dom} f$ and any $y_0 \in Y$, the condition (4.1), i.e., $f(x_0) \in y_0 + (-\infty, +\infty)k_0 - K$, is automatically satisfied. Thus, Theorem 4.1 also generalizes and improves [2, Theorem 3.1].

(2) If we put $y_0 = f(x_0)$ in Theorem 4.1, then conditions (4.1) and (4.2) can be replaced by one condition that $f(X) \cap (f(x_0) - \epsilon k_0 - K) = \emptyset$.

(3) In fact, the condition (4.1) can be replaced by the following condition: $S(x_0) \cap (y_0 + (-\infty, +\infty)k_0 - K) \neq \emptyset$.

(4) In general, the condition that there exist $y_0 \in Y$ and $\epsilon > 0$ such that $f(X) \cap (y_0 - \epsilon k_0 - K) = \emptyset$ is strictly weaker than one that f is $\leq_K$-bounded from below, i.e., there exists $w_0 \in Y$ such that $f(x) \geq_K w_0$, $\forall x \in X$.

Theorem 4.5 (Vectorial Caristi's fixed point theorem) *Let $(X, \mathcal{U}), p, Y, K, k_0, \varphi$ and f be as in Theorem 4.1. Also, let $x_0 \in X, y_0 \in Y$ and $\epsilon > 0$ be as in Theorem 4.1. Assume that a set-valued map $T : X \to 2^X \backslash \{\emptyset\}$ satisfies the following condition:*

(C1) $f(y) + \frac{p(x,y)}{\varphi \circ f(x)} k_0 \leq_K f(x), \forall x \in \mathrm{dom} f, \forall y \in Tx$.

Then either $Tx_0 = \{x_0\}$ with $p(x_0, x_0) = 0$, or there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{p(x_0,\bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $T\bar{x} = \{\bar{x}\}$ *with* $p(\bar{x}, \bar{x}) = 0$.

Assume that $T : X \to 2^X \backslash \{\emptyset\}$ satisfies the following condition:

(C2) $\forall x \in \mathrm{dom} f$, $\exists y \in Tx$ *such that* $f(y) + \frac{p(x,y)}{\varphi \circ f(x)} k_0 \leq_K f(x)$.

Then either $x_0 \in Tx_0$ with $p(x_0, x_0) = 0$, or there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{p(x_0,\bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $\bar{x} \in T\bar{x}$ *with* $p(\bar{x}, \bar{x}) = 0$.

Proof We first prove the conclusion concerning set-valued maps satisfying (C1). By Theorem 4.1, either

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \not\leq_K f(x_0), \quad \forall x \in X \backslash \{x_0\}, \tag{4.19}$$

or there exists $\bar{x} \in X$ such that

$$f(\bar{x}) + \frac{p(x_0, \bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0) \tag{4.20}$$

and

$$f(x) + \frac{p(\bar{x}, x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \quad \forall x \in X \backslash \{\bar{x}\}. \tag{4.21}$$

According to the above two cases, we state the proof.

Case 1 Assume that x_0 satisfies (4.19). By (C1),

$$f(y)+\frac{p(x_0,y)}{\varphi\circ f(x_0)}k_0\leq_K f(x_0),\quad \forall y\in Tx_0. \tag{4.22}$$

By (4.19) and (4.22), we have $y=Tx_0$ and $Tx_0=\{x_0\}$ with $p(x_0,x_0)=0$.

Case 2 Assume that x_0 does not satisfy (4.19). Then there exists $\bar{x}\in X$ satisfying (4.20) and (4.21). By (C1),

$$f(y)+\frac{p(\bar{x},y)}{\varphi\circ f(\bar{x})}k_0\leq_K f(\bar{x}),\quad \forall y\in T\bar{x}. \tag{4.23}$$

By (4.21) and (4.23), we have $T\bar{x}=\{\bar{x}\}$ and $p(\bar{x},\bar{x})=0$. Obviously, $\bar{x}$ satisfies (i) and (ii). Thus, we complete the proof under the condition (C1). Similarly, we can prove the corresponding conclusion under the condition (C2). □

Theorem 4.6 (Vectorial Takahashi's nonconvex minimization theorem) *Let $(X,\mathcal{U})$, p, Y, K, k_0, φ and f be as in Theorem* 4.1. *Also, let $x_0\in X, y_0\in Y$ and $\epsilon>0$ be as in Theorem* 4.1. *Assume that f satisfies the following condition*:

(C3) *for each $u\in X$ such that $f(X)\cap(f(u)-K)\neq\{f(u)\}$, there exists $v\in X$, $v\neq u$ such that*

$$f(v)+\frac{p(u,v)}{\varphi\circ f(u)}k_0\leq_K f(u).$$

Then either $f(X)\cap(f(x_0)-K)=\{f(x_0)\}$ or there exists $\bar{x}\in X$ such that

(i) $f(\bar{x})+\frac{p(x_0,\bar{x})}{\varphi\circ f(x_0)}k_0\leq_K f(x_0)$;

(ii) $f(X)\cap(f(\bar{x})-K)=\{f(\bar{x})\}$.

Proof Assume that $f(X)\cap(f(x_0)-K)\neq\{f(x_0)\}$. By (C3), we know that

$$S(x_0):=\left\{x\in X: f(x)+\frac{p(x_0,x)}{\varphi\circ f(x_0)}k_0\leq_K f(x_0)\right\}\supset S(x_0)\backslash\{x_0\}\neq\emptyset.$$

Take any fixed $x_0'\in S(x_0)\backslash\{x_0\}$. Then by (4.1), we have

$$f(x_0')\in f(x_0)-K\subset y_0+(-\infty,+\infty)k_0-K. \tag{4.24}$$

By (4.2), $f(X)\cap(y_0-\epsilon k_0-K)=\emptyset$. Certainly,

$$f(S(x_0))\cap(y_0-\epsilon k_0-K)=\emptyset. \tag{4.25}$$

Let $\{x_n\}\subset S(x_0)$ such that $p(x_n,x_m)\to 0$ $(m>n\to\infty)$ and $f(x_{n+1})\leq_K f(x_n)$ for every n. Since $(X,\mathcal{U})$ is sequentially complete w.r.t. $(p,f\downarrow)$, there exists $\bar{x}\in X$ such that $x_n\to\bar{x}$ in X. Since $S(x_0)$ is sequentially closed w.r.t. $(p,f\downarrow)$, we have $\bar{x}\in S(x_0)$. Thus, we have shown that the uniform subspace

$$(S(x_0),\mathcal{U}|S(x_0)) \text{ is sequentially complete w.r.t. } (p,f\downarrow). \tag{4.26}$$

Moreover, for any $x\in S(x_0)$, $S(x)\subset S(x_0)$. Also, it is easy to show that

$$S(x),\ \text{as a subset of } (S(x_0),\mathcal{U}|S(x_0)),\ \text{is sequentially closed w.r.t. } (p,f\downarrow). \tag{4.27}$$

Assume that the conclusion is not true. That is, for any $x\in S(x_0)$,

$$f(X)\cap(f(x)-K)\neq\{f(x)\}.$$

Then by (C3), there exists $v \in X$, $v \neq x$ such that

$$f(v) + \frac{p(x,v)}{\varphi \circ f(x)} k_0 \leq_K f(x), \quad \text{i.e., } S(x)\backslash\{x\} \neq \emptyset.$$

We define a set-valued map $T : S(x_0) \to 2^{S(x_0)}\backslash\{\emptyset\}$ as follows:

$$T(x) = S(x)\backslash\{x\}, \quad \forall x \in S(x_0). \tag{4.28}$$

Obviously, T satisfies the condition (C1):

$$f(y) + \frac{p(x,y)}{\varphi \circ f(x)} k_0 \leq_K f(x), \quad \forall x \in S(x_0),\ \forall y \in T(x).$$

By (4.24)–(4.27), we may substitute X by $S(x_0)$ in Theorem 4.5. Thus, there exists $\bar{x} \in S(x_0)$ such that $T\bar{x} = \{\bar{x}\}$, which contradicts (4.28). □

Theorem 4.7 *Theorems* 4.1, 4.5 *and* 4.6 *are equivalent to each other.*

Proof It is sufficient to prove that Theorem 4.6 implies Theorem 4.1. Assume that there exists $x' \in X$, $x' \neq x_0$ such that

$$f(x') + \frac{p(x_0,x')}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0).$$

Then $S(x_0) \supset S(x_0)\backslash\{x_0\} \neq \emptyset$. As shown in the proof of Theorem 4.6, take any fixed $x_0' \in S(x_0)\backslash\{x_0\}$. Then, by (4.1), we have

$$f(x_0') \in y_0 + (-\infty, +\infty)k_0 - K. \tag{4.29}$$

By (4.2),

$$f(S(x_0)) \cap (y_0 - \epsilon k_0 - K) = \emptyset. \tag{4.30}$$

As done in the proof of Theorem 4.6, we can deduce that $(S(x_0), \mathcal{U}|S(x_0))$ is sequentially complete w.r.t. $(p, f\downarrow)$ and for any $x \in S(x_0)$, $S(x) \subset S(x_0)$ is sequentially closed w.r.t. $(p, f\downarrow)$. Combining this with (4.29) and (4.30), we may substitute X by $S(x_0)$ in Theorem 4.6. Thus, there exists $x^* \in S(x_0)$ such that

$$f(S(x_0)) \cap (f(x^*) - K) = \{f(x^*)\}. \tag{4.31}$$

Now we claim that there exists $\bar{x} \in S(x_0)$ such that

$$f(x) + \frac{p(\bar{x},x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \quad \forall x \in X\backslash\{\bar{x}\},$$

i.e., $\bar{x}$ satisfies (i) and (ii) in Theorem 4.1. Assume the contrary. There exists $x' \in X$, $x' \neq x^*$ such that

$$f(x') + \frac{p(x^*,x')}{\varphi \circ f(x^*)} k_0 \leq_K f(x^*). \tag{4.32}$$

From this, $x' \in S(x^*)$. Combining this with $x^* \in S(x_0)$, we have

$$x' \in S(x_0). \tag{4.33}$$

By (4.31)–(4.33), we have

$$f(x') = f(x^*). \tag{4.34}$$

By (4.32), (4.34) and $k_0 \notin -K$, we have

$$p(x^*, x') = 0. \tag{4.35}$$

Since $x' \in S(x_0)$, by the assumption on the contrary, there exists $x'' \in X$, $x'' \neq x'$ such that

$$f(x'') + \frac{p(x', x'')}{\varphi \circ f(x')} k_0 \leq_K f(x').$$

Thus, $x'' \in S(x') \subset S(x^*) \subset S(x_0)$. Hence

$$f(x'') + \frac{p(x^*, x'')}{\varphi \circ f(x^*)} k_0 \leq_K f(x^*).$$

Combining this with $x'' \in S(x_0)$ and (4.31), we have $f(x'') = f(x^*)$ and hence

$$p(x^*, x'') = 0. \tag{4.36}$$

By (4.35), (4.36) and (p3) in Definition 3.1, we have $x' = x''$, contradicting $x'' \neq x'$. □

Remark 4.8 Concerning Theorems 4.5 and 4.6, we also have the corresponding remarks like Remarks 4.2–4.4.

5 Some Particular Versions of Vectorial EVP

By Theorems 4.1, 4.5 and 4.6, we can deduce a number of particular versions of EVP, Caristi's fixed point theorem and Takahashi's nonconvex minimization theorem. Here we only give some particular versions of Theorem 4.1 (i.e., vectorial EVP).

In particular, if $Y = \mathbb{R}$, $K = [0, +\infty)$ and $k_0 = 1$ in Theorem 4.1, then we have the following scalar-valued version of EVP.

Theorem 5.1 *Let $(X, \mathcal{U})$ be a separated uniform space, p be a p-distance on X and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ be a nondecreasing function. Let $f : X \to (-\infty, \infty]$ be a bounded from below, proper function such that for any $x \in X$,*

$$S(x) := \left\{ x' \in X : f(x') + \frac{p(x, x')}{\varphi \circ f(x)} \leq f(x) \right\}$$

is sequentially closed w.r.t. $(p, f\downarrow)$ and $(X, \mathcal{U})$ is sequentially complete w.r.t. $(p, f\downarrow)$. Then for any $x_0 \in \mathrm{dom} f$, either

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} > f(x_0), \quad \forall x \in X \backslash \{x_0\},$$

or there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{p(x_0, \bar{x})}{\varphi \circ f(x_0)} \leq f(x_0)$;

(ii) $f(x) + \frac{p(\bar{x}, x)}{\varphi \circ f(\bar{x})} > f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

If we assume that f is sequentially lower monotone w.r.t. p, then for a sequence $\{x_n\} \subset S(x)$ satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty), x_n \to \bar{x}$ in X and $f(x_{n+1}) \leq f(x_n)$, we have $f(\bar{x}) \leq f(x_n), \forall n \in \mathbb{N}$. Since $x_n \in S(x)$, we have

$$f(x_n) + \frac{p(x, x_n)}{\varphi \circ f(x)} \leq f(x).$$

Thus, for any $n \in \mathbb{N}$,

$$\begin{aligned} f(\bar{x}) + \frac{p(x, \bar{x})}{\varphi \circ f(x)} &\leq f(\bar{x}) + \frac{p(x, x_n)}{\varphi \circ f(x)} + \frac{p(x_n, \bar{x})}{\varphi \circ f(x)} \\ &\leq f(x_n) + \frac{p(x, x_n)}{\varphi \circ f(x)} + \frac{p(x_n, \bar{x})}{\varphi \circ f(x)} \end{aligned}$$

$$\leq f(x) + \frac{p(x_n, x)}{\varphi \circ f(x)}. \tag{5.1}$$

Since p is a p-distance, $p(x_n, x_m) \to 0$ ($m > n \to \infty$) and $x_n \to \bar{x}$ imply that $p(x_n, \bar{x}) \to 0$. Letting $n \to \infty$ in (5.1), we have

$$f(\bar{x}) + \frac{p(x, \bar{x})}{\varphi \circ fx)} \leq f(x), \quad \text{i.e., } \bar{x} \in S(x).$$

Thus, $S(x)$ is sequentially closed w.r.t. $(p, f \downarrow)$. Hence in Theorem 5.1, the condition that for any $x \in X$, $S(x)$ is sequentially closed w.r.t. $(p, f \downarrow)$ can be replaced by one that f is sequentially lower monotone w.r.t. p (see [46, Theorem 4.2]).

Corollary 5.2 *Let $(X,\mathcal{U}), p$ and φ be the same as in Theorem* 5.1*. Let $f : X \to (-\infty, +\infty]$ be a bounded from below, proper function which is sequentially lower monotone w.r.t. p. Then the result of Theorem* 5.1 *holds.*

In Theorem 4.1, if we assume that $k_0 \in K\backslash - \mathrm{vcl}(K)$ and f is $\leq_K$-bounded from below, i.e., there exists $w_0 \in Y$ such that $f(x) \geq_K w_0$ for any $x \in X$, then the conditions (4.1) and (4.2) in Theorem 4.1 can be removed and the result remains true.

Theorem 5.3 *Let $(X,\mathcal{U}), p, Y, K, k_0, \varphi$ and f be the same as in Theorem* 4.1*. Moreover, let $k_0 \in K\backslash - \mathrm{vcl}(K)$ and f be $\leq_K$-bounded from below. Then for any $x_0 \in \mathrm{dom}f$, either*

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \not\leq_K f(x_0), \quad \forall x \in X\backslash\{x_0\},$$

or there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{p(x_0,\bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{p(\bar{x},x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \forall x \in X\backslash\{\bar{x}\}$.

Proof Since f is $\leq_K$-bounded from below, we may assume that there exists $w_0 \in Y$ such that $f(x) \geq_K w_0$ for any $x \in X$, i.e., $f(X) \subset (w_0 + K) \cup \{\infty\}$. We assert that for any $x_0 \in \mathrm{dom}f$, there exists $\epsilon > 0$ such that

$$f(X) \cap (f(x_0) - \epsilon k_0 - K) = \emptyset. \tag{5.2}$$

If not, assume that there exists $x_0 \in \mathrm{dom}f$ such that

$$f(X) \cap (f(x_0) - nk_0 - K) \neq \emptyset, \quad \forall n \in \mathbb{N}.$$

Since $f(X) \subset (w_0 + K) \cup \{\infty\}$, we have

$$(w_0 + K) \cap (f(x_0) - nk_0 - K) \neq \emptyset, \quad \forall n \in \mathbb{N}.$$

Hence there exists $k_n, k'_n \in K$ such that

$$w_0 + k_n = f(x_0) - nk_0 - k'_n, \quad \forall n \in \mathbb{N}.$$

From this,

$$k_0 + \frac{w_0 - f(x_0)}{n} = \frac{-k_n - k'_n}{n} \in -K, \quad \forall n \in \mathbb{N}.$$

This leads to that $k_0 \in -\mathrm{vcl}(K)$, contradicting the assumption that $k_0 \in K\backslash - \mathrm{vcl}(K)$. Thus, (5.2) holds. Now, by Theorem 4.1 and Remark 4.4 (2), we obtain the result immediately. □

Let $(X,\mathcal{U})$ be a separated uniform space with the topology generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of pseudo-metrics (see [35]) and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of positive real numbers. Define $p(x,y) = \sup_{\lambda\in\Lambda}\alpha_\lambda q_\lambda(x,y)$ for any $x,y\in X$. Then it is easy to verify that p is a p-distance on X with $p(x,x)=0$ for any $x\in X$ (see [46, Example 2.4]). Thus, from Theorem 5.3, we can deduce a vectorial version of [35, Theorem 2] as follows.

Corollary 5.4 *Let $(X,\mathcal{U})$ be a separated uniform space with the topology generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of pseudo-metrics, $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of positive real numbers and $(X,\mathcal{U})$ be sequentially complete w.r.t. $p := \sup_{\lambda\in\Lambda}\alpha_\lambda q_\lambda$. Let Y be a real linear space pre-ordered by a convex cone K containing* 0, *$k_0\in K\backslash -\mathrm{vcl}(K)$ and $\varphi : (Y,\leq_K)\to(0,+\infty)$ be a nondecreasing function. Let $f : X\to Y^\bullet$ be a $\leq_K$-bounded from below, proper function such that for any $x\in X$,*

$$S(x) := \left\{x'\in X : f(x') + \frac{1}{\varphi\circ f(x)}\sup_{\lambda\in\Lambda}\alpha_\lambda q_\lambda(x,x')k_0 \leq_K f(x)\right\}$$

is sequentially closed w.r.t. p. Then for any $x_0\in\mathrm{dom}f$, there exists $\bar{x}\in X$ such that

(i) $f(\bar{x}) + \frac{1}{\varphi\circ f(x_0)}\sup_{\lambda\in\Lambda}\alpha_\lambda q_\lambda(x_0,\bar{x})k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{1}{\varphi\circ f(\bar{x})}\sup_{\lambda\in\Lambda}\alpha_\lambda q_\lambda(\bar{x},x)k_0 \not\leq_K f(\bar{x}), \forall x\in X\backslash\{\bar{x}\}$.

Moreover, if K is k_0-closed, then (i) *and* (ii) *can be rewritten as follows*:

(i)′ $f(\bar{x}) + \frac{1}{\varphi\circ f(x_0)}\alpha_\lambda q_\lambda(x_0,\bar{x})k_0 \leq_K f(x_0), \forall\lambda\in\Lambda$;

(ii)′ $\forall x\in X\backslash\{\bar{x}\}, \exists\,\mu\in\Lambda$ *such that* $f(x) + \frac{1}{\varphi\circ f(\bar{x})}\alpha_\mu q_\mu(\bar{x},x)k_0 \not\leq_K f(\bar{x})$.

Obviously, Corollary 5.4 extends [20, Corollary 2], [21, Theorem 3], [26, Theorem 3], [43, Corollary 3.1] and [44, Theorem 3.2]. In 1996, using families of quasi-metrics, Fang [16] introduced F-type topological spaces and established an EVP on F-type topological spaces, where the perturbation involves the family of quasi-metrics. Later Hamel and Löhne (see [23]) proved that a family of quasi-metrics is also another possibility to generate a uniform structure and that the class of F-type topological spaces coincides with the class of separated uniform spaces. Moreover, they obtained some interesting extensions of Fang's result. Using p-distances, we can also obtain some new results in the direction. As in [16, 22, 23], let $(X,\mathcal{U})$ be a separated uniform space (i.e., an F-type topological space) with the topology generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasi-metrics and $h:\Lambda\to(0,+\infty)$ be a nonincreasing function. Define $p(x,y) := \sup_{\lambda\in\Lambda}h(\lambda)q_\lambda(x,y), \forall x,y\in X$. Then it is easy to verify that p is a p-distance on X with $p(x,x)=0$ for any $x\in X$ (see [46, Example 2.3]). By Theorem 5.3, we obtain the following invariant of [23, Corollary 11].

Corollary 5.5 *Let $(X,\mathcal{U})$ be a separated uniform space* (*equivalently, an F-type topological space*) *with the topology generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasi-metrics, $h:\Lambda\to(0,+\infty)$ be a nondecreasing function and $(X,\mathcal{U})$ be sequentially complete w.r.t. $p := \sup_{\lambda\in\Lambda}h(\lambda)q_\lambda$. Let Y,K,k_0 and φ be the same as in Corollary* 5.4. *Let $f : X\to Y^\bullet$ be a $\leq_K$-bounded from below, proper function such that for any $x\in X$,*

$$S(x) := \left\{x' : f(x') + \frac{1}{\varphi\circ f(x)}\sup_{\lambda\in\Lambda}h(\lambda)q_\lambda(x,x')k_0 \leq_K f(x)\right\}$$

is sequentially closed w.r.t. p. Then for any $x_0\in\mathrm{dom}f$, there exists $\bar{x}\in X$ such that

(i) $f(\bar{x}) + \frac{1}{\varphi\circ f(x_0)}\sup_{\lambda\in\Lambda}h(\lambda)q_\lambda(x_0,\bar{x})k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{1}{\varphi \circ f(\bar{x})} \sup_{\lambda \in \Lambda} h(\lambda) q_\lambda(\bar{x}, x) k_0 \not\leq_K f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

Moreover, if K is k_0-closed, then (i) *and* (ii) *can be rewritten as follows*:

(i)′ $f(\bar{x}) + \frac{1}{\varphi \circ f(x_0)} h(\lambda) q_\lambda(x_0, \bar{x}) k_0 \leq_K f(x_0), \forall \lambda \in \Lambda$;

(ii)′ $\forall x \in X \backslash \{\bar{x}\}, \exists \mu \in \Lambda$ *such that* $f(x) + \frac{1}{\varphi \circ f(\bar{x})} h(\mu) q_\mu(\bar{x}, x) k_0 \not\leq_K f(\bar{x})$.

Remark 5.6 In Theorem 5.3 and in Corollaries 5.4 and 5.5, if the condition that $k_0 \in K \backslash -\mathrm{vcl}(K)$ is replaced by one that $k_0 \in \mathrm{cor}(K)$, then the condition that f is $\leq_K$-bounded from below can be replaced by a weaker condition (4.2), i.e., there exist $y_0 \in Y$ and $\epsilon > 0$ such that $f(X) \cap (y_0 - \epsilon k_0 - K) = \emptyset$. Thus, the above results also extend [2, Theorem 3.1].

Remark 5.7 Here the condition that $(X, \mathcal{U})$ is sequentially complete w.r.t. $p := \sup_{\lambda \in \Lambda} \alpha_\lambda q_\lambda$ (or $p := \sup_{\lambda \in \Lambda} h(\lambda) q_\lambda$) may be strictly weaker than one that $(X, \mathcal{U})$ is sequentially complete. Similarly, the condition that $S(x)$ is sequentially closed with w.r.t. p may be strictly weaker than one that $S(x)$ is sequentially closed. For example, let X be a complete convex space which is not sequentially complete (on the existence of such a space, see [39, Example 5.1.12], [40, Example 1], [41, Example 3.1]), $\{\| \cdot \|_\lambda\}_{\lambda \in \Lambda}$ be a family of semi-norms generating the topology on X and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ be a family of positive real numbers. Define a p-distance p on X as follows: $p(x, y) := \sup_{\lambda \in \Lambda} \alpha_\lambda \|x - y\|_\lambda$, $\forall x, y \in X$. Then a sequence $\{x_n\}$ in X satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ is a locally Cauchy sequence (see [43]). Since X is locally complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. That is, X is sequentially complete w.r.t. p, but it is not sequentially complete. Besides, $S(x)$ being locally closed implies that $S(x)$ is sequentially closed w.r.t. p. And there exists a locally closed set which is not sequentially closed (see, for example, [41, Example 3.1]). Hence, a set which is sequentially closed w.r.t. p, may be not sequentially closed. In fact, for any locally convex space X (even the space is not locally complete), provided that $S \subset X$ is a locally complete convex set containing 0, then X is sequentially complete w.r.t. p, where $p(x, y) := p_S(x - y), \forall x, y \in X$. Here, p_S denotes the Minkowski functional of S and clearly $p = p_S$ is a p-distance on X.

Corollary 5.8 (Vectorial versions of [43, Corollary 3.1] and [44, Theorem 3.2]) *Let X be a locally complete locally convex* (*resp. p-convex*) *space with the topology generated by a family $\{\| \cdot \|_\lambda\}_{\lambda \in \Lambda}$ of semi-norms* (*resp. p-homogeneous F-pseudonorms*) *and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ be a family of positive real numbers. Let Y, K, k_0 and φ be the same as in Corollary* 5.4. *Let $f : X \to Y^\bullet$ be a $\leq_K$-bounded from below, proper function such that for any $x \in X$,*

$$S(x) := \left\{ x' \in X : f(x') + \frac{1}{\varphi \circ f(x)} \sup_{\lambda \in \Lambda} \alpha_\lambda \|x - x'\|_\lambda k_0 \leq_K f(x) \right\}$$

is locally closed. Then for any $x_0 \in \mathrm{dom} f$, there exists an $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{1}{\varphi \circ f(x_0)} \sup_{\lambda \in \Lambda} \alpha_\lambda \|x_0 - \bar{x}\|_\lambda k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{1}{\varphi \circ f(\bar{x})} \sup_{\lambda \in \Lambda} \alpha_\lambda \|(\bar{x} - x\|_\lambda k_0 \not\leq_K f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

Corollary 5.9 (Vectorial versions of [43, Theorem 3.2] and [44, Theorem 3.1′]) *Let X be a locally convex* (*resp. p-convex*) *space, $S \subset X$ be a locally complete bounded convex* (*resp. p-convex*) *set containing* 0 *and p_S be the Minkowski functional of S. Let Y, K, k_0 and φ be the same as in Corollary* 5.4. *Let $f : X \to Y^\bullet$ be a $\leq_K$-bounded from below, proper function such that for any $x \in X$,*

$$S(x) := \left\{ x' \in X : f(x') + \frac{1}{\varphi \circ f(x)} p_S(x - x') k_0 \leq_K f(x) \right\}$$

is locally closed. Then for any $x_0 \in \mathrm{dom} f$*, there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + \frac{1}{\varphi \circ f(x_0)} p_S(x_0 - \bar{x})k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{1}{\varphi \circ f(\bar{x})} p_S(\bar{x} - x)k_0 \not\leq_K f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

Remark 5.10 As pointed out by Hamel (see [22, Theorem 9]), in Theorem 5.3 and in Corollaries 5.4, 5.5, 5.8 and 5.9, if Y is a topological vector space and $k_0 \in K \backslash -\mathrm{cl}(K)$, then the condition that $f : x \to Y^\bullet$ is $\leq_K$-bounded from below can be replaced by a weaker condition: there exists a bounded set $M \subset Y$ such that $\{f(x) : x \in \mathrm{dom} f\} \subset M + K := \{m + k : m \in M, k \in K\}$. We shall see that the above weaker condition can be further weakened. Since $k_0 \notin -\mathrm{cl}(K)$, by the Hahn–Banach separation theorem, there exists $l \in K^+$ such that $l(k_0) > 0$. We denote the set consisting of such linear functionals l by $K^+_{k_0}$ (see [31]). A subset M of Y is said to be l-lower bounded if $l(M) = \{l(m) : m \in M\}$ is a lower bounded scalar set, i.e., there exists $\alpha \in \mathbb{R}$ such that $l(M) \geq \alpha$. In fact, the requirement that $M \subset Y$ is bounded can be replaced by one that $M \subset Y$ is l-lower bounded for some $l \in K^+_{k_0}$.

Theorem 5.11 *Let* $(X, \mathcal{U})$ *be a separated uniform space and* p *be a p-distance such that* $(X, \mathcal{U})$ *is sequentially complete w.r.t.* p*. Let* Y *be a topological vector space pre-ordered by a convex cone* K *containing* 0*,* $k_0 \in K \backslash -\mathrm{cl}(K)$ *and* $\varphi : (Y, \leq_K) \to (0, +\infty)$ *be a nondecreasing function. Let* $f : X \to Y^\bullet$ *be a proper function satisfying the following conditions*:

(a) *for any* $x \in X$,

$$S(x) := \left\{ x' \in X : f(x') + \frac{p(x, x')}{\varphi \circ f(x)} k_0 \leq_K f(x) \right\}$$

is sequentially closed w.r.t. p;

(b) *there exists an* l*-lower bounded set* M*, where* $l \in K^+_{k_0}$*, such that* $\{f(x) : x \in \mathrm{dom} f\} \subset M + K$.

Then for any $x_0 \in \mathrm{dom} f$*, either*

$$f(x) + \frac{p(x_0, x)}{\varphi \circ f(x_0)} k_0 \not\leq_K f(x_0), \quad \forall x \in X \backslash \{x_0\},$$

or there exists $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + \frac{p(x_0, \bar{x})}{\varphi \circ f(x_0)} k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{p(\bar{x}, x)}{\varphi \circ f(\bar{x})} k_0 \not\leq_K f(\bar{x}), \forall x \in X \backslash \{\bar{x}\}$.

Proof By Theorem 4.1 and Remark 4.4 (2), we only need to prove that for any $x_0 \in \mathrm{dom} f$, there exists $\epsilon > 0$ such that $f(X) \cap (f(x_0) - \epsilon k_0 - K) = \emptyset$. Assume the contrary. There exists $x_0 \in \mathrm{dom} f$ such that

$$f(X) \cap (f(x_0) - nk_0 - K) \neq \emptyset, \quad \forall n \in \mathbb{N}.$$

Thus,

$$(M + K) \cap (f(x_0) - nk_0 - K) \neq \emptyset, \quad \forall n \in \mathbb{N}.$$

Hence there exist $m_n \in M$, $k_n, k'_n \in K$ such that

$$m_n + k_n = f(x_0) - nk_0 - k'_n, \quad \forall n \in \mathbb{N}.$$

From this, we have

$$\frac{m_n}{n} - \frac{f(x_0)}{n} + k_0 = \frac{-k_n - k'_n}{n} \in -K.$$

As M is l-lower bounded, there exists $\alpha \in \mathbb{R}$ such that $l(M) \geq \alpha$. Thus,

$$\frac{\alpha}{n} - \frac{l(f(x_0))}{n} + l(k_0) \leq \frac{l(m_n)}{n} - \frac{l(f(x_0))}{n} + l(k_0) \leq 0. \tag{5.3}$$

Letting $n \to \infty$ in (5.3), we have $l(k_0) \leq 0$. However, by $l \in K_{k_0}^{+}$, we also have $l(k_0) > 0$, which is a contradiction. □

Remark 5.12 From Theorem 5.11, we know that in Corollaries 5.4, 5.5, 5.8 and 5.9, if Y is a topological vector space and $k_0 \in K\backslash - \mathrm{cl}(K)$, then the condition that $f : X \to Y^{\bullet}$ is $\leq_K$-bounded from below can be replaced by the following weaker condition: there exists an l-bounded set $M \subset Y$, where $l \in K_{k_0}^{+}$, such that $\{f(x) : x \in \mathrm{dom} f\} \subset M + K$. In particular, we can obtain the following variant of [9, Theorem 5].

Corollary 5.13 *Let $(X,\mathcal{U}), \{q_\lambda\}_{\lambda\in\Lambda}$ and $h : \Lambda \to (0,+\infty)$ be the same as in Corollary 5.5. Let Y be a locally convex topological vector space, $K \subset Y$ be a convex cone with a base, $k_0 \in K\backslash\{0\}$ and $\varphi : (Y, \leq_K) \to (0,+\infty)$ be a nondecreasing function. Let $f : X \to Y^{\bullet}$ satisfy the following conditions:*

(a) *for any $x \in X$,*

$$S(x) := \left\{x' \in X : f(x') + \frac{1}{\varphi \circ f(x)} \sup_{\lambda\in\Lambda} h(\lambda) q_\lambda(x,x') k_) \leq_K f(x)\right\}$$

is sequentially closed w.r.t. p;

(b) *there exists an l-lower bounded set $M \subset Y$, where $l \in K_{k_0}^{+}$, such that $\{f(x) : x \in \mathrm{dom} f\} \subset M + K$.*

Then for any $x_0 \in \mathrm{dom} f$, there exists $\bar{x} \in X$ such that

(i) $f(\bar{x}) + \frac{1}{\varphi\circ f(x_0)} \sup_{\lambda\in\Lambda} h(\lambda) q_\lambda(x_0,\bar{x}) k_0 \leq_K f(x_0)$;

(ii) $f(x) + \frac{1}{\varphi\circ f(\bar{x})} \sup_{\lambda\in\Lambda} h(\lambda) q_\lambda(\bar{x},x) k_0 \not\leq_K f(\bar{x}),\ \forall x \in X\backslash\{\bar{x}\}$.

Proof Since K has a base, $K^{+i} := \{l \in Y^* : l(k) > 0,\ \forall k \in K\backslash\{0\}\} \neq \emptyset$. First we assert that $k_0 \notin -\mathrm{cl}(K)$. If not, assume that $-k_0 \in \mathrm{cl}(K)$. Then there exists a net $\{k_\delta\} \subset K$ such that $k_\delta \to -k_0$ in Y. Take any fixed $l \in K^{+i}$. Then $l(-k_0) = \lim l(k_\delta) \geq 0$, which leads to that $l(k_0) \leq 0$. However, by $l \in K^{+i}$ and $k_0 \in K\backslash\{0\}$, we have $l(k_0) > 0$, which is a contradiction. Thus, we have shown that $k_0 \in K\backslash -\mathrm{cl}(K)$. Now by Theorem 5.11 and Remark 5.12, we obtain the result. □

In [8, 20, 22, 32, 34, 36], sequentially lower monotone functions have been considered. Such a function class is more extensive than lower semi-continuous function class (see [8, 36]). In the following, we study vectorial EVP for sequentially lower K-monotone w.r.t. p functions. We observe that in Theorems 4.1 and 5.3, the condition that $S(x)$ is sequentially closed w.r.t. $(p, f\downarrow)$ can be replaced by one that f is sequentially lower K-monotone w.r.t. p.

Theorem 5.14 *Let $(X,\mathcal{U})$ be a separated uniform space and p be a p-distance on X. Let Y be a real linear space pre-ordered by a convex cone K containing 0, $k_0 \in K\backslash - K$ such that K is k_0-closed and $\varphi : (Y, \leq_K) \to (0,+\infty)$ be a nondecreasing function. Let $f : X \to Y^{\bullet}$ be sequentially lower K-monotone w.r.t. p and $(X,\mathcal{U})$ be sequentially complete w.r.t. $(p, f\downarrow)$. Let $x_0 \in X, y_0 \in Y$ and $\epsilon > 0$ such that (4.1) and (4.2) in Theorem 4.1 hold. Then the result of Theorem 4.1 holds.*

Proof It is sufficient to show that for any $x \in X$, $S(x)$ is sequentially closed w.r.t. $(p, f\downarrow)$. Let $\{x_n\} \subset S(x)$ satisfy $p(x_n, x_m) \to 0$ $(m > n \to \infty)$, $x_n \to \bar{x}$ in X and $f(x_{n+1}) \leq_K f(x_n)$ for every n. We will show that $\bar{x} \in S(x)$. Since f is sequentially lower K-monotone w.r.t. p, we have

$$f(\bar{x}) \leq_K f(x_n), \quad \forall n \in \mathbb{N}. \tag{5.4}$$

By $x_n \in S(x)$, we have

$$f(x_n) + \frac{p(x, x_n)}{\varphi \circ f(x)} k_0 \leq_K f(x). \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$f(\bar{x}) + \frac{p(x, x_n)}{\varphi \circ f(x)} k_0 \leq_K f(x), \quad \forall n \in \mathbb{N}. \tag{5.6}$$

By (5.6), we have

$$\begin{aligned} f(\bar{x}) + \frac{p(x, \bar{x})}{\varphi \circ f(x)} k_0 &\leq_K f(\bar{x}) + \frac{p(x, x_n)}{\varphi \circ f(x)} k_0 + \frac{p(x_n, \bar{x})}{\varphi \circ f(x)} k_0 \\ &\leq_K f(x) + \frac{p(x_n, \bar{x})}{\varphi \circ f(x)} k_0, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.7}$$

By (p2) in Definition 3.1, the assumption that $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ and $x_n \to \bar{x}$ implies that $p(x_n, \bar{x}) \to 0$ $(n \to \infty)$. And since K is k_0-closed, letting $n \to \infty$ in (5.7), we have

$$f(\bar{x}) + \frac{p(x, \bar{x})}{\varphi \circ f(x)} k_0 \leq_K f(x), \quad \text{i.e., } \bar{x} \in S(x). \qquad \square$$

We see that under the assumption that K is k_0-closed, in Theorems 5.3, 5.11 and Corollaries 5.4, 5.5, 5.13, the condition that $S(x)$ is sequentially closed w.r.t. p can be replaced by the condition that f is sequentially lower K-monotone w.r.t. p. And in Corollaries 5.8 and 5.9, the condition that $S(x)$ is locally closed can be replaced by the condition that f is locally sequentially lower K-monotone, i.e., for a sequence $\{x_n\} \subset X$ which is locally convergent to $\bar{x}$ and satisfies $f(x_{n+1}) \leq_K f(x_n)$, we have $f(\bar{x}) \leq_k f(x_n)$ for every n. Particularly, inspired by [22], we have the following corollary.

Corollary 5.15 ([22, Theorem 9]) *Let* $(X, \mathcal{U})$, $\{q_\lambda\}_{\lambda \in \Lambda}$ *and* h *be the same as in Corollary* 5.5. *Let* Y *be a topological vector space,* $K \subset Y$ *be a convex cone containing* 0, $k_0 \in K \backslash -\mathrm{cl}(K)$ *such that* K *is* k_0*-closed and* $\varphi : (Y, \leq_K) \to (0, +\infty)$ *be a nondecreasing function. Let* $f : X \to Y^\bullet$ *be a proper function satisfying the following conditions*:

(a) f *is sequentially lower* K*-monotone w.r.t.* $p := \sup_{\lambda \in \Lambda} h(\lambda) q_\lambda$;

(b) *there exists an* l*-lower bounded set* M, *where* $l \in K_{k_0}^+$, *such that* $\{f(x) : x \in \mathrm{dom} f\} \subset M + K$.

Then for any $x_0 \in \mathrm{dom} f$, *there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + \frac{1}{\varphi \circ f(x_0)} h(\lambda) q_\lambda(x_0, \bar{x}) k_0 \leq_K f(x_0)$, $\forall \lambda \in \Lambda$;

(ii) $\forall x \in X \backslash \{\bar{x}\}, \exists\, \mu \in \Lambda$ *such that* $f(x) + \frac{1}{\varphi \circ f(\bar{x})} h(\mu) q_\mu(\bar{x}, x) k_0 \not\leq_K f(\bar{x})$.

Corollary 5.16 ([22, Theorem 10]) *Let* $(X, \mathcal{U})$, $\{q_\lambda\}_{\lambda \in \Lambda}$ *and* h *be the same as in Corollary* 5.5. *Let* Y *be a real linear space pre-ordered by a convex cone* K *containing* 0, $k_0 \in K \backslash -K$ *such that* K *is* k_0*-closed and* $\varphi : (Y, \leq_K) \to (0, +\infty)$ *be a nondecreasing function. Let* $f : X \to Y^\bullet$ *be a proper function satisfying the following conditions*:

(a) *f is sequentially K-monotone w.r.t.* $p := \sup_{\lambda\in\Lambda} h(\lambda)q_\lambda$;

(b) *there exists* $y_0 \in Y$ *and* $\epsilon > 0$ *such that* $f(X) \cap (y_0 - \epsilon k_0 - K) = \emptyset$.

Then for any $x_0 \in X$ *with* $f(x_0) \in y_0 + (-\infty, +\infty)k_0 - K$, *there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) + \frac{1}{\varphi\circ f(x_0)} h(\lambda)q_\lambda(x_0, \bar{x})k_0 \leq_K f(x_0),\ \forall\lambda \in \Lambda$;

(ii) $\forall x \in X\backslash\{\bar{x}\}, \exists\, \mu \in \Lambda$ *such that* $f(x) + \frac{1}{\varphi\circ f(\bar{x})} h(\mu)q_\mu(\bar{x}, x)k_0 \not\leq_K f(\bar{x})$.

Problem The authors don't know whether it is necessary to assume that K is k_0-closed in Theorem 5.14 and in Corollaries 5.15 and 5.16.

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