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Some Remarks on One-dimensional Functions and Their Riemann–Liouville Fractional Calculus

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Abstract A one-dimensional continuous function of unbounded variation on $[0, 1]$ has been constructed. The length of its graph is infinite, while part of this function displays fractal features. The Box dimension of its Riemann–Liouville fractional integral has been calculated.

Keywords Box dimension, Hausdorff dimension, fractional integral, fractal function

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1 Introduction

Many phenomena display fractal features when plotted as functions of time. Examples include financial series, bioelectric recordings, levels of reservoirs and prices on the stock market, at least when recorded over fairly long time spans. Such functions are often continuous everywhere but differentiable nowhere. For example, Weierstrass function defined as [1]

$$W(x) = \sum_{j=1}^{\infty} \lambda^{-\alpha j} \sin(\lambda^j x), \quad 0 < \alpha < 1, \lambda > 4$$

is a prototype example. The importance of studying continuous functions whose graphs display fractal features, such as Weierstrass function, was emphasized by Falconer [1], Poincaré [2] and Mandelbrot [3] long time ago. Because they are often lack of differentiability, we apply fractional calculus, such as the Riemann–Liouville (R–L) fractional calculus [4] or the Weyl–Marchaud fractional calculus [9] on them. In recent years, fractal dimensions of fractional calculus of certain functions have been discussed by many scholars, for example, [5–8, 10–13]. Most of the people pay attention to special functions such as Weierstrass function and Besicovitch function. In the present paper, we deal with some one-dimensional functions and their R–L fractional calculus.

In Section 2, we prove that both of the Box dimension and the Hausdorff dimension of a continuous function of bounded variation is 1. And the R–L fractional integral of a function of bounded variation is still a function of bounded variation. In Sections 3 and 4, we construct a one-dimensional function whose graph has infinite length, i.e., the variation of this function is infinite. The Box dimension of this function and its R–L fractional integral are 1. Certain graphs of the R–L fractional calculus of such function has been given.

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We denote $R_f[a,b]$ for the maximum range of $f(x)$ over a closed interval $[a,b]$, i.e.,

$$R_f[a,b] = \sup_{a<x,y<b} |f(x)-f(y)|.$$

Let $f(x)$ be a function on $[0,1]$ and $\{x_i\}_{i=1}^n$ be arbitrary points satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. If $\sup_{(x_0,x_1,\ldots,x_n)} \sum_{k=1}^n |f(x_k) - f(x_{k-1})| < C$, we say that $f(x)$ is of bounded variation on I. Let $\mathrm{BV}(I)$ denote the functions of bounded variation on I. In the present paper, I is the unit interval $[0,1]$, and all functions are defined and continuous on I. The upper, lower Box dimension, the Box dimension and the Hausdorff dimension are defined as [1]. If the Box dimension of a function is 1, we say that it is one-dimensional function.

Throughout the present paper, let $\Gamma(f,I)$ denote the graph of $f(x)$ on I. By C we denote a positive constant that may have different values at different occurrences, even in the same line.

2 The R–L Fractional Integral of Functions of Bounded Variation on *I*

Lemma 2.1 *Let $f: I \to \mathbb{R}$ be finite. Suppose that $\delta \in (0, \frac{1}{2})$, and n is the least integer greater than or equal to δ^{-1}. If $N_\delta f$ is the number of squares of the δ-mesh that intersect $\Gamma(f,I)$, then*

$$N_\delta f \geq n \tag{2.1}$$

and

$$N_\delta f \leq 2n + \delta^{-1} \sum_{i=1}^n R_f[(i-1)\delta, i\delta]. \tag{2.2}$$

Proof From the definition of $f(x)$ and $N_\delta f$, the number of mesh squares of side δ in the column above the interval $[(i-1)\delta, i\delta]$ that intersect $\Gamma(f,I)$ is at least 1. Summing over all such intervals leads to (2.1). So does (2.2). □

By Lemma 2.1, we have

Theorem 2.2 ([5]) *If $f(x) \in \mathrm{BV}(I)$, $\dim_B(f,I) = 1$.*

Definition of the R–L fractional integral is given as

Definition 2.3 ([9]) *Let $f(x) \in C_{[0,1]}$ and $v > 0$. For $t \in [0,1]$, we call*

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt$$

the R–L *integral of $f(x)$ of order v.*

With Definition 2.3 and [5], we have the following conclusion.

Theorem 2.4 *If $f(x) \in \mathrm{BV}(I)$, $D^{-v} f(x) \in \mathrm{BV}(I)$ for any positive number v.*

3 An Example of One-dimensional Function of Unbounded Variation on *I*

In this section, we give the construction and graphs of a one-dimensional function of unbounded variation on I.

3.1 Construction

Let $f_1(x)$ be composed of two line segments, $I_1^{(1)}$ and $I_2^{(1)}$. Write

$$\theta_1 = \arccos \frac{1 - \frac{1}{2}}{2 \cdot 1}.$$

The starting point of I_1 is $x_1^{(1)} = (0,0)$, and the end point of I_1 is $x_2^{(1)} = (1 \cdot \cos\theta_1, 1 \cdot \sin\theta_1) = (\frac{1}{4}, \sin\theta_1)$. The starting point of I_2 is $x_2^{(1)}$, and the end point of I_2 is $x_3^{(1)} = (2 \cdot \cos\theta_1, 0) = (\frac{1}{2}, 0)$. The graph of $f_1(x)$ is given in Figure 1.

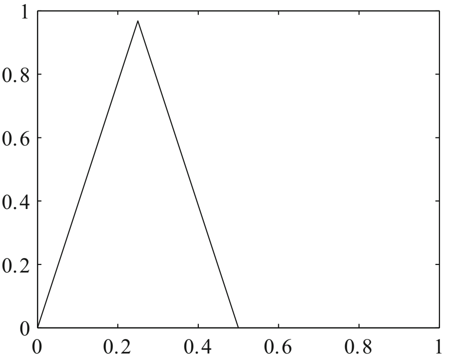

Figure 1 $f_1(x)$

Let $f_n(x)(n \geq 2, n \in \mathbb{N})$ be composed of $2n$ line segments, $I_1^{(n)}$, $I_2^{(n)}, \ldots$, and $I_{2n}^{(n)}$. Write $\theta_n = \arccos \frac{\frac{1}{n} - \frac{1}{n+1}}{2 \cdot \frac{1}{n}}$. The starting point of $I_1^{(n)}$ is $x_1^{(n)} = (0,0)$ and the end point of $I_1^{(n)}$ is $x_2^{(n)} = (\frac{1}{4}, \sin\theta_1)$. The starting point of $I_2^{(n)}$ is $x_2^{(n)}$, and the end point of $I_2^{(n)}$ is $x_3^{(n)} = (\frac{1}{2}, 0)$. The starting point of $I_3^{(n)}$ is $x_3^{(n)}$ and, the end point of $I_3^{(n)}$ is $x_4^{(n)} = (2\cos\theta_1 + \frac{1}{2}\cos\theta_2, \frac{1}{2}\sin\theta_2), \ldots$. The starting point of $I_{2n}^{(n)}$ is

$$x_{2n}^{(n)} = \left(2\sum_{i=1}^{n-1} \frac{1}{i}\cos\theta_i + \frac{1}{n}\cos\theta_n, \frac{1}{n}\sin\theta_n\right),$$

and the end point of $I_{2n}^{(n)}$ is $x_{2n+1}^{(n)} = (1 - \frac{1}{n+1}, 0)$. Define

$$M(x) = \lim_{n\to\infty} f_n(x). \tag{3.1}$$

The graphs of $M(x)$ is given in Figure 2.

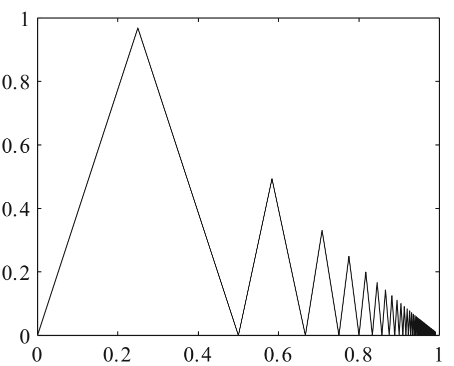

Figure 2 $M(x)$

We give the expression of $M(x)$. If $x \in [0, 1/4), M(x) = \sqrt{15}x$; if $x \in [1/4, 1/2), M(x) = -\sqrt{15}(x - 1/2)$; if $x \in [1/2, 7/12), M(x) = \sqrt{35}(x - 1/2)$; if $x \in [7/12, 2/3), M(x) = -\sqrt{35}(x - 2/3), \ldots$. Let

$$a_n = n/(n+1), \quad b_n = (n+1)/(n+2), \quad c_n = 1/(n+1) \ (n \geq 0).$$

If $x \in [a_n, (b_n + a_n)/2)$,

$$M(x) = \frac{\sqrt{4c_n^2 - (b_n - a_n)^2}}{(b_n - a_n)}(x - a_n);$$

if $x \in [(b_n + a_n)/2, b_n)$,

$$M(x) = -\frac{\sqrt{4c_n^2 - (b_n - a_n)^2}}{(b_n - a_n)}(x - b_n).$$

3.2 Fractal Dimensions of $\Gamma(M, I)$

From the construction of $M(x)$, we have the following two theorems.

Theorem 3.1 *$M(x)$ has infinite length and is of unbounded variation on I.*

Proof $M(x)$ is composed of line segments $I_1, I_2, \ldots, I_{2n+1}, I_{2n+2}, \ldots$. The length of each segment are $1, 1, 1/2, 1/2, \ldots, 1/n, 1/n, \ldots$, respectively. Thus the length of $\Gamma(M, I)$ is

$$2(1 + 2^{-1} + 3^{-1} + \cdots + n^{-1} + \cdots) = 2\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$$

This shows that the length of $\Gamma(M, I)$ is infinite. It is obvious that the variation of $M(x)$ on I is larger than $\sum_{n=1}^{\infty} n^{-1} = \infty$. So $M(x)$ has unbounded variation on I. □

Theorem 3.2 *Both of the Box dimension and the Hausdorff dimension of $M(x)$ are 1. Furthermore, the one-dimensional Hausdorff measure of $\Gamma(M, I)$ is infinite.*

Proof Let $0 \le \delta < 1, \delta^{-1} \le n \le 1 + \delta^{-1}$. From [1, (3.4)], $\underline{\dim}_B \Gamma(M, I) \ge 1$. Since $1/n \le \delta$, the number of squares of the δ-mesh that intersect $M(x)$ are at most $2n + 2\delta^{-1}\sum_{i=1}^{n} i^{-1} + 2\delta^{-1}$. Thus

$$\frac{\log N_\delta(F)}{-\log\delta} \le \frac{\log[2n + 2\delta^{-1}(\log(n+1) + 1)]}{-\log\delta},$$

and

$$\lim_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \le 1.$$

This shows $\dim_B \Gamma(M, I) = 1$.

The Hausdorff dimension of F of a subset of $\mathbb{R}^2$ is no more than its Box dimension. So it holds $\dim_H \Gamma(M, I) \le \dim_B \Gamma(M, I) = 1$. This shows that the Hausdorff dimension of $\Gamma(M, I)$ is 1. For arbitrary subsets of $\mathbb{R}^n$, the n-dimensional Hausdorff measure is, to within a constant multiple, just the n-dimensional Lebesgue measure [1]. Since the Lebesgue measure of $\Gamma(M, I)$ is infinite by Theorem 3.1, one-dimensional Hausdorff measure of $\Gamma(M, I)$ is also infinite. So the graph of $\Gamma(M, I)$ is not a 1-set by [1, Subsection 2.2]. □

From Subsection 3.1, it is easy to get the following proposition and remark.

Proposition 3.3 *$M(x)$ is bounded and differentiable almost all on I.*

Remark 3.4 The fine structure of $\Gamma(M, I)$ is reflected in the irregularities at all scales near point $(1, 0)$; nevertheless, this intricate structure stems from a basically simple construction. Whilst it is reasonable to call $\Gamma(M, I)$ a curve, it is much too irregular to have tangents of points near $(1, 0)$ in the classical sense. Let A be the set of points which $M(x)$ is not differentiable on. Then the Lebesgue measure of A is 0. Thus the geometry of $\Gamma(M, I)$ is not easily described in classical terms. On the one hand, by Theorem 3.1, $\Gamma(M, I)$ has infinite length. On the other hand, $\Gamma(M, I)$ occupies zero area in the plane, so neither length nor area provides a very useful description of the size of $\Gamma(M, I)$. Points near $(1, 0)$ of $\Gamma(M, I)$ display fractal feature.

4 The R–L Fractional Integral of $M(x)$

Now we discuss the R–L fractional integral of $M(x)$. $D^{-v}M(x)$ is defined as (3.1).

4.1 Graphs of the R–L Fractional Integral of $M(x)$

We give the graph of $D^{-0.1}M(x)$ as Figure 3. It seems smoother than the graph of $M(x)$.

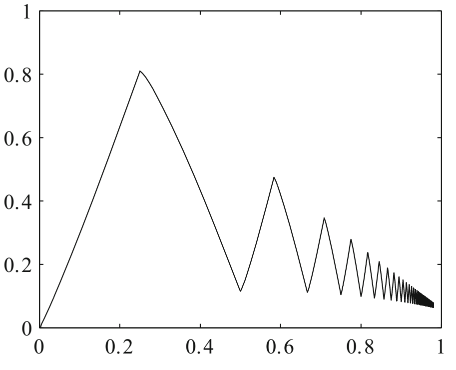

Figure 3 $D^{-0.1}M(x)$

Figure 4 shows the graph of $D^{-0.3}M(x)$.

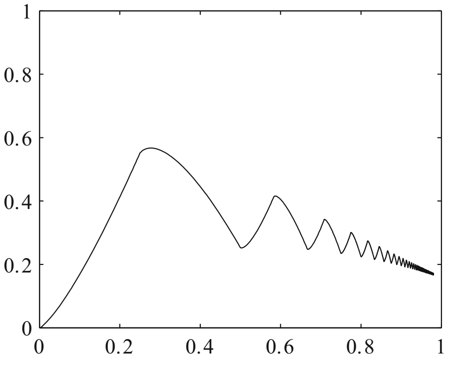

Figure 4 $D^{-0.3}M(x)$

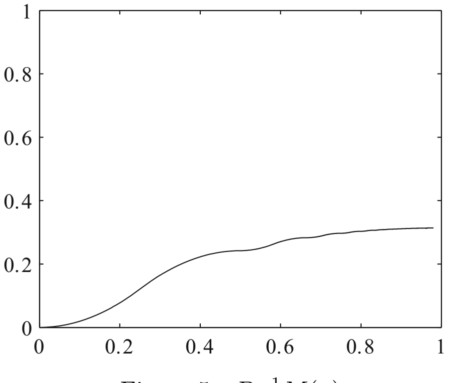

Figure 5 $D^{-1}M(x)$

Figure 5 shows the graph of $D^{-1}M(x)$ which is the Riemann integral of $M(x)$.

From the figures above, we find the bigger v is, the smoother the graph of $D^{-v}M(x)$ is. This observation is consolidated in the general result of the classical Riemann integral.

4.2 The Expression of the R–L Integral of $M(x)$ of Order v

Here we give the expression of $D^{-v}M(x), v \in (0,1)$. If $x \in [0,1/4)$, from Subsection 3.1,

$$D^{-v}M(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} M(t)dt = \frac{\sqrt{15}}{\Gamma(v+2)} x^{v+1}. \tag{4.1}$$

If $x \in [1/4, 1/2)$,

$$D^{-v}M(x) = \frac{\sqrt{15}}{\Gamma(v+2)} x^{v+1} - 2\frac{\sqrt{15}}{\Gamma(v+2)} \left(x - \frac{1}{4}\right)^{v+1}; \tag{4.2}$$

if $x \in [1/2, 7/12)$,

$$\begin{aligned} D^{-v}M(x) = {} & \frac{\sqrt{15}}{\Gamma(v+2)} x^{v+1} - 2\frac{\sqrt{15}}{\Gamma(v+2)} \left(x - \frac{1}{4}\right)^{v+1} + \frac{\sqrt{15}}{\Gamma(v+2)} \left(x - \frac{1}{2}\right)^{v+1} \\ & + \frac{\sqrt{35}}{\Gamma(v+2)} \left(x - \frac{1}{2}\right)^{v+1}; \end{aligned}$$

if $x \in [7/12, 2/3)$,

$$\begin{aligned} D^{-v}M(x) = {} & \frac{\sqrt{15}}{\Gamma(v+2)} x^{v+1} - 2\frac{\sqrt{15}}{\Gamma(v+2)} \left(x - \frac{1}{4}\right)^{v+1} + \frac{\sqrt{15}}{\Gamma(v+2)} \left(x - \frac{1}{2}\right)^{v+1} \\ & + \frac{\sqrt{35}}{\Gamma(v+2)} \left(x - \frac{1}{2}\right)^{v+1} - 2\frac{\sqrt{35}}{\Gamma(v+2)} \left(x - \frac{7}{12}\right)^{v+1}, \ldots. \end{aligned}$$

Let $a_n = n/(n+1), b_n = (n+1)/(n+2), c_n = 1/(n+1) (n \geq 0)$. If $x \in [a_n, (b_n + a_n)/2)$,

$$\begin{aligned} D^{-v}M(x) = {} & \frac{1}{\Gamma(v+2)} \sum_{i=0}^{n-1} \frac{\sqrt{4c_i^2 - (b_i - a_i)^2}}{b_i - a_i} \left((x - a_i)^{v+1} - 2\left(x - \frac{a_i + b_i}{2}\right)^{v+1} \right. \\ & \left. + (x - b_i)^{v+1} \right) + \frac{\sqrt{4c_n^2 - (b_n - a_n)^2}}{\Gamma(v+2)(b_n - a_n)} (x - b_n)^{v+1}; \end{aligned}$$

if $x \in [(b_n + a_n)/2, b_n)$,

$$\begin{aligned} D^{-v}M(x) = {} & \frac{1}{\Gamma(v+2)} \sum_{i=0}^{n-1} \frac{\sqrt{4c_i^2 - (b_i - a_i)^2}}{b_i - a_i} \left((x - a_i)^{v+1} - 2\left(x - \frac{a_i + b_i}{2}\right)^{v+1} \right. \\ & \left. + (x - b_i)^{v+1} \right) + \frac{\sqrt{4c_n^2 - (b_n - a_n)^2}}{\Gamma(v+2)(b_n - a_n)} \left((x - b_n)^{v+1} - 2\left(x - \frac{b_n + a_n}{2}\right)^{v+1} \right). \end{aligned}$$

The expression of $D^{-v}M(x)$ seems more complicated than that of $M(x)$ itself.

4.3 Fractal Dimensions of $D^{-v}M(x)$

The following theorem gives the Box dimension and the Hausdorff dimension of $D^{-v}M(x)$.

Theorem 4.1 *$D^{-v}M(x)$ is finite and differentiable at any points belong to $[0,1)$, and*

$$\dim_H \Gamma(D^{-v}M, I) = \dim_B \Gamma(D^{-v}M, I) = 1, \quad 0 < v < 1. \tag{4.3}$$

Proof Let $0 < v \leq 1$,

$$|D^{-v}M(x)| = \left|\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}M(t)dt\right| \leq \frac{1}{\Gamma(v+1)}x^v \max_{x\in I}|M(x)| \leq \frac{1}{\Gamma(v+1)}.$$

This shows $D^{-v}M(x)$ is finite.

From (4.1) and (4.2), by simple calculation, $D^{-v}M(x)$ is differentiable at any points belong to $[0, 1/2)$. When $n \geq 2$, $D^{-v}M(x)$ is differentiable at any points belong to $[0, (n-1)/n)$. If n tenders to $\infty, D^{-v}M(x)$ is differentiable at any points belong to $[0, 1)$.

Let $h > 0$ and $x, x+h \in [0, 1/4)$. There exists a certain $\xi \in (0, 1/4)$ satisfying

$$|D^{-v}M(x+h) - D^{-v}M(x)| \leq \left|\frac{d}{dx}D^{-v}M(\xi)\right|h \leq \frac{\sqrt{15}}{\Gamma(v+1)}h.$$

When $h > 0$ and $x, x+h \in [1/4, 1/2)$,

$$|D^{-v}M(x+h) - D^{-v}M(x)| \leq 3\frac{\sqrt{15}}{\Gamma(v+1)}h.$$

Write $a_n = n/(n+1)$ and $b_n = (n+1)/(n+2)$. Let $h > 0$ and $x, x+h \in [a_n, (b_n + a_n)/2)$,

$$|D^{-v}M(x+h) - D^{-v}M(x)| \leq \frac{10(n+2)}{\Gamma(v+1)}h.$$

When $h > 0$ and $x, x+h \in [(b_n + a_n)/2, b_n)$,

$$|D^{-v}M(x+h) - D^{-v}M(x)| \leq \frac{14(n+2)}{\Gamma(v+1)}h.$$

Write $\delta_{2n} = [a_n, (b_n + a_n)/2), \delta_{2n+1} = [(b_n + a_n)/2, b_n)$ and $\Delta = \{\delta_i\}_{i=1}^{\infty}$. For any $x, x+h \in \delta_i$, there exists a positive constant C which satisfies that

$$|D^{-v}M(x+h) - D^{-v}M(x)| \leq C|M(x+h) - M(x)|.$$

Let $N_\delta(D^{-v}M(x))$ be the smallest number of sets of diameter at most δ which can cover $D^{-v}M(x)$. From Theorem 3.1,

$$\begin{aligned}\overline{\dim}_B\Gamma(D^{-v}M, I) &= \overline{\lim_{\delta\to 0}}\frac{\log N_\delta\Gamma(D^{-v}M, I)}{-\log\delta} \leq \lim_{\delta\to 0}\frac{\log N_\delta\Gamma(M, I)}{-\log\delta} \\ &= \dim_B\Gamma(M, I) = 1.\end{aligned}$$

This shows the Box dimension of $D^{-v}M(x)$ is 1. Since the Box dimension of a curve is no less than the Hausdorff dimension of this curve,

$$1 \leq \dim_H\Gamma(M(x), I) \leq \dim_B\Gamma(M(x), I) = 1.$$

Thus (4.3) holds. □

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