Acta Mathematica Sinica, English Series
Nov., 2013, Vol. 29, No. 11, pp. 2209–2218
Published online: February 6, 2013
DOI: 10.1007/s10114-013-1536-2
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Universal Inequalities for Lower Order Eigenvalues of Self-Adjoint Operators and the Poly-Laplacian

He Jun SUN
Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094*, P. R. China*
E-mail: *hejunsun@163.com*

Ling Zhong ZENG
Department of Mathematics, Faculty of Science and Engineering, Saga University, Saga 840-8502*, Japan*
E-mail: *lzzeng8@gmail.com*

Abstract In this paper, we first establish an abstract inequality for lower order eigenvalues of a self-adjoint operator on a Hilbert space which generalizes and extends the recent results of Cheng et al. (*Calc. Var. Partial Differential Equations*, **38**, 409–416 (2010)). Then, making use of it, we obtain some universal inequalities for lower order eigenvalues of the biharmonic operator on manifolds admitting some special functions. Moreover, we derive a universal inequality for lower order eigenvalues of the poly-Laplacian with any order on the Euclidean space.

Keywords Eigenvalue, self-adjoint operator, biharmonic operator, poly-Laplacian, Riemannian manifold

MR(2010) Subject Classification 35P15, 58C40

1 Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. The Dirichlet eigenvalue problem of the poly-Laplacian with any order is described by

$$\begin{cases} (-\Delta)^l u = \lambda u, \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \cdots = \left.\dfrac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Δ is the Laplacian and ν denotes the outward unit normal vector field of $\partial\Omega$. Problem (1.1) is called the Dirichlet Laplacian problem when $l = 1$ and the clamped plate problem when $l = 2$.

In this paper, we are concerned about inequalities for lower order eigenvalues of self-adjoint operators and problem (1.1). To begin with, we give a brief review of related results. For the Dirichlet Laplacian problem, Payne et al. [22] proved that its lower order eigenvalues satisfy

$$\lambda_2 + \lambda_3 \le 6\lambda_1 \tag{1.2}$$

Received September 22, 2011, accepted August 20, 2012
The first author is supported by National Natural Science Foundation of China (Grant No. 11001130)

for $\Omega \subset \mathbb{R}^2$. This led to the famous Payne–Pólya–Weinberger conjecture for $\Omega \subset \mathbb{R}^n$. In 1993, Ashbaugh and Benguria [2] established the following universal inequality

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_{i+1} \leq \left(1+\frac{4}{n}\right)\lambda_1 \tag{1.3}$$

for $\Omega \subset \mathbb{R}^n$. For more references on the solution of this conjecture, we refer the readers to [3, 4, 10, 15, 21, 24]. In 2008, Sun et al. [25] further derived some universal inequalities for lower order eigenvalues of the Dirichlet Laplacian problem on bounded domains in a complex projective space and a unit sphere. Chen and Cheng [6] proved (1.3) still holds when Ω is a bounded domain in a complete Riemannian manifold isometrically minimally immersed in $\mathbb{R}^n$.

Γ_i is usually used to denote the i-th eigenvalue of the clamped plate problem. In 1998, Ashbaugh [1] announced the following interesting inequalities without proofs

$$\sum_{i=1}^{n}(\Gamma_{i+1}^{\frac{1}{2}}-\Gamma_1^{\frac{1}{2}}) \leq 4\Gamma_1^{\frac{1}{2}} \tag{1.4}$$

and

$$\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1) \leq 24\Gamma_1. \tag{1.5}$$

In 2010, for a bounded domain Ω in an n-dimensional complete Riemannian manifold M, Cheng et al. [8] proved

$$\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}} \leq \left[(2n+4)\Gamma_1^{\frac{1}{2}}+n^2H_0^2\right]^{\frac{1}{2}}\left(4\Gamma_1^{\frac{1}{2}}+n^2H_0^2\right)^{\frac{1}{2}}, \tag{1.6}$$

where H_0 is a nonnegative constant which only depends on M and Ω. When M is an n-dimensional complete minimal submanifold in a Euclidean space, (1.6) implies

$$\sum_{i=1}^{n}(\Gamma_{i+1}-\Gamma_1)^{\frac{1}{2}} \leq [8(n+2)\Gamma_1]^{\frac{1}{2}}. \tag{1.7}$$

The proofs of (1.4) and (1.5) were given by Cheng et al. [9]. In fact, they considered problem (1.1) and proved

$$\sum_{i=1}^{n}(\lambda_{i+1}^{\frac{1}{l}}-\lambda_1^{\frac{1}{l}})^{l-1} \leq (2l)^{l-1}\lambda_1^{\frac{l-1}{l}} \tag{1.8}$$

for $l \geq 2$, and

$$\sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1) \leq 4l(2l-1)\lambda_1. \tag{1.9}$$

It is easy to find that (1.8) and (1.9) respectively become (1.4) and (1.5) when $l=2$. Moreover, (1.9) covers (1.3) when $l=1$. In 2011, Jost et al. [18] derived the inequality

$$\sum_{i=2}^{n+1}\lambda_i+\sum_{i=1}^{n-1}\frac{2(l-1)i}{2l+i-1}(\lambda_{n+1-i}-\lambda_1) \leq (n+4l(2l-1))\lambda_1, \tag{1.10}$$

which covers (1.3) when $l=1$ and improves (1.5) when $l=2$.

It is natural to consider whether these inequalities can be deduced to self-adjoint operators on Hilbert spaces. Harrell and Davies [12] first realized that some results (e.g. the PPW inequality) of Payne et al. [22] for higher order eigenvalues of the Laplcian rely on some facts involving auxiliary operators and their commutators. Some inequalities for higher order eigenvalues of the Laplacian, biharmonic operator and the poly-Laplacian have been deduced to self-adjoint operators on Hilbert spaces (see [5, 13, 14, 16, 17, 19, 23]). To the authors' knowledge, fewer inequalities for lower order eigenvalues of self-adjoint operators on Hilbert spaces have been obtained by using purely algebraic arguments.

The structure of this paper is as follows. In Section 2, we establish an abstract inequality (2.2) for lower order eigenvalues of self-adjoint operators on Hilbert spaces, which extends the recent work of Cheng et al. [8]. The latter sections are devoted to deriving some explicit inequalities for lower order eigenvalues of problem (1.1) on some Riemannian manifolds. In fact, Cheng et al. [8] considered Riemannian manifolds isometrically immersed in a Euclidean space. We study eignvalues of biharmonic operator on manifolds admitting some special functions in Section 3. The first kind is Riemannian manifolds admitting spherical eigenmaps. As we know, any compact homogeneous Riemannian manifold admits eigenmaps for the first eigenvalue of the Laplacian (see [20]). And another kind is Riemannian manifolds admitting some functions $f_r : M \to \mathbb{R}$ such that

$$\begin{cases} \langle \nabla f_r, \nabla f_s \rangle = \delta_{rs}, \\ \Delta f_r = 0. \end{cases} \tag{1.11}$$

Product manifolds of Euclidean spaces with any complete manifolds satisfy this condition (see [11, 26]). Utilizing Theorem 2.1, we derive some universal inequalities for lower order eigenvalues of the biharmonic operator on these manifolds in Theorem 3.1. In Section 4, a universal inequality for the poly-Laplacian with any order on a bounded domain in $\mathbb{R}^n$ is obtained, which covers (1.7) when $l = 2$.

2 Self-Adjoint Operators on Hilbert Spaces

In this section, we establish an abstract inequality which relates lower order eigenvalues of self-adjoint operators to two collections of auxiliary operators and their commutators.

Theorem 2.1 *Let $\mathcal{H}$ be a complex Hilbert space with a given inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Let $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathcal{D}$ which is semibounded beblow and has a discrete spectrum $\mu_1 \le \mu_2 \le \cdots$. Let $\{T_i : \mathcal{D} \to \mathcal{H}\}_{i=1}^n$ be a collection of skew-symmetric operators and $\{B_i : A(\mathcal{D}) \to \mathcal{H}\}_{i=1}^n$ be a collection of symmetric operators which leave $\mathcal{D}$ invariant. Denote by $\{u_i\}_{i=1}^\infty$ the normalized eigenvectors corresponding to the i-th eigenvalues μ_i of A. This family of eigenvectors are further assumed to be an orthonormal basis for $\mathcal{H}$. If the operators $\{B_i\}_{i=1}^n$ satisfy*

$$\langle B_i u_1, u_{j+1} \rangle, \quad \text{for } 1 \le j < i \le n, \tag{2.1}$$

we have

$$\sum_{i=1}^n (\mu_{i+1} - \mu_1)^{\frac{1}{2}} \langle [T_i, B_i] u_1, u_1 \rangle \le 2 \Bigg\{ \sum_{i=1}^n \langle [A, B_i] u_1, B_i u_1 \rangle \sum_{i=1}^n \| T_i u_1 \|^2 \Bigg\}^{\frac{1}{2}}, \tag{2.2}$$

where $[A, B] = AB - BA$ is called the commutator of operators A and B.

Proof We consider the vectors ϕ^i given by

$$\phi^i = B_i u_1 - a^i u_1, \tag{2.3}$$

where $a^i = \langle B_i u_1, u_1\rangle$. Then, according to (2.1), it is easy to check that

$$\langle \phi^i, u_{j+1}\rangle = 0, \quad \text{for } 0 \le j < i \le n. \tag{2.4}$$

Moreover, (2.4) yields

$$\langle \phi^i, B_i u_1\rangle = \|\phi^i\|^2. \tag{2.5}$$

Since (2.4) holds, we can take ϕ^i as a trial vector in the Rayleigh–Ritz ratio and get

$$\mu_{i+1} \le \frac{\langle A\phi^i, \phi^i\rangle}{\langle \phi^i, \phi^i\rangle}. \tag{2.6}$$

It follows from (2.4) and (2.5) that

$$\langle A\phi^i, \phi^i\rangle = \langle [A, B_i]u_1, \phi^i\rangle + \langle B_i A u_1, \phi^i\rangle = \langle [A, B_i]u_1, \phi^i\rangle + \mu_1\|\phi^i\|^2. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$(\mu_{i+1} - \mu_1)\|\phi^i\|^2 \le \langle [A, B_i]u_1, \phi^i\rangle. \tag{2.8}$$

As we know, $\langle\cdot,\cdot\rangle$ is taken to be linear in its first argument and conjugate linear in its second argument. Since

$$\langle [A, B_i]u_1, u_1\rangle = \langle B_i u_1, A u_1\rangle - \langle B_i A u_1, u_1\rangle = 0,$$

we have

$$\langle [A, B_i]u_1, \phi^i\rangle = \langle [A, B_i]u_1, B_i u_1\rangle. \tag{2.9}$$

Hence, substituting (2.9) into (2.8), we get

$$(\mu_{i+1} - \mu_1)\|\phi^i\|^2 \le \langle [A, B_i]u_1, B_i u_1\rangle. \tag{2.10}$$

Since

$$\overline{a^i}\langle T_i u_1, u_1\rangle = -\langle B_i u_1, u_1\rangle\overline{\langle T_i u_1, u_1\rangle} = -a^i\overline{\langle T_i u_1, u_1\rangle},$$

we obtain

$$\mathrm{Re}\ \overline{a^i}\langle T_i u_1, u_1\rangle = 0. \tag{2.11}$$

Moreover, we have

$$2\mathrm{Re}\langle T_i B_i u_1, u_1\rangle = \langle T_i B_i u_1, u_1\rangle + \langle u_1, T_i B_i u_1\rangle = \langle [T_i, B_i]u_1, u_1\rangle. \tag{2.12}$$

Hence, taking the real parts in both sides of the following equality

$$-2\langle T_i u_1, \phi^i\rangle = 2\langle u_1, T_i B_i u_1\rangle + 2\overline{a^i}\langle T_i u_1, u_1\rangle,$$

and utilizing (2.11) and (2.12), we obtain

$$\langle [T_i, B_i]u_1, u_1\rangle = -2\mathrm{Re}\langle \phi^i, T_i u_1\rangle. \tag{2.13}$$

Multiplying both sides of (2.13) by $(\mu_{i+1} - \mu_1)^{\frac{1}{2}}$ and using (2.10), we deduce

$$\begin{aligned}(\mu_{i+1} - \mu_1)^{\frac{1}{2}}\langle [T_i, B_i]u_1, u_1\rangle &= -2(\mu_{i+1} - \mu_1)^{\frac{1}{2}}\mathrm{Re}\langle \phi^i, T_i u_1\rangle \\ &\le \delta(\mu_{i+1} - \mu_1)\|\phi^i\|^2 + \frac{1}{\delta}\|T_i u_1\|^2 \\ &\le \delta\langle [A, B_i]u_1, B_i u_1\rangle + \frac{1}{\delta}\|T_i u_1\|^2, \end{aligned} \tag{2.14}$$

where δ is a positive constant. Taking sum on i from 1 to n in (2.14), we have

$$\delta^2 \sum_{i=1}^{n} \langle [A, B_i]u_1, B_i u_1\rangle - \delta \sum_{i=1}^{n} (\mu_{i+1} - \mu_1)^{\frac{1}{2}} \langle [T_i, B_i]u_1, u_1\rangle + \sum_{i=1}^{n} \|T_i u_1\|^2 \geq 0. \tag{2.15}$$

The left-hand side of (2.15) is a quadratic polynomial of δ. From (2.8) and (2.9), we know that $\langle [A, B_i]u_1, B_i u_1\rangle \geq 0$ for $i = 1, \ldots, n$. Therefore, we know that its discriminant must be nonpositive. This yields (2.2). □

3 The Biharmonic Operator on Riemannian Manifolds Admitting Some Special Functions

In this section, we obtain some universal inequalities for lower order eigenvalues of the biharmonic operator on manifolds admitting some special functions.

Theorem 3.1 *Let M be an n-dimensional complete Riemannian manifold. Denote by Γ_i the i-th eigenvalue of the clamped plate problem on a bounded domain Ω of M.*

(i) *Suppose that M admits a spherical eigenmap φ. Namely, the components $\varphi_1, \ldots, \varphi_{m+1}$ of map $\varphi : M \to \mathbb{S}^m(1)$ are all eigenfuctions corresponding to the same eigenvalue λ of the Laplacian on M, where $\mathbb{S}^m(1)$ is an m-dimensional unite sphere. Then we have*

$$\sum_{i=1}^{n} (\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} \leq n\big[\big(\lambda + 6\Gamma_1^{\frac{1}{2}}\big)\big(\lambda + 4\Gamma_1^{\frac{1}{2}}\big)\big]^{\frac{1}{2}} \tag{3.1}$$

and

$$\sum_{i=1}^{n} \big[(\Gamma_{i+1} - \Gamma_1)^{\frac{1}{2}} - 5\Gamma_1^{\frac{1}{2}}\big] \leq n\lambda. \tag{3.2}$$

(ii) *If there exist m functions $f_r : M \to \mathbb{R}$ such that* (1.11) *holds, then*

$$\sum_{r=1}^{m} (\Gamma_{r+1} - \Gamma_1)^{\frac{1}{2}} \leq 2(2m+4)^{\frac{1}{2}} \Gamma_1^{\frac{1}{2}}. \tag{3.3}$$

Proof (i) Since $\varphi_1, \ldots, \varphi_{m+1}$ are the components of an eigenmap, it holds

$$\begin{cases} -\Delta\varphi_\alpha = \lambda\varphi_\alpha, \\ \displaystyle\sum_{\alpha=1}^{m+1} |\nabla\varphi_\alpha|^2 = \lambda, \\ \displaystyle\sum_{\alpha=1}^{m+1} \varphi_\alpha^2 = 1. \end{cases} \tag{3.4}$$

In order to make use of Theorem 2.1, we construct some functions satisfying (2.1) by using $\{\varphi_\alpha\}_{\alpha=1}^{m+1}$. We consider an $(m+1)\times(m+1)$ matrix $Q = \big(\int_\Omega \varphi_\alpha u_1 u_{\beta+1}\big)_{(m+1)\times(m+1)}$. According to the QR-factorization theorem, we know that there exists an orthogonal $(m+1) \times (m+1)$ matrix $P = \big(p_{\alpha\beta}\big)_{(m+1)\times(m+1)}$ such that $U = PQ$ is an upper triangle matrix. Namely, we have

$$\sum_{\gamma=1}^{m+1} p_{\alpha\gamma} \int_\Omega \varphi_\gamma u_1 u_{\beta+1} = 0, \quad \text{for } 1 \leq \beta < \alpha \leq m+1.$$

Define functions ψ_α by

$$\psi_\alpha = \sum_{\gamma=1}^{m+1} p_{\alpha\gamma}\varphi_\gamma.$$

Thus we infer

$$\int_\Omega \psi_\alpha u_1 u_{\beta+1} = 0, \quad \text{for } 1 \le \beta < \alpha \le m+1. \tag{3.5}$$

Moreover, because P is an orthogonal matrix, it follows from (3.4) that

$$\begin{cases} -\Delta\psi_\alpha = \lambda\psi_\alpha, \\ \sum_{\alpha=1}^{m+1} |\nabla\psi_\alpha|^2 = \lambda, \\ \sum_{\alpha=1}^{m+1} \psi_\alpha^2 = 1. \end{cases} \tag{3.6}$$

According to (3.5), taking $A = \Delta^2$, $B_\alpha = \psi_\alpha$ and $T_\alpha = \big[\Delta, \psi_\alpha\big]$ in Theorem 2.1, we derive

$$\begin{aligned} &\sum_{\alpha=1}^{m+1} (\Gamma_{\alpha+1} - \Gamma_1)^{\frac{1}{2}} \langle [[\Delta, \psi_\alpha], \psi_\alpha] u_1, u_1 \rangle \\ &\quad \le 2\bigg\{ \sum_{\alpha=1}^{m+1} \langle [\Delta^2, \psi_\alpha] u_1, \psi_\alpha u_1 \rangle \sum_{\alpha=1}^{m+1} \|[\Delta, \psi_\alpha] u_1\|^2 \bigg\}^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

Now we need to calculate and estimate both sides of (3.7). A straightforward calculation gives

$$\begin{aligned} &\langle [\Delta^2, \psi_\alpha] u_1, \psi_\alpha u_1 \rangle \\ &\quad = \int_\Omega u_1 \psi_\alpha [u_1 \Delta^2 \psi_\alpha + 2\Delta\psi_\alpha \Delta u_1 + 2\langle \nabla\psi_\alpha, \nabla\Delta u_1 \rangle + 2\langle \nabla\Delta\psi_\alpha, \nabla u_1 \rangle \\ &\qquad + 2\Delta\langle \nabla\psi_\alpha, \nabla u_1 \rangle] \\ &\quad = \int_\Omega \big[u_1^2 (\Delta\psi_\alpha)^2 + 4u_1 \Delta\psi_\alpha \langle \nabla\psi_\alpha, \nabla u_1 \rangle + 4\langle \nabla\psi_\alpha, \nabla u_1 \rangle^2 - 2|\nabla\psi_\alpha|^2 u_1 \Delta u_1\big]. \end{aligned} \tag{3.8}$$

Moreover, using the Cauchy–Schwarz inequality, one gets

$$\int_\Omega |\nabla u_1|^2 = -\int_\Omega u_1 \Delta u_1 \le \bigg(\int_\Omega u_1^2\bigg)^{\frac{1}{2}} \bigg[\int_\Omega (-\Delta u_1)^2\bigg]^{\frac{1}{2}} = \Gamma_1^{\frac{1}{2}}. \tag{3.9}$$

Then it follows from (3.6), (3.8) and (3.9) that

$$\begin{aligned} \sum_{\alpha=1}^{m+1} \langle [\Delta^2, \psi_\alpha] u_1, \psi_\alpha u_1 \rangle &= \lambda^2 \int_\Omega u_1^2 + 4\int_\Omega \sum_{\alpha=1}^{m+1} \langle \nabla\psi_\alpha, \nabla u_1 \rangle^2 - 2\lambda \int_\Omega u_1 \Delta u_1 \\ &\le \lambda^2 + 4\int_\Omega \sum_{\alpha=1}^{m+1} |\nabla\psi_\alpha|^2 |\nabla u_1|^2 - 2\lambda \int_\Omega u_1 \Delta u_1 \\ &\le \lambda^2 + 6\lambda\Gamma_1^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

At the same time, it follows from (3.6) and (3.9) that

$$\begin{aligned} \sum_{\alpha=1}^{m+1} \|\big[\Delta, \psi_\alpha\big] u_1\|^2 &= \int_\Omega \sum_{\alpha=1}^{m+1} \big[u_1^2 (\Delta\psi_\alpha)^2 + 4\langle \nabla\psi_\alpha, \nabla u_1 \rangle^2 + 4u_1 \Delta\psi_\alpha \langle \nabla\psi_\alpha, \nabla u_1 \rangle\big] \\ &\le \lambda^2 + 4\lambda\Gamma_1^{\frac{1}{2}}. \end{aligned} \tag{3.11}$$

On the other hand, according to the definition of the commutator, it holds

$$\begin{aligned}
&\langle[[\Delta,\psi_\alpha],\psi_\alpha]u_1,u_1\rangle\\
&\quad=\int_\Omega u_1\big[\Delta(\psi_\alpha^2u_1)-2\psi_\alpha\Delta(\psi_\alpha u_1)+\psi_\alpha^2\Delta u_1\big]\\
&\quad=\int_\Omega\big\{2u_1\Delta(\psi_\alpha^2u_1)-2u_1\psi_\alpha\big[\psi_\alpha\Delta u_1+2\langle\nabla\psi_\alpha,\nabla u_1\rangle+u_1\Delta\psi_\alpha\big]\big\}\\
&\quad=-2\int_\Omega\big[u_1\psi_\alpha\langle\nabla\psi_\alpha,\nabla u_1\rangle+u_1^2\psi_\alpha\Delta\psi_\alpha\big]\\
&\quad=2\int_\Omega u_1^2|\nabla\psi_\alpha|^2.
\end{aligned}\tag{3.12}$$

Moreover, since

$$\sum_{j=1}^n|\nabla\psi_j|^2\le\sum_{\alpha=1}^{m+1}|\nabla\psi_\alpha|^2=\lambda,$$

we observe that

$$|\nabla\psi_j|^2\le\frac{\lambda}{n},\quad\text{for } j=1,\ldots,n.\tag{3.13}$$

Therefore, making use of (3.13), we can deduce

$$\begin{aligned}
&\sum_{\alpha=1}^{m+1}(\Gamma_{\alpha+1}-\Gamma_1)^{\frac12}|\nabla\psi_\alpha|^2\\
&\quad\ge\sum_{i=1}^n(\Gamma_{i+1}-\Gamma_1)^{\frac12}|\nabla\psi_i|^2+(\Gamma_{n+1}-\Gamma_1)^{\frac12}\sum_{k=n+1}^{m+1}|\nabla\psi_k|^2\\
&\quad=\sum_{i=1}^n(\Gamma_{i+1}-\Gamma_1)^{\frac12}|\nabla\psi_i|^2+(\Gamma_{n+1}-\Gamma_1)^{\frac12}\sum_{j=1}^n\left(\frac{\lambda}{n}-|\nabla\psi_j|^2\right)\\
&\quad\ge\sum_{i=1}^n(\Gamma_{i+1}-\Gamma_1)^{\frac12}|\nabla\psi_i|^2+\sum_{j=1}^n(\Gamma_{j+1}-\Gamma_1)^{\frac12}\left(\frac{\lambda}{n}-|\nabla\psi_j|^2\right)\\
&\quad=\frac{\lambda}{n}\sum_{i=1}^n(\Gamma_{i+1}-\Gamma_1)^{\frac12}.
\end{aligned}\tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\begin{aligned}
\sum_{\alpha=1}^{m+1}(\Gamma_{\alpha+1}-\Gamma_1)^{\frac12}\langle[[\Delta,\psi_\alpha],\psi_\alpha]u_1,u_1\rangle&=2\sum_{\alpha=1}^{m+1}(\Gamma_{\alpha+1}-\Gamma_1)^{\frac12}\int_\Omega u_1^2|\nabla\psi_\alpha|^2\\
&\ge\frac{2\lambda}{n}\sum_{i=1}^n(\Gamma_{i+1}-\Gamma_1)^{\frac12}.
\end{aligned}\tag{3.15}$$

Substituting (3.10), (3.11) and (3.15) into (3.7), we can infer (3.1).

From $\big[(\lambda+6\Gamma_1^{\frac12})(\lambda+4\Gamma_1^{\frac12})\big]^{\frac12}\le\lambda+5\Gamma_1^{\frac12}$, we know that (3.2) is true.

(ii) Similar to the proof of (i), using the QR-factorization theorem, we can construct functions $\{h_r\}_{r=1}^m$ by using $\{f_r\}_{r=1}^m$, such that h_r satisfy

$$\begin{cases}\displaystyle\int_\Omega h_ru_1u_{t+1}=0,\quad 1\le t<r\le m,\\ \langle\nabla h_r,\nabla h_s\rangle=\delta_{rs},\\ \Delta h_r=0.\end{cases}\tag{3.16}$$

Therefore, taking $A=\Delta^2$, $B_r=h_r$ and $T_r=\big[\Delta,h_r\big]$ in Theorem 2.1, we get

$$\sum_{r=1}^{m}(\Gamma_{r+1}-\Gamma_1)^{\frac{1}{2}}\langle[[\Delta,h_r],h_r]u_1,u_1\rangle \\ \le 2\bigg\{\sum_{r=1}^{m}\langle[\Delta^2,h_r]u_1,h_ru_1\rangle\sum_{r=1}^{m}\|[\Delta,h_r]u_1\|^2\bigg\}^{\frac{1}{2}}. \tag{3.17}$$

According to (3.16), $\{\nabla h_r\}_{r=1}^m$ is a set of orthonormal vector fields. Hence, we have

$$\sum_{r=1}^{m}\langle\nabla h_r,\nabla u_1\rangle^2\le|\nabla u_1|^2. \tag{3.18}$$

Using (3.9), (3.16) and (3.18), we obtain

$$\begin{aligned}&\sum_{r=1}^{m}\langle[\Delta^2,h_r]u_1,h_ru_1\rangle\\&=\int_\Omega\sum_{r=1}^{m}\big[u_1^2(\Delta h_r)^2+4u_1\Delta h_r\langle\nabla h_r,\nabla u_1\rangle+4\langle\nabla h_r,\nabla u_1\rangle^2-2|\nabla h_r|^2u_1\Delta u_1\big]\\&=\int_\Omega\sum_{r=1}^{m}\big[4\langle\nabla h_r,\nabla u_1\rangle^2-2|\nabla h_r|^2u_1\Delta u_1\big]\\&\le(2m+4)\int_\Omega|\nabla u_1|^2\\&\le(2m+4)\Gamma_1^{\frac{1}{2}}\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\sum_{r=1}^{m}\|\big[\Delta,h_r\big]u_1\|^2&=\int_\Omega\sum_{r=1}^{m}\big[u_1^2(\Delta h_r)^2+4\langle\nabla h_r,\nabla u_1\rangle^2+4u_1\Delta h_r\langle\nabla h_r,\nabla u_1\rangle\big]\\&\le 4\Gamma_1^{\frac{1}{2}}.\end{aligned} \tag{3.20}$$

Moreover, according to (3.12) and (3.16), the term on the left-hand side of (3.17) is

$$\langle[[\Delta,h_r],h_r]u_1,u_1\rangle=2\int_\Omega u_1^2|\nabla h_r|^2=2. \tag{3.21}$$

Substituting (3.19)–(3.21) into (3.17), we derive (3.3). □

4 The Poly-Laplacian with Any Order on the Euclidean Space

In this section, we obtain a universal inequality for problem (1.1). Moreover, it covers (1.7) of [8] when $l=2$.

Theorem 4.1 *Let Ω be a bounded domain in $\mathbb{R}^n$. Denote by λ_i the i-th eigenvalue of problem* (1.1)*. Then we have*

$$\sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\le\big[4l(n+2l-2)\big]^{\frac{1}{2}}\lambda_1^{\frac{1}{2}}. \tag{4.1}$$

Proof Similar to the proof of Theorem 3.1, by utilizing the QR-factorization theorem, one can prove that there exists a set of Cartesian coordinate system $(x_1,\dots,x_n)$ of $\mathbb{R}^n$ such that the following orthogonality conditions are satisfied:

$$\int_\Omega x_i u_1 u_{j+1} = 0 \quad \text{for } 1 \le j < i \le n. \tag{4.2}$$

Namely, $B_i = x_i$ $(i = 1, \dots, n)$ satisfy (2.1). Therefore, taking $A = (-\Delta)^l$, $B_i = x_i$ and $T_i = \frac{\partial}{\partial x_i}$ in Theorem 2.1, we have

$$\begin{aligned}\sum_{i=1}^n (\mu_{i+1} - \mu_1)^{\frac{1}{2}} \left\langle \left[\frac{\partial}{\partial x_i}, x_i\right] u_1, u_1 \right\rangle \\ \le 2\left\{ \sum_{i=1}^n \langle [(-\Delta)^l, x_i] u_1, x_i u_1 \rangle \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} u_1 \right\|^2 \right\}^{\frac{1}{2}}.\end{aligned} \tag{4.3}$$

Utilizing

$$\int_\Omega |\nabla^k u_1|^2 \le \lambda_1^{\frac{k}{l}} \tag{4.4}$$

(cf. [7]) and

$$(-\Delta)^l (x_i u_1) = x_i (-\Delta)^l u_1 - 2l \nabla (-\Delta)^l u_1 \cdot \nabla x_i,$$

we have

$$\begin{aligned}&\sum_{i=1}^n \langle [(-\Delta)^l, x_i] u_1, x_i u_1 \rangle \\ &= -2l \sum_{i=1}^n \int_\Omega x_i u_1 \nabla (-\Delta)^l u_1 \cdot \nabla x_i \\ &= l \sum_{i=1}^n \int_\Omega u_1 (-\Delta)^{l-1} u_1 - 2l(l-1) \sum_{i=1}^n \int_\Omega (-\Delta)^{l-2} u_1 \nabla x_i \cdot \nabla(\nabla u_1 \cdot \nabla x_i) \\ &= l \sum_{i=1}^n \int_\Omega |\nabla^{l-1} u_1|^2 - 2l(l-1) \int_\Omega (-\Delta)^{l-2} u_1 \Delta u_1 \\ &= l(n + 2l - 2) \int_\Omega |\nabla^{l-1} u_1|^2 \\ &\le l(n + 2l - 2) \lambda_1^{\frac{l-1}{l}}.\end{aligned} \tag{4.5}$$

At the same time, it is not difficult to get

$$\left\langle \left[\frac{\partial}{\partial x_i}, x_i\right] u_1, u_1 \right\rangle = 1 \tag{4.6}$$

and

$$\sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} u_1 \right\|^2 = \int_\Omega |\nabla u_1|^2 \le \lambda_1^{\frac{1}{l}}. \tag{4.7}$$

Substituting (4.5)–(4.7) into (4.3), we obtain (4.1). □

References

[1] Ashbaugh, M. S.: Isoperimetric and universal inequalities for eigenvalues. In: Spectral theory and geometry (E. B. Davies and Yu Safarov eds.), London Math. Soc. Lecture Notes, Vol. 273, Cambridge Univ. Press, Cambridge, 1999, 95–139

[2] Ashbaugh, M. S., Benguria, R. D.: More bounds on eigenvalue ratios for Dirichlet Laplacians in n dimension. *SIAM J. Math. Anal.*, **24**, 1622–1651 (1993)

[3] Ashbaugh, M. S., Benguria, R. D.: A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions. *Ann. of Math.*, **135**(3), 601–628 (1992)

[4] Ashbaugh, M. S., Benguria, R. D.: A second proof of the Payne–Pólya–Weinberger conjecture. *Comm. Math. Phys.*, **147**, 181–190 (1992)

[5] Ashbaugh, M. S., Hermi, L.: A unified approach to universal inequalities for eigenvalues of elliptic operators. *Pacific J. Math.*, **217**, 201–219 (2004)

[6] Chen, D. G., Cheng, Q.-M.: Extrinsic estimates for eigenvalues of the Laplace operator. *J. Math. Soc. Japan*, **60**, 325–339 (2008)

[7] Chen, Z. C., Qian, C. L.: Estimates for discrete spectrum of Laplacian operator with any order. *J. Univ. Sci. Technol. China*, **20**, 259–266 (1990)

[8] Cheng, Q.-M., Huang, G. Y., Wei, G. X.: Estimates for lower order eigenvalues of a clamped plate problem. *Calc. Var. Partial Differential Equations*, **38**, 409–416 (2010)

[9] Cheng, Q.-M., Ichikawa, T., Mametsuka, S.: Inequalities for eigenvalues of Laplacian with any order. *Commun. Contemp. Math.*, **11**, 639–655 (2009)

[10] Chiti, G.: A bound for the ratio of the first two eigenvalues of a membrane. *SIAM J. Math. Anal.*, **14**, 1163–1167 (1983)

[11] do Carmo, M. P., Wang, Q. L., Xia, C. Y.: Inequalities for eigenvalues of elliptic operators in divergence form on Riemannian manifolds. *Ann. Mat. Pura Appl.*, **189**, 643–660 (2010)

[12] Harrell II, E. M.: General bounds for the eigenvalues of Schrödinger operators. In: Maximum Principles and Eigenvalue Problems in Partial Differential Equations (P. W. Schaefer eds.), Pitman Research Notes in Mathematics Series, Vol. 175, Longman Scientific and Technical, Harlow, Essex, United Kingdom, 1988, 146–166

[13] Harrell II, E. M., Michel, P. L.: Commutator bounds for eigenvalues, with applications to spectral geometry. *Comm. Partial Differential Equations*, **19**, 2037–2055 (1994)

[14] Harrell II, E. M., Stubbe, J.: On trace identities and universal eigenvalue estimates for some partial differential operators. *Trans. Amer. Math. Soc.*, **349**, 1797–1809 (1997)

[15] Hile, G. N., Protter, M. H.: Inequalities for eigenvalues of the Laplacian. *Indiana Univ. Math. J.*, **29**, 523–538 (1980)

[16] Hook, S. M.: Domain independent upper bounds for eigenvalues of elliptic operators. *Trans. Amer. Math. Soc.*, **318**, 615–642 (1990)

[17] Ilias, S., Makhoul, O.: Universal inequalities for the eigenvalues of a power of the Laplacian operator. *Manuscripta Math.*, **132**, 75–102 (2010)

[18] Jost, J., Li-Jost, X. Q., Wang, Q. L., et al.: Universal bounds for eigenvalues of the polyharmonic operator. *Trans. Amer. Math. Soc.*, **363**, 1821–1854 (2011)

[19] Levitin, M., Parnovski, L.: Commutators, spectral trace identities, and universal estimates for eigenvalues. *J. Funct. Anal.*, **192**, 425–445 (2002)

[20] Li, P.: Eigenvalue estimates on homogeneous manifolds. *Comment. Math. Helv.*, **55**, 347–363 (1980)

[21] Marcellini, P.: Bounds for the third membrane eigenvalue. *J. Differential Equations*, **37**, 438–443 (1980)

[22] Payne, L. E., Pólya, G., Weinberger, H. F.: On the ratio of consecutive eigenvalues. *J. Math. Phys.*, **35**, 289–298 (1956)

[23] Soufi, A. E., Harrell II, E. M., Ilias, S.: Universal inequalities for the eigenvalues of Laplace and Schrödinger operators on submanifolds. *Trans. Amer. Math. Soc.*, **361**, 2337–2350 (2009)

[24] Sun, H. J.: Yang-type inequalities for weighted eigenvalues of a second order uniformly elliptic operator with a nonnegative potential. *Proc. Amer. Math. Soc.*, **38**, 2827–2838 (2010)

[25] Sun, H. J., Cheng, Q.-M., Yang, H. C.: Lower order eigenvalues of Dirichlet Laplacian. *Manuscripta Math.*, **125**, 139–156 (2008)

[26] Wang, Q. L., Xia, C. Y.: Inequalities for eigenvalues of a clamped plate problem. *Calc. Var. Partial Differential Equations*, **40**, 273–289 (2011)