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Homoclinic Solutions in Periodic Nonlinear Difference Equations with Superlinear Nonlinearity

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Abstract In this paper, we consider the existence of homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity. The classical Ambrosetti–Rabinowitz superlinear condition is improved by a general superlinear one. The proof is based on the critical point theory in combination with periodic approximations of solutions.

Keywords Homoclinic solution, periodic nonlinear difference equation, superlinear nonlinearity, critical point theory, periodic approximation

MR(2010) Subject Classification 39A11

1 Introduction

Wave propagation in nonlinear periodic lattices is associated with a host of exciting phenomena that have no counterpart whatsoever in bulk media. Perhaps, the most intriguing entities that can exist in such systems are discrete self-localized state — better known as discrete solitions. By their very nature, these intrinsically localized models represent collective excitations of the chain as a whole, and are the outcome of the balance between nonlinear and linear coupling effects [8]. One of the simplest lattice models, which deserves special attention, is represented by Discrete Nonlinear Schrödinger (DNLS for short) equations [9]. In the past decade, the existence of discrete solitons of the DNLS equations has drawn a great deal of interest. To mention a few, see [1–3, 7, 14, 16]. Among the methods used are the principle of anticontinuity [2, 16], variational methods [1, 3], centre manifold reduction [14], and Nehari manifold approach [18]. However, most of the existing literature is devoted to the DNLS equations with constant coefficients. Results on such DNLS equations have been summarized in the reviews [4, 9, 10]. And the experimental observations of two-dimensional discrete solitons have been reported in [6, 11, 12, 15].

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Recently, the DNLS equations with periodic coefficients have been considered in the physics literature, for example, [22]. Moreover, Gorbach and Johansson [13] reported results on numerical simulation of discrete gap solitons (a special discrete soliton which is defined later) in a particular periodic DNLS equation. Nonlinearity has a crucial impact on the properties of DNLS equations, and the most popular nonlinearity is the so-called Kerr, or cubic, nonlinearity. In recent years, one can see a growing interest to the wide class of superlinear nonlinearities [7, 9].

Assume that T is a positive integer. In this paper, we will consider the following periodic nonlinear difference equation

$$Lu_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $f_n(u)$ depends T-periodically on n, i.e., $f_{n+T}(u) = f_n(u)$ for $n \in \mathbb{Z}$, and is continuous in u with superlinear nonlinearity, L is a Jacobi operator (see [23]) given by

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n,$$

where $\{a_n\}$, $\{b_n\}$ are real valued T-periodic sequences.

Since $f_n(0) = 0$, $u_n \equiv 0$ is a solution of (1.1), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1.1) is homoclinic (to 0) if

$$\lim_{|n|\to\infty} u_n = 0. \tag{1.2}$$

In addition, if $u_n \not\equiv 0$, then u is called a nontrivial homoclinic solution. We are interested in the existence of the nontrivial homoclinic solutions for (1.1). This problem appears when we look for the discrete solitons of the periodic DNLS equation

$$\mathrm{i}\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n \psi_n - f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is the discrete Laplacian in one spatial dimension, the given sequence $\{\varepsilon_n\}$ of real numbers and the sequence $\{f_n(\cdot)\}$ of functions from $\mathbb{C}$ into $\mathbb{C}$, are assumed to be T-periodic in n, i.e., $\varepsilon_{n+T} = \varepsilon_n$ and $f_{n+T} = f_n$. Moreover, the nonlinearity f_n is supposed to be gauge invariant, i.e.,

$$f_n(\mathrm{e}^{\mathrm{i}\theta} u) = \mathrm{e}^{\mathrm{i}\theta} f_n(u), \quad \theta \in \mathbb{R},$$

and in addition, $f_n(u) \geq 0$ for $u \geq 0$. Since solitons are spatially localized time-periodic solutions and decay to zero at infinity. Thus ψ_n has the form

$$\psi_n = u_n \mathrm{e}^{-\mathrm{i}\omega t},$$

and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.3) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z} \tag{1.4}$$

and (1.2) holds. Therefore, the problem on the existence of solitons of the DNLS equation (1.3) has been reduced to that on the existence of homoclinic solutions of (1.4), which is a special case of (1.1) with $a_n \equiv -1$ and $b_n = 2 + \varepsilon_n$.

Since the operator L is a bounded and self-adjoint operator in the space l^2 of two-sided infinite sequences, we consider (1.1) as a nonlinear equation in l^2 with (1.2) being satisfied automatically. The spectrum $\sigma(L)$ of L has a band structure, i.e., $\sigma(L)$ is a union of a finite number of closed intervals [23]. Thus the complement $\mathbb{R}\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite. The solitons of (1.3) with the temporal frequency ω belonging to a spectral gap, in particular to a finite gap, are of considerable importance. Such solitons are called gap solitons.

We fix a spectral gap and denote it by (α, β). When $f_n(u) = \chi_n u^3$ where $\{\chi_n\}$ is a positive T-periodic sequence, (1.1) was considered in [18, 19]. It was shown that, if $\omega \in (\alpha, \beta)$ and $\beta \neq +\infty$, then (1.1) has a nontrivial solution in l^2. As an extension of the cubic form nonlinearity, Pankov also considered (1.1) in [19] when the nonlinearity f_n satisfies the following superlinear conditions.

(i) There exist $p > 2$ and $c > 0$ such that

$$0 \leq f_n(u) \leq c|u|^{p-1}$$

near $u = 0$.

(ii) There exists $\nu > 2$ such that

$$0 < \nu F_n(u) \leq f_n(u)u, \quad u \neq 0,$$

where $F_n(u)$ is the primitive function of $f_n(u)$ given by

$$F_n(u) = \int_0^u f_n(s)ds.$$

As we know, the condition (ii) is often called as Ambrosetti–Rabinowitz superlinear condition [25], which played an important role in the existence of homoclinic solutions of (1.1). The aim of this paper is to establish the existence of nontrivial solutions of (1.1) in l^2 under the assumptions that f_n satisfies more general superlinear conditions than (i) and (ii). In fact, we have the following theorem.

Theorem 1.1 *Assume that $\omega \in (\alpha, \beta)$, $f_n(u)$ is continuous in u, $f_{n+T}(u) = f_n(u)$ for any $n \in \mathbb{Z}$ and $u \in \mathbb{R}$, $f_n(u) = o(u)$ as $u \to 0$. And for each $n \in \mathbb{Z}$, the following conditions hold.*

(H1) *$f_n(u)u > 0$ for $u \neq 0$ and $\lim_{|u|\to\infty} f_n(u)/u = +\infty$.*

(H2) *$f_n(u)u - 2F_n(u) > 0$ for $u \neq 0$, $f_n(u)u - 2F_n(u) \to \infty$ as $|u| \to +\infty$, and*

$$\limsup_{u\to 0} \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} = p_n < \infty.$$

If $\beta \neq +\infty$, then equation (1.1) has at least a nontrivial solution u in l^2. Moreover, the solution decays exponentially at infinity. That is, there exist two positive constants C and τ such that

$$|u_n| \leq C\mathrm{e}^{-\tau|n|}, \quad n \in \mathbb{Z}. \tag{1.5}$$

Theorem 1.1 gives a sufficient condition on the existence of nontrivial solutions of (1.1) in l^2. We notice that, with part of these conditions being violated, (1.1) has no nontrivial solutions in l^2. In fact, we have the following proposition.

Proposition 1.2 *Assume that $\omega \in (\alpha, \beta)$, f_n satisfies the conditions in Theorem 1.1. If $\beta = +\infty$, then equation (1.1) has no nontrivial solutions in l^2.*

Remark 1.3 It is easy to see that the condition (i) implies $f_n(u) = o(u)$ as $u \to 0$, and Ambrosetti–Rabinowitz superlinear condition (ii) implies $F(u) \geq c|u|^\nu$ for $|u| \geq 1$ where c is a positive constant, so $\lim_{|u|\to+\infty} f_n(u)/u = +\infty$. Moreover,

$$0 \leq \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} \leq \frac{f_n^2(u)}{f_n(u)u - \frac{2}{\nu} f_n(u)u} = \frac{\nu}{\nu-2} \frac{f_n(u)}{u} \to 0 \quad \text{as } u \to 0.$$

Thus conditions (i) and (ii) imply that (H1), (H2) hold with $p_n = 0$, and Theorem 1.1 improved Theorems 5.1 and 7.1 in [19].

Example 1.4 Let $\{c_n\}$ be a positive T-periodic sequence, and

$$f_n(u) = \begin{cases} 0, & u = 0, \\ \dfrac{c_n u}{1 - \ln|u|}, & 0 < |u| \leq 1, \\ c_n u(1 + \ln|u|), & |u| > 1. \end{cases}$$

Then f_n satisfies all conditions of Theorem 1.1 with

$$p_n = \lim_{u\to 0} \frac{f_n^2(u)}{f_n(u)u - 2F_n(u)} = \lim_{u\to 0} \frac{2f_n(u)f_n'(u)}{f_n'(u)u - f_n(u)} = \lim_{u\to 0} \frac{2c_n(2 - \ln|u|)}{1 - \ln|u|} = 2c_n.$$

Clearly, f_n does not satisfy (i) and (ii).

For the case when f_n is saturable, i.e., asymptotically linear, we refer to [20, 27, 28] for the existence of solutions of (1.1) in l^2.

The main idea in this paper is an application of linking theorem combined with an approximation technique. This idea has been employed in [19]. We mention that critical point theory is a powerful tool to deal with the periodic solutions and the boundary value problems of differential equations [17, 21, 24] and is used to study periodic solutions and boundary value problems of discrete systems in recent years [26, 29, 30].

The remaining of this paper is organized as follows. First, in Section 2, we establish the variational framework associated with (1.1) and transfer the problem of the existence of homoclinic solutions of (1.1) into that of the existence of critical points of the corresponding functional. We also recall some basic results from critical point theory. Then, in Section 3, by establishing some technical lemmas, we give the proofs of the main results.

2 Preliminaries

In this section, we first establish the variational framework associated with (1.1).

On the Hilbert space $E = l^2$, we consider the functional

$$J(u) = \frac{1}{2}(Lu - \omega u, u) - \sum_{n=-\infty}^{+\infty} F_n(u),$$

where $(\cdot, \cdot)$ is the inner product in l^2 defined by

$$(u, v) = \sum_{n=-\infty}^{+\infty} u_n v_n, \quad u, v \in E,$$

and the norm $\|\cdot\|$ is given by

$$\|u\| = \left(\sum_{n=-\infty}^{+\infty} u_n^2\right)^{\frac{1}{2}}, \quad u \in E.$$

Then $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), v\rangle = (Lu - \omega u, v) - \sum_{n=-\infty}^{+\infty} f_n(u_n)v_n, \quad u, v \in E. \tag{2.1}$$

Equation (2.1) implies that (1.1) is the corresponding Euler–Lagrange equation for J. Therefore, we have reduced the problem of finding a nontrivial solution of (1.1) in l^2 to that of seeking a nonzero critical point of the functional J on E.

Let S be the set of all two-sided sequences, that is,

$$S = \{u = \{u_n\} \,|\, u_n \in \mathbb{R},\ n \in \mathbb{Z}\}.$$

Then S is a vector space with $au + bv = \{au_n + bv_n\}$ for $u,\ v \in S,\ a,\ b \in \mathbb{R}$.

For any fixed positive integer k, we define the subspace E_k of S as

$$E_k = \{u = \{u_n\} \in S \,|\, u_{n+2kT} = u_n, n \in \mathbb{Z}\}.$$

Obviously, E_k is isomorphic to $\mathbb{R}^{2kT}$ and hence E_k can be equipped with the inner product $(\cdot,\cdot)_k$ and norm $\|\cdot\|_k$ as

$$(u, v)_k = \sum_{n=-kT}^{kT-1} u_n v_n, \quad u, v \in E_k$$

and

$$\|u\|_k = \left(\sum_{n=-kT}^{kT-1} u_n^2\right)^{\frac{1}{2}}, \quad u \in E_k,$$

respectively. We also define a norm $\|\cdot\|_{k\infty}$ in E_k by

$$\|u\|_{k\infty} = \max\{|u_n| : -kT \le n \le kT - 1\}, \quad u \in E_k.$$

Consider the functional J_k on E_k defined by

$$J_k(u) = \frac{1}{2}(Lu - \omega u, u)_k - \sum_{n=-kT}^{kT-1} F_n(u_n). \tag{2.2}$$

Then

$$\langle J_k'(u), v\rangle = (Lu - \omega u, v)_k - \sum_{n=-kT}^{kT-1} f_n(u_n)v_n, \quad u, v \in E_k. \tag{2.3}$$

Since the coefficients of the operator L are T-periodic, it is easy to see that the critical points of J_k in E_k are exactly $2kT$-periodic solutions of equation (1.1).

Let L_k be the operator L acting in E_k. It follows from the spectral theory of Jacobi operators [23] that $\sigma(L_k) \subset \sigma(L)$ and hence $\|L_k\| \le \|L\|$.

Let E_k^+ and E_k^- be the positive and negative spectral subspaces of the operator $L_k - \omega$ in E_k, respectively. Similarly, we denote E^+ and E^- as the positive and negative spectral subspaces of $L - \omega$ in E, respectively. Let δ be the distance from ω to the spectrum $\sigma(L)$, that is,

$$\delta = \min\{\omega - \alpha, \beta - \omega\}.$$

Then, we have

$$\pm(Lu - \omega u, u) \ge \delta\|u\|^2, \quad u \in E^{\pm} \tag{2.4}$$

and

$$\pm(L_k u-\omega u, u)_k \geq \delta\|u\|_k^2, \quad u \in E_k^{\pm}. \tag{2.5}$$

In order to obtain the existence of critical points of J_k on E_k, for the convenience of the readers, we cite some basic notations and some known results from critical point theory.

Let H be a Hilbert space and $C^1(H,\mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H.

Definition 2.1 *Let $J \in C^1(H,\mathbb{R})$. A sequence $\{x_j\} \subset H$ is called a Palais–Smale sequence* (P.S. *sequence for short) for J if $\{J(x_j)\}$ is bounded and $J'(x_j) \to 0$ as $j \to \infty$. We say that J satisfies the Palais–Smale condition* (P.S. *condition for short) if any* P.S. *sequence for J possesses a convergent subsequence.*

Let B_r be the open ball in H with radius r and center 0 and let ∂B_r denote its boundary. The following lemma is taken from [21] and will play an important role in the proofs of our main results.

Lemma 2.2 (Linking theorem) *Let H be a real Hilbert space and $H = H_1 \oplus H_2$, where H_1 is a finite-dimensional subspace of H. Assume that $J \in C^1(H,\mathbb{R})$ satisfies the* P.S. *condition and the following two conditions.*

(J_1) There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap H_2} \geq a$.

(J_2) There exist an $e \in \partial B_1 \cap H_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$, where $Q \triangleq (\bar{B}_{R_0} \cap H_1) \oplus \{re | 0 < r < R_0\}$.

Then J possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$c = \inf_{h\in\Gamma} \max_{x\in\bar{Q}} J(h(x)),$$

where $\Gamma = \{h \in C(\bar{Q}, H) : h|_{\partial Q} = \mathrm{id}_{\partial Q}\}$ and $\mathrm{id}_{\partial Q}$ is the identity operator on ∂Q.

3 Proofs of Main Results

In this section, we will give proofs of Theorem 1.1 and Proposition 1.2. In order to complete the proof of Theorem 1.1, we need to establish some lemmas.

Lemma 3.1 *Under the assumptions of Theorem* 1.1*, the functional J_k satisfies the* P.S. *condition.*

Proof Let $\{u^{(j)}\} \subset E_k$ be a P.S. sequence for J_k. We need to show that $\{u^{(j)}\}$ has a convergent subsequence. Since E_k is finite dimensional, it suffices to show that $\{\|u^{(j)}\|_k\}$ is bounded. Choose $M > 0$ such that $|J_k(u^{(j)})| \leq M$ for $j \in \mathbb{N}$. By (H1), there exists a constant R such that

$$F_n(u) \geq \frac{1}{2}(\|L-\omega\|+1)u^2, \quad n \in \mathbb{N},\ |u| \geq R. \tag{3.1}$$

Let

$$Q_k^{(j)} = \{n \in \mathbb{Z} : |u_n^{(j)}| \geq R, -kT \leq n \leq kT-1\},$$
$$R_k^{(j)} = \{n \in \mathbb{Z} : |u_n^{(j)}| < R, -kT \leq n \leq kT-1\}.$$

Then by (2.2), we have

$$\sum_{n=-kT}^{kT-1} F_n(u_n^{(j)}) = \frac{1}{2}(Lu^{(j)}-\omega u^{(j)}, u^{(j)})_k - J_k(u^{(j)}) \leq \frac{1}{2}\|L_k-\omega\|\|u^{(j)}\|_k^2 + M. \tag{3.2}$$

Since

$$\begin{aligned}\sum_{n=-kT}^{kT-1} F_n(u_n^{(j)}) &= \sum_{n\in Q_k^{(j)}} F_n(u_n^{(j)}) + \sum_{n\in R_k^{(j)}} F_n(u_n^{(j)})\\ &\geq \frac{1}{2}(\|L-\omega\|+1)\sum_{n\in Q_k^{(j)}}(u_n^{(j)})^2 + \sum_{n\in R_k^{(j)}} F_n(u_n^{(j)})\\ &= \frac{1}{2}(\|L-\omega\|+1)\sum_{n=-kT}^{kT-1}(u_n^{(j)})^2\\ &\quad + \sum_{n\in R_k^{(j)}}\left[F_n(u_n^{(j)}) - \frac{1}{2}(\|L-\omega\|+1)(u_n^{(j)})^2\right]\\ &\geq \frac{1}{2}(\|L-\omega\|+1)\|u^{(j)}\|_k^2 + 2kTm, \end{aligned}\tag{3.3}$$

where

$$m = \min\left\{F_n(u) - \frac{1}{2}(\|L-\omega\|+1)u^2 : n\in\mathbb{Z}, |u|\leq R\right\}.$$

By the fact that $f_n(u) = o(u)$ as $u\to 0$, and the periodic behavior of $f_n(u)$ in n, we see that $m<0$. Noticing that $\|L_k-\omega\|\leq\|L-\omega\|$, (3.2) and (3.3) imply

$$\|u^{(j)}\|_k^2 \leq 2(M-2kTm).$$

This implies that $\{\|u^{(j)}\|_k\}$ is bounded and the proof is completed. □

Lemma 3.2 *Under the assumptions of Theorem* 1.1*, there exists $k_0\in\mathbb{N}$ such that J_k has at least a nonzero critical point $u^{(k)}$ in E_k for each $k\geq k_0$. Moreover, $0<J_k(u^{(k)})\leq M_0$ for some constant M_0 independent of k.*

Proof Let P^+ be the spectral projector in E corresponding to $\sigma(L)\cap[\beta,+\infty)$ and P_k^+ be the spectral projector in E_k corresponding to $\sigma(L_k)\cap[\beta,+\infty)$. Then P^+ and P_k^+ have the same form (see Bruno et al. [5] or Pankov [19])

$$P^+u_n = \sum_{n=-\infty}^{+\infty} K(m,n)u_m \quad\text{and}\quad P_k^+u_n = \sum_{n=-\infty}^{+\infty} K(m,n)u_m,\tag{3.4}$$

where $K(m,n)$ satisfies

$$|K(m,n)|\leq C\mathrm{e}^{-\mu|m-n|},\quad m,n\in\mathbb{Z}$$

for some positive constants C and μ.

Let $e=\{e_n\}\in E^+$ be an arbitrary unit vector and $\varepsilon=\min\{1/8,\delta(64\|L-\omega\|)^{-1}\}$. Then there exists a large positive integer N such that

$$8C\sqrt{2TN}\mathrm{e}^{-NT} < \varepsilon(1-\mathrm{e}^{-\mu})\quad\text{and}\quad \sum_{n=-NT}^{NT-1} e_n^2 > \left(1-\frac{\varepsilon}{2}\right)^2.\tag{3.5}$$

Let $k_0=2N$. For $k\geq k_0$, set

$$e^{(k)} = \|P_k^+S_ke\|_k^{-1}P_k^+S_ke\in E_k^+,$$

where S_k is the "periodization" operator, that is, $S_k: E\to E_k$ such that

$$S_ke_n = e_n,\quad -kT\leq n\leq kT-1.$$

In the following, we will use Lemma 2.2 to finish the proof.

Let $H = E_k$, $H_1 = E_k^-$, and $H_2 = E_k^+$. Since $\beta \neq +\infty$, $H_2 \neq \{0\}$. By Lemma 3.1, J_k satisfies the P.S. condition.

First, we show that J_k satisfies the condition (J_1) in Lemma 2.2. In fact, since $f_n(u) = o(u)$ as $u \to 0$, there exists a positive constant ρ such that

$$0 \le F_n(u) \le \frac{1}{4}\delta u^2, \quad n \in \mathbb{Z},\ |u| \le \rho. \tag{3.6}$$

Then, for $u \in H_2$ with $\|u\|_k \le \rho$,

$$\begin{aligned} J_k(u) &= \frac{1}{2}((L_k - \omega)u, u)_k - \sum_{n=-kT}^{kT-1} F_n(u_n) \\ &\ge \frac{1}{2}\delta\|u\|_k^2 - \sum_{n=-kT}^{kT-1} \frac{1}{4}\delta u_n^2 \\ &= \frac{1}{4}\delta\|u\|_k^2. \end{aligned}$$

Take $a = \frac{1}{4}\delta\rho^2$. Then $J|_{\partial B_\rho \cap H_2} \ge a$ and hence J_k satisfies the condition (J_1) of Lemma 2.2.

Next we prove that J satisfies the condition (J_2) of Lemma 2.2. It follows from (3.4) that

$$e_n = \sum_{m=-\infty}^{+\infty} K(m,n)e_m \quad \text{and} \quad P_k^+ S_k e_n = \sum_{m=-\infty}^{+\infty} K(m,n) S_k e_m.$$

Then from (3.5), for $-NT \le n \le NT - 1$, we have

$$\begin{aligned} |P_k^+ S_k e_n - e_n| &= \Bigg| \sum_{\{m \ge kT\} \cup \{m \le -kT-1\}} K(m,n)(S_k e_m - e_m) \Bigg| \\ &\le 2 \sum_{|m| \ge kT} |K(m,n)| \\ &\le 2C \sum_{|m| \ge kT} \mathrm{e}^{-\mu|m-n|} \\ &\le 4C \sum_{m=NT}^{+\infty} \mathrm{e}^{-\mu m} \\ &= \frac{4C\mathrm{e}^{-NT}}{1 - \mathrm{e}^{-\mu}} \\ &< \frac{\varepsilon}{2\sqrt{2NT}}. \end{aligned}$$

Let

$$W_N = \{n \in \mathbb{Z}, -NT \le n \le NT - 1\}, \quad W_k = \{n \in \mathbb{Z}, -kT \le n \le kT - 1\}.$$

Then

$$\begin{aligned} \Bigg(\sum_{n \in W_N} (P_k^+ S_k e_n)^2 \Bigg)^{\frac{1}{2}} &\ge \Bigg(\sum_{n \in W_N} e_n^2 \Bigg)^{\frac{1}{2}} - \Bigg(\sum_{n \in W_N} (P_k^+ S_k e_n - e_n)^2 \Bigg)^{\frac{1}{2}} \\ &\ge 1 - \frac{\varepsilon}{2} - \sqrt{2NT} \cdot \frac{\varepsilon}{2\sqrt{2NT}} \\ &= 1 - \varepsilon. \end{aligned}$$

Since $\|P_k^+ S_k e\|_k \le \|S_k e\|_k \le 1$, we see that

$$\sum_{n\in W_N} (e_n^{(k)})^2 \ge \sum_{n\in W_N} (P_k^+ S_k e_n)^2 \ge (1-\varepsilon)^2 \ge \left(\frac{7}{8}\right)^2 > \frac{3}{4}, \tag{3.7}$$

and

$$\sum_{n\in W_k\setminus W_N} (e_n^{(k)})^2 = 1 - \sum_{n\in W_N} (e_n^{(k)})^2 \le 1-(1-\varepsilon)^2 \le 2\varepsilon \le \frac{\delta}{32\|L-\omega\|}. \tag{3.8}$$

For any $z \in H_1$ and $r > 0$, let $u = z + re^{(k)}$. Noticing that $(z, e^{(k)})_k = 0$, we have

$$\begin{aligned}
\sum_{n\in W_N} u_n^2 &= \sum_{n\in W_N} (z_n^2 + 2rz_n e_n^{(k)} + r^2 (e_n^{(k)})^2) \\
&\ge \sum_{n\in W_N} 2rz_n e_n^{(k)} + r^2 \sum_{n\in W_N} (e_n^{(k)})^2 \\
&= -\sum_{n\in W_k\setminus W_N} 2rz_n e_n^{(k)} + r^2 \sum_{n\in W_N} (e_n^{(k)})^2 \\
&\ge -\sum_{n\in W_k\setminus W_N} \frac{\delta}{16\|L-\omega\|} z_n^2 - \sum_{n\in W_k\setminus W_N} \frac{16\|L-\omega\| r^2}{\delta} (e_n^{(k)})^2 + \frac{3}{4} r^2 \\
&\ge -\frac{\delta}{16\|L-\omega\|}\|z\|_k^2 - \frac{16\|L-\omega\| r^2}{\delta}\frac{\delta}{32\|L-\omega\|} + \frac{3}{4} r^2 \\
&= -\frac{\delta}{16\|L-\omega\|}\|z\|_k^2 + \frac{1}{4} r^2.
\end{aligned} \tag{3.9}$$

By (H1), there exists a positive constant R_1 such that

$$F_n(u) \ge 4\|L-\omega\| u^2, \quad n \in \mathbb{N},\ |u| \ge R_1.$$

Let

$$\begin{aligned}
Q_N &= \{n \in \mathbb{Z} : |u_n| \ge R_1, -NT \le n \le NT-1\}, \\
R_N &= \{n \in \mathbb{Z} : |u_n| < R_1, -NT \le n \le NT-1\},
\end{aligned}$$

and

$$m_1 = \min\{F_n(u) - 4\|L-\omega\| u^2 : n \in \mathbb{Z},\ |u| \le R_1\}.$$

Notice the fact that $f_n(u) = o(u)$ as $u \to 0$, and $f_n(u)$ is T-periodic in n, we see that $m_1 < 0$. By (3.9), we have

$$\begin{aligned}
\sum_{n=-NT}^{NT-1} F_n(u_n) &= \sum_{n\in Q_N} F_n(u_n) + \sum_{n\in R_N} F_n(u_n) \\
&\ge \sum_{n\in Q_N} 4\|L-\omega\| u_n^2 + \sum_{n\in R_N} F_n(u_n) \\
&= 4\|L-\omega\| \sum_{n\in W_N} u_n^2 + \sum_{n\in R_N} \left(F_n(u_n) - 4\|L-\omega\| u_n^2\right) \\
&\ge 4\|L-\omega\| \left(-\frac{\delta}{16\|L-\omega\|}\|z\|_k^2 + \frac{1}{4} r^2\right) + 2NTm_1 \\
&= -\frac{\delta}{4}\|z\|_k^2 + \|L-\omega\| r^2 + 2NTm_1.
\end{aligned}$$

This implies that

$$\begin{aligned} J_k(z+re^{(k)}) &= \frac{1}{2}((L-\omega)u,u)_k - \sum_{n\in W_k} F_n(u_n) \\ &\le \frac{1}{2}((L-\omega)z,z)_k + \frac{1}{2}r^2((L-\omega)e^{(k)},e^{(k)})_k - \sum_{n\in W_N} F_n(u_n) \\ &\le -\frac{1}{2}\delta\|z\|_k^2 + \frac{1}{2}\|L-\omega\|r^2 \\ &\quad - \left(-\frac{\delta}{4}\|z\|_k^2 + \|L-\omega\|r^2 + 2NTm_1\right) \\ &= -\frac{1}{4}\delta\|z\|_k^2 - \frac{1}{2}\|L-\omega\|r^2 - 2NTm_1. \end{aligned} \tag{3.10}$$

Since $J_k(z) \le 0$ for $z \in H_1$, it follows from (3.10) that there exists some positive constant $R_2 > \rho$ such that, for any $u \in \partial Q$, $J(u) \le 0$, where $Q = (\bar{B}_{R_2} \cap H_1) \oplus \{re^{(k)} : 0 < r < R_2\}$. So we have verified the condition (J_2) of Lemma 2.2.

Now that we have verified all assumptions of Lemma 2.2, we know that J_k possesses a critical value $\alpha_k \ge a$, where

$$\alpha_k = \inf_{h\in\Gamma}\max_{u\in\bar{Q}} J(h(u)) \quad \text{and} \quad \Gamma = \{h \in C(\bar{Q},H) : h|_{\partial\bar{Q}} = \mathrm{id}_{\partial\bar{Q}}\}.$$

A critical point $u^{(k)}$ of J_k corresponding to α_k is nonzero as $\alpha_k \ge a > 0$. It is also clear from (3.10) that

$$J_k(u^{(k)}) \le M_0 \triangleq -2NTm_1. \tag{3.11}$$

Obviously, $M_0 > 0$ is independent of k. □

Lemma 3.3 *There exist positive constants ξ and η such that*

$$\xi \le \|u^{(k)}\|_{k\infty} \le \eta \tag{3.12}$$

holds for every critical point $u^{(k)}$ obtained by Lemma 3.2, *of J_k in E_k with $k \ge k_0$, where k_0 is defined in Lemma* 3.2.

Proof Let $k \ge k_0$. Since $u^{(k)}$ is a critical point of J_k, by (2.2), (2.3) and (3.11), we have

$$J_k(u^{(k)}) = \sum_{n=-kT}^{kT-1}\left(\frac{1}{2}f_n(u_n^{(k)})u_n^{(k)} - F_n(u_n^{(k)})\right) \le M_0. \tag{3.13}$$

From (H2), there exists a positive constant η such that

$$\frac{1}{2}f_n(u)u - F_n(u) > M_0, \quad n \in \mathbb{Z},\ |u| > \eta,$$

then (3.13) implies that $|u_n^{(k)}| \le \eta$ for $n \in \mathbb{Z}$, that is

$$\|u^{(k)}\|_{k\infty} \le \eta. \tag{3.14}$$

On the other hand, let $u^{(k)} = u^{(k)+} + u^{(k)-}$, where $u^{(k)+} \in E_k^+$ and $u^{(k)-} \in E_k^-$. Then

$$\delta\|u^{(k)+}\|_k^2 \le ((L-\omega)u^{(k)}, u^{(k)+}) = \sum_{n=-kT}^{kT-1} f_n(u_n^{(k)})u_n^{(k)+} \tag{3.15}$$

and

$$\delta\|u^{(k)-}\|_k^2 \le -((L-\omega)u^{(k)}, u^{(k)-}) = -\sum_{n=-kT}^{kT-1} f_n(u_n^{(k)})u_n^{(k)-}. \tag{3.16}$$

Since $\lim_{u\to 0} f_n(u)/u = 0$, then there exists a positive constant ξ such that

$$|f_n(u)| \le \frac{\delta}{2}|u|, \quad n \in \mathbb{Z},\ |u| \le \xi. \tag{3.17}$$

We claim that

$$\|u^{(k)}\|_{k\infty} \ge \xi. \tag{3.18}$$

In fact, if (3.18) is not true, then by (3.17), $|f_n(u_n^{(k)})| \le (\delta/2)|u_n^{(k)}|$. By adding (3.15) with (3.16), we get

$$\begin{aligned}
\delta\|u^{(k)}\|_k^2 &\le \sum_{n=-kT}^{kT-1} f_n(u_n^{(k)})(u_n^{(k)+} - u_n^{(k)-}) \\
&\le \left(\sum_{n=-kT}^{kT-1} f_n^2(u_n^{(k)})\right)^{\frac{1}{2}} \left(\sum_{n=-kT}^{kT-1} (u_n^{(k)+} - u_n^{(k)-})^2\right)^{\frac{1}{2}} \\
&\le \frac{\delta}{2}\left(\sum_{n=-kT}^{kT-1} (u_n^{(k)})^2\right)^{\frac{1}{2}} (\|u^{(k)+}\|_k^2 + \|u^{(k)-}\|_k^2)^{\frac{1}{2}} \\
&= \frac{\delta}{2}\|u^{(k)}\|_k^2.
\end{aligned}$$

This is a contradiction and (3.18) holds. The proof is completed. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let k_0 be the integer obtained in Lemma 3.2. For every $k \ge k_0$, assume that $u^{(k)} = \{u_n^{(k)}\} \in E_k$ is a critical point obtained by Lemma 3.2. By Lemma 3.3, there exists an $n_k \in \mathbb{Z}$ such that

$$|u_{n_k}^{(k)}| \ge \xi. \tag{3.19}$$

Notice that

$$a_n u_{n+1}^{(k)} + a_{n-1}u_{n-1}^{(k)} + (b_n - \omega)u_n^{(k)} = f_n(u_n^{(k)}), \quad n \in \mathbb{Z}. \tag{3.20}$$

By the periodicity of the coefficients in (3.20), we see that $\{u_{n+T}^{(k)}\}$ is also a solution of (3.20). Making some shifts if necessary, without loss of generality, we can assume that $0 \le n_k \le T-1$ in (3.19). Moreover, passing to a subsequence of $\{u^{(k)}\}$ if necessary, we can also assume that $n_k = n^*$ for $k \ge k_0$ and some integer n^* such that $0 \le n^* \le T-1$. It follows from (3.12) and (3.19) that we can choose a subsequence, still denoted by $\{u^{(k)}\}$, such that

$$u_n^{(k)} \to v_n \quad \text{as } k \to \infty \text{ for } n \in \mathbb{Z}.$$

Then $v = \{v_n\}$ is a nonzero sequence as (3.19) implies $|v_{n^*}| \ge \xi$. For each $n \in \mathbb{Z}$, letting $k \to \infty$ in (3.20) gives us

$$a_n v_{n+1} + a_{n-1}v_{n-1} + (b_n - \omega)v_n = f_n(v_n),$$

that is, $v = \{v_n\}$ is a solution of (1.1). It remains to show that $v = \{v_n\} \in l^2$. By using (3.15) and (3.16), we have

$$\delta\|u^{(k)}\|_k^2 \le \left(\sum_{n=-kT}^{kT-1} f_n^2(u_n^{(k)})\right)^{\frac{1}{2}} \|u^{(k)}\|_k.$$

This implies that

$$\|u^{(k)}\|_k^2 \le \delta^{-2} \sum_{n=-kT}^{kT-1} f_n^2(u_n^{(k)}). \tag{3.21}$$

From (H2), there exists a positive constant ζ such that

$$f_n^2(u) \le \zeta(f_n(u)u - 2F_n(u)), \quad n \in \mathbb{Z},\ |u| \le \eta. \tag{3.22}$$

Therefore, from (3.21),

$$\|u^{(k)}\|_k^2 \le \delta^{-2} \sum_{n=-kT}^{kT-1} \zeta\big(f_n(u_n^{(k)})u_n^{(k)} - 2F_n(u_n^{(k)})\big) = \frac{2\zeta}{\delta^2} J_k(u^{(k)}) \le \frac{2\zeta}{\delta^2} M_0. \tag{3.23}$$

For each $s \in \mathbb{N}$, let $k > \max\{s, k_0\}$. Then it follows from (3.23) that

$$\sum_{n=-s}^{s} \big(u_n^{(k)}\big)^2 \le \|u^{(k)}\|_k^2 \le \frac{2\zeta}{\delta^2} M_0.$$

Letting $k \to \infty$ gives us $\sum_{n=-s}^{s} v_n^2 \le 2\zeta\delta^{-2} M_0$. By the arbitrariness of s, we know that $v = \{v_n\} \in l^2$.

Finally, similar to the proof of [19, Theorem 6.1], we show that v satisfies that (1.5).

In fact, for $n \in \mathbb{Z}$, let

$$w_n = \begin{cases} -\dfrac{f_n(v_n)}{v_n}, & \text{if } v_n \ne 0, \\ 0, & \text{if } v_n = 0. \end{cases}$$

Then

$$\tilde{L} v_n = \omega v_n, \tag{3.24}$$

where

$$\tilde{L} v_n = L v_n + w_n v_n.$$

Clearly, $\lim_{|n|\to\infty} w_n = 0$. Thus, the multiplication by w_n is a compact operator in l^2, which implies that

$$\sigma_{\text{ess}}(\tilde{L}) = \sigma_{\text{ess}}(L),$$

where σ_{ess} stands for the essential spectrum. (3.24) means that $v = \{v_n\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_{\text{ess}}(\tilde{L})$ of the operator $\tilde{L}$. (1.5) follows from the standard theorem on exponential decay for such eigenfunctions (see [23]).

Now the proof of Theorem 1.1 is complete. □

Proof of Proposition 1.2 By way of contradiction, we assume that (1.1) has a nontrivial solution $u = \{u_n\} \in l^2$. Then u is a nonzero critical point of J, and

$$\langle J'(u), u\rangle = ((L - \omega)u, u) - \sum_{n=-\infty}^{\infty} f_n(u_n)u_n = 0.$$

Since $\beta = +\infty$, then $E = E^-$ and $\delta = \omega - \alpha$. By (2.4) and (H1), the above equality implies that

$$\langle J'(u), u\rangle \le ((L - \omega)u, u) \le -(\omega - \alpha)\|u\|^2 < 0.$$

This is a contradiction and the proof is complete. □

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