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Exponential Polynomials as Solutions of Certain Nonlinear Difference Equations

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Abstract Recently, C.-C. Yang and I. Laine have investigated finite order entire solutions f of non-linear differential-difference equations of the form $f^n + L(z, f) = h$, where $n \geq 2$ is an integer. In particular, it is known that the equation $f(z)^2 + q(z)f(z+1) = p(z)$, where $p(z), q(z)$ are polynomials, has no transcendental entire solutions of finite order. Assuming that $Q(z)$ is also a polynomial and $c \in \mathbb{C}$, equations of the form $f(z)^n + q(z)\mathrm{e}^{Q(z)}f(z+c) = p(z)$ do posses finite order entire solutions. A classification of these solutions in terms of growth and zero distribution will be given. In particular, it is shown that any exponential polynomial solution must reduce to a rather specific form. This reasoning relies on an earlier paper due to N. Steinmetz.

Keywords Convex hull, difference equation, entire solution, exponential polynomial, Nevanlinna theory

MR(2000) Subject Classification 39B32, 30D35

1 Introduction

In [1] Yang investigates transcendental finite order entire solutions f of nonlinear differential equations of the form

$$L(f) - p(z)f^n = h(z), \tag{1.1}$$

where $L(f)$ is a linear differential polynomial in f with polynomial coefficients, $p(z)$ is a non-vanishing polynomial, $h(z)$ is entire, and $n \geq 4$ an integer. In particular, it is proved in [1] that f has to be unique, unless $L(f) \equiv 0$. Furthermore, if h has only a few distinct zeros in the sense that $\overline{N}(r, 1/h) = o(T(r, h))$, then (1.1) does not admit a transcendental entire solution f. We recall that the order $\sigma(f)$ and the hyper-order $\sigma_2(f)$ of a meromorphic f are given by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r},$$

while

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/f)}{\log r} \quad \text{and} \quad \lambda_2(f) = \limsup_{r\to\infty} \frac{\log\log N(r, 1/f)}{\log r}$$

stand for the corresponding convergence exponents of zeros of f.

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In [2] the ideas of Yang are developed further and applied, for example, to meromorphic solutions f of (1.1), where $p(z), h(z)$ are now allowed to be meromorphic, and the coefficients of $L(f)$ are meromorphic functions of growth $S(r, f)$. In particular, if f has only a few poles in the sense that $N(r, f) = S(r, f)$, then some conclusions about the uniqueness of f as well as the exact form of (1.1) are made in [2].

Yang and Laine [3] then investigated finite order entire solutions f of nonlinear differential-difference equations of the form

$$f^n + L(z, f) = h(z), \tag{1.2}$$

where $L(z, f)$ is a linear differential-difference polynomial in f with meromorphic coefficients of growth $S(r, f)$, $h(z)$ is meromorphic, and $n \geq 2$ is an integer. In particular, it is shown in [3] that the equation

$$f(z)^2 + q(z)f(z+1) = p(z), \tag{1.3}$$

where $p(z), q(z)$ are polynomials, admits no transcendental entire solutions of finite order.

The aim of this paper is to classify the finite order entire (meromorphic) solutions f of

$$f(z)^n + q(z)\mathrm{e}^{Q(z)}f(z+c) = P(z), \tag{1.4}$$

where $q(z), Q(z), P(z)$ are polynomials, $n \geq 2$ is an integer and $c \in \mathbb{C} \setminus \{0\}$. Supposing that f is a meromorphic solution of (1.4), we note that if z_0 is a pole of f of multiplicity $k \geq 1$, then, by (1.4), $z_0 + c$ is a pole of f of multiplicity $\geq nk$. Continuing inductively, we see that $z_m = z_0 + mc$ is a pole of f of multiplicity $\geq n^m k$. If $n \geq 2$, then the hyper-exponent of convergence $\lambda_2(\{z_m\})$ of the sequence $\{z_m\}$ satisfies $\lambda_2(\{z_m\}) \geq 1$. Since $\lambda_2(\{z_m\}) \leq \sigma_2(f)$ is true in general, we conclude that every meromorphic solution f of (1.4) with $\sigma_2(f) < 1$ is entire in the case $n \geq 2$.

Entire finite order solutions of (1.4) do exist, while a solution does not have to be unique. For example, $f_1(z) = \mathrm{e}^z + 1$ and $f_2(z) = \mathrm{e}^z - 1$ both solve

$$f(z)^2 - 2\mathrm{e}^z f(z - \log 2) = 1. \tag{1.5}$$

Although we are interested in finite order entire solutions of (1.4), we note that (1.4) possesses infinite order entire solutions also. For example, the function $f(z) = \mathrm{e}^{\mathrm{e}^z} + \mathrm{e}^{-\mathrm{e}^z}$ solves $f(z)^2 - f(z + \log 2 + \pi\mathrm{i}) = 2$.

We will show that every entire finite order solution f of (1.4) satisfies $\sigma(f) = \deg(Q)$. Hence it seems plausible that such an f could be an exponential polynomial of the form

$$f(z) = P_1(z)\mathrm{e}^{Q_1(z)} + \cdots + P_k(z)\mathrm{e}^{Q_k(z)}, \tag{1.6}$$

where P_j's and Q_j's are polynomials in z. It turns out that every solution f of the form (1.6) reduces to a function that belongs to one of the following two classes of transcendental entire functions:

$$\Gamma_1 = \{h = \mathrm{e}^{\alpha(z)} + d : d \in \mathbb{C} \text{ and } \alpha \text{ polynomial}, \alpha \neq \text{const.}\},$$
$$\Gamma_0 = \{h = \mathrm{e}^{\alpha(z)} : \alpha \text{ polynomial}, \alpha \neq \text{const.}\}.$$

For example, the solutions f_1, f_2 of (1.5) belong to $\Gamma_1 \setminus \Gamma_0$. There are solutions in Γ_0 also. Indeed, the function $f(z) = \mathrm{e}^{z^2}$ solves

$$f(z)^2 - \mathrm{e}^{z^2-2z-1} f(z+1) = 0. \tag{1.7}$$

We note that if $f \in \Gamma_1 \setminus \Gamma_0$, then the second main theorem of Nevanlinna yields $\sigma(f) = \lambda(f) = \deg(\alpha)$, and hence f has infinitely many zeros. We state the findings in this paper as follows.

Theorem 1.1 *Let $n \geq 2$ be an integer, let $c \in \mathbb{C} \setminus \{0\}$, and let $q(z), Q(z), P(z)$ be polynomials such that $Q(z)$ is not a constant and $q(z) \not\equiv 0$. Then we identify the finite order entire solutions f of equation* (1.4) *as follows:*

(a) *Every solution f satisfies $\sigma(f) = \deg Q$ and is of mean type.*

(b) *Every solution f satisfies $\lambda(f) = \sigma(f)$ if and only if $P(z) \not\equiv 0$.*

(c) *A solution f belongs to Γ_0 if and only if $P(z) \equiv 0$. In particular, this is the case if $n \geq 3$.*

(d) *If a solution f belongs to Γ_0 and if g is any other finite order entire solution to* (1.4), *then $f = \eta g$, where $\eta^{n-1} = 1$.*

(e) *If f is an exponential polynomial solution of the form* (1.6), *then $f \in \Gamma_1$. Moreover, if $f \in \Gamma_1 \setminus \Gamma_0$, then $\sigma(f) = 1$.*

The assumption $n \geq 2$ in (a) is necessary since $f(z) = \mathrm{e}^z$ is a solution of $f(z) + f(z+\pi\mathrm{i}) = 0$. If $n \geq 3$, then (1.4) may possess more than one entire solution with finite order. For example, the functions $f_1(z) = \mathrm{e}^z$ and $f_2(z) = -\mathrm{e}^z$ both solve $f(z)^3 - \mathrm{e}^{2z-1} f(z+1) = 0$. Solutions in $\Gamma_1 \setminus \Gamma_0$ and in Γ_0 are possible, as is shown by means of (1.5) and (1.7), respectively. As for the sharpness of (e), we note that if $f \in \Gamma_0$, then $\sigma(f) = 1$ does not hold in general by (1.7).

The remainder of this paper is organized as follows. Section 2 contains auxiliary results most of which can be found in the existing literature. Rather straightforward proofs for (a)–(d) are then given in Section 3. The proof of (e), to be carried out in Sections 4–5, is more involved forming the core of this paper. The key idea is to use the value distribution results for exponential polynomials due to Steinmetz [4].

2 Auxiliary Results

The following auxiliary results will be instrumental in proving Theorem 1.1.

Lemma 2.1 ([5, Corollary 2.5]) *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let c be a fixed non-zero complex constant. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.2 ([5, Theorem 2.1]) *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let c be a fixed non-zero complex constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.3 ([6, Lemma 2.2]) *Let $T : (0,+\infty) \to (0,+\infty)$ be a non-decreasing continuous function, $s > 0$, $0 < \alpha < 1$, and let $F \subset \mathbb{R}_+$ be the set of all r such that*

$$T(r) \leq \alpha T(r+s).$$

If the logarithmic measure of F is infinite, then

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \infty.$$

Lemma 2.4 *Let $f(z) \not\equiv 0$ and $p(z)$ be polynomials, $n \geq 1$ be an integer, and let $c \neq 0$ be a constant. If*

$$f(z)^n = p(z)f(z+c) \tag{2.1}$$

for all $z \in \mathbb{C}$, then $f(z)$ and $p(z)$ are constants.

Proof If $f(z+c)$ has a zero that is not a zero of $f(z)$, we arrive at a contradiction by (2.1). Hence every zero of $f(z+c)$ must be a zero of $f(z)$, but possibly with a different multiplicity. In other words, every distinct zero of $f(z)$ must be a zero of $f(z-c)$. Since $c \neq 0$ and since $f(z)$ has at most finitely many (distinct) zeros, we have a contradiction. Hence $f(z)$ cannot have any zeros, in which case $f(z)$ and $p(z)$ are constants. □

3 Proof of Theorem 1.1 (a)–(d)

Proof of (a) Suppose there is an entire solution f of (1.4) with $\sigma(f) < \infty$. Using Lemma 2.1 in (1.4), we conclude that

$$\begin{aligned}
nm(r,f) &= m(r, P(z) - q(z)\mathrm{e}^{Q(z)}f(z+c)) \\
&\leq m(r,P) + m(r,q) + m(r,\mathrm{e}^Q) + m(r,f(z+c)) + O(1) \\
&\leq m(r,\mathrm{e}^Q) + m\left(r, \frac{f(z+c)}{f(z)}\right) + m(r,f) + O(\log r) \\
&\leq m(r,\mathrm{e}^Q) + m(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r),
\end{aligned}$$

that is, $(n-1)m(r,f) \leq m(r,\mathrm{e}^{Q(z)}) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r)$. Since $n \geq 2$, this shows that $\sigma(f) \leq \deg(Q)$, see (4.1) below. Furthermore, equation (1.4) gives

$$\begin{aligned}
m(r,\mathrm{e}^Q) &= m\left(r, \frac{P(z) - f(z)^n}{q(z)f(z+c)}\right) \\
&\leq m\left(r, \frac{P(z)}{q(z)f(z+c)}\right) + m\left(r, \frac{f(z)^n}{q(z)f(z+c)}\right) + O(1) \\
&\leq 2m\left(r, \frac{f(z)}{f(z+c)}\right) + m\left(r, \frac{1}{f}\right) + (n-1)m(r,f) + O(\log r) \\
&\leq nT(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r),
\end{aligned}$$

which shows that $\deg(Q) \leq \sigma(f)$. Hence $\sigma(f) = \deg(Q)$. As for the type of f, it is clear that $\limsup_{r\to\infty} T(r,f) \cdot r^{-\deg(Q)} \in (0,\infty)$. □

Proof of (b) Suppose that f is an entire solution of (1.4) with $\sigma(f) < \infty$. We prove $\lambda(f) < \sigma(f) \Longleftrightarrow P(z) \equiv 0$, which is logically equivalent to (b).

Let $\lambda(f) < \sigma(f)$, and suppose on the contrary to the assertion that $P(z) \not\equiv 0$. We can factorize f as

$$f(z) = T(z)\mathrm{e}^{\alpha(z)}, \tag{3.1}$$

where $T(z)$ and $f(z)$ have the same zeros, if any, and $\alpha(z)$ is a polynomial. By (a) and the assumptions, we have

$$\lambda(f) = \sigma(T) < \sigma(f) = \deg(\alpha) = \deg(Q) = q.$$

From equation (1.4), we get

$$T(z)^n \mathrm{e}^{n\alpha(z)} + q(z)\mathrm{e}^{Q(z)+\alpha(z+c)}T(z+c) = P(z). \tag{3.2}$$

Note that

$$\deg(Q(z) + \alpha(z+c)) = \deg(n\alpha(z)) = q$$

by (3.2). Let $\lambda(f) < \beta < \sigma(f)$. Then

$$N\left(r, \frac{1}{P(z) - q(z)\mathrm{e}^{Q(z)+\alpha(z+c)}T(z+c)}\right) = N\left(r, \frac{1}{T(z)^n \mathrm{e}^{n\alpha(z+c)}}\right) = O(r^{\beta}),$$

which is impossible by the second main theorem for three small target functions [7, Theorem 2.5]. Hence $P(z) \equiv 0$.

Conversely, suppose that $P(z) \equiv 0$, and factorize f as in (3.1). By (3.2) it follows that

$$\frac{T(z)^n}{q(z)T(z+c)} = -\mathrm{e}^{Q(z)+\alpha(z+c)-n\alpha(z)}. \tag{3.3}$$

If $T(z)$ has infinitely many zeros, then there exists a zero z_0 of $T(z)$ such that none of the points $z_m = z_0 + mc$, $m \in \mathbb{N} \cup \{0\}$, is a zero of $q(z)$. If z_0 is of multiplicity $k \geq 1$, then, by (3.3), $z_0 + c$ is a zero of $T(z)$ of multiplicity nk. Continuing inductively, we see that z_m is a zero of $T(z)$ of multiplicity $n^m k$. Since $n \geq 2$, the sequence of zeros (counting multiplicities) is of infinite convergence exponent. This is a contradiction, and hence $\lambda(f) = 0 < q = \sigma(f)$ by (a). $\square$

Proof of (c) If $f \in \Gamma_0$ is a solution of (1.4), then $P(z) \equiv 0$ by (b).

Conversely, suppose that $P(z) \equiv 0$, and factorize f as in (3.1). The proof of (b) shows that $T(z)$ must be a polynomial. From (3.3) we now have that $\frac{T(z)^n}{q(z)T(z+c)}$ must be a constant. Finally, Lemma 2.4 implies that $T(z)$ and $q(z)$ must both be constants. This shows that $f \in \Gamma_0$.

Finally assume that $n \geq 3$. If $P(z) \not\equiv 0$, then the second fundamental theorem for three small target functions and Lemma 2.3 yield

$$\begin{aligned} nT(r,f) = T(r,f^n) &\leq \overline{N}(r,f^n) + \overline{N}\left(r, \frac{1}{f^n}\right) + \overline{N}\left(r, \frac{1}{f^n - P(z)}\right) + O(\log r) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f(z+c)}\right) + O(\log r) \\ &\leq 2T(r,f) + O(r^{\sigma(f)-1+\epsilon}) + O(\log r), \end{aligned}$$

contradicting $n \geq 3$. Hence $P(z) \equiv 0$, and so $f \in \Gamma_0$. $\square$

Proof of (d) Since $f \in \Gamma_0$, we have $P(z) \equiv 0$, and so $g \in \Gamma_0$ as well. Substituting $f(z) = \mathrm{e}^{\alpha(z)}$, resp. $g(z) = \mathrm{e}^{A(z)}$, into (1.4), we obtain

$$\mathrm{e}^{n\alpha(z)-\alpha(z+c)} = \mathrm{e}^{nA(z)-A(z+c)}.$$

Taking logarithms, denoting $S(z) = A(z) - \alpha(z)$, and assuming that the polynomial $S(z) \not\equiv 0$, we conclude that for some integer k,

$$n = \frac{S(z+c)}{S(z)} + k\frac{2\pi \mathrm{i}}{S(z)}.$$

If $S(z)$ is non-constant, then letting $|z| \to \infty$, we obtain $n = 1$, which is a contradiction. Therefore $S = 2\pi\mathrm{i}\frac{k}{n-1}$ is a constant, and we immediately see that $(\mathrm{e}^{S(z)})^{n-1} \equiv 1$. □

4 On Exponential Polynomials

Let $P(z) = a_q z^q + \cdots + a_0$, where $a_q \neq 0$. It is well known [7, p. 7] that

$$T(r, \mathrm{e}^{P(z)}) = |a_q|\frac{r^q}{\pi} + o(r^q). \tag{4.1}$$

This elementary observation generalizes to exponential polynomials f of the form (1.6). Indeed, following the reasoning in [4], suppose that the polynomials $Q_j(z)$ in (1.6) are pairwise different and normalized by $Q_j(0) = 0$. The representation (1.6) is then uniquely determined, and the functions $P_j(z)\mathrm{e}^{Q_j(z)}$ are linearly independent. Let

$$q = \max\{\deg(Q_j) : Q_j(z) \not\equiv 0\},$$

and let $w_1, \dots, w_m$ $(m \leq k)$ be pairwise different leading coefficients of the polynomials $Q_j(z)$ of maximum degree q. Then f can be written in the form

$$f(z) = H_0(z) + H_1(z)\mathrm{e}^{w_1 z^q} + \cdots + H_m(z)\mathrm{e}^{w_m z^q}, \tag{4.2}$$

where the functions $H_j(z)$ are either exponential polynomials of degree $< q$ or ordinary polynomials in z. Due to the construction, we have $H_j(z) \not\equiv 0$ for $1 \leq j \leq m$. It follows that $H_0(z) \equiv 0$ if and only if all polynomials $Q_j(z)$ in (1.6) are of maximal degree.

The convex hull of a set $W \subset \mathbb{C}$, denoted by $\mathrm{co}(W)$, is the intersection of all convex sets containing W. If W contains only finitely many elements, then $\mathrm{co}(W)$ is obtained as an intersection of finitely many closed half-planes, and hence $\mathrm{co}(W)$ is either a compact polygon (with a non-empty interior) or a line segment. We denote the perimeter of $\mathrm{co}(W)$ by $C(\mathrm{co}(W))$. If $\mathrm{co}(W)$ is a line segment, then $C(\mathrm{co}(W))$ equals to twice the length of this line segment. Through the rest of the paper, we fix the notation for $W = \{\overline{w_1}, \dots, \overline{w_m}\}$, $W_0 = \{0, \overline{w_1}, \dots, \overline{w_m}\}$ and $Q(z) = b_q z^q + \cdots + b_0$.

Theorem 4.1 ([4, Satz 1]) *Let f be given by* (4.2)*. Then*

$$T(r, f) = C(\mathrm{co}(W_0))\frac{r^q}{2\pi} + o(r^q). \tag{4.3}$$

Note that growth rates (4.1) and (4.3) coincide in the special case when $f(z) = \mathrm{e}^{P(z)}$. This is a consequence of the fact that the convex hull of the two-element set $\{0, a_q\}$ is just the line segment $[0, a_q]$.

Repeated application of (4.3) yields the following immediate consequence.

Corollary 4.2 *Let $q \in \mathbb{N}$, $a_0(z), \dots, a_n(z)$ be either exponential polynomials of degree $< q$ or ordinary polynomials in z, and let $b_1, \dots, b_n \in \mathbb{C} \setminus \{0\}$ be distinct constants. Then*

$$\sum_{j=1}^{n} a_j(z)\mathrm{e}^{b_j z^q} = a_0(z)$$

holds only when $a_0(z) \equiv a_1(z) \equiv \cdots \equiv a_n(z) \equiv 0$.

We note that Corollary 4.2 is a special case of Borel–Nevanlinna theorem on linear combinations of entire functions, see [8, pp. 70, 108] and [9, p. 77].

Theorem 4.3 ([4, Satz 2]) *Let f be given by* (4.2). *If $H_0(z) \not\equiv 0$, then*

$$m\left(r, \frac{1}{f}\right) = o(r^q), \tag{4.4}$$

while if $H_0(z) \equiv 0$, then

$$N\left(r, \frac{1}{f}\right) = C(\mathrm{co}(W))\frac{r^q}{2\pi} + o(r^q). \tag{4.5}$$

Taking $n = 2$ in the proof of (a) in Section 3 and making use of (4.1), (4.3) and (4.4), we have the following consequence.

Corollary 4.4 *Let f be given by* (4.2), *where $H_0(z) \not\equiv 0$. If f is a solution of* (1.4), *where $n = 2$, then $C(\mathrm{co}(W_0)) = 2|b_q|$.*

The next result deals with exponential polynomial solutions of (1.4) in the case when $\mathrm{co}(W_0)$ is a line segment.

Lemma 4.5 *Let f be given by* (4.2), *and suppose that f is a solution of* (1.4), *where $n = 2$. If the points $0, w_1, \ldots, w_m$ are collinear, then $m = 1$.*

Proof Assume on the contrary to the assertion that $m \geq 2$. For each $i \in \{1, \ldots, m\}$, we may write $w_i = k_i w$, where the constants $k_i \in \mathbb{R} \setminus \{0\}$ are distinct and $w \in \mathbb{C} \setminus \{0\}$. Moreover, we may suppose that $k_i > k_j$ for $i > j$. Equation (1.4) can be written as

$$\sum_{i=0}^{m}\sum_{j=0}^{m} H_i(z)H_j(z)\mathrm{e}^{(k_i+k_j)wz^q} + q(z)\mathrm{e}^{Q(z)}\sum_{h=0}^{m} H_h(z+c)\mathrm{e}^{k_h w(z+c)^q} = P(z), \tag{4.6}$$

where $k_0 = 0$. By Corollary 4.2 there exists an index $j \in \{0, 1, \ldots, m\}$ such that $2k_1 w = b_q + k_j w$, for otherwise $H_1(z) \equiv 0$, which is a contradiction.

Case 1 If $j = 1$, then $b_q = k_1 w$. In the case $H_0(z) \equiv 0$ at least one term of the form $H_i(z)^2\mathrm{e}^{2k_i wz^q}$ remains in the left-hand side of (4.6), since $m \geq 2$. Corollary 4.2 then implies that $H_i(z) \equiv 0$, which a contradiction. In the case $H_0(z) \not\equiv 0$ at least one term of the form $2H_0(z)H_i(z)\mathrm{e}^{k_i wz^q}$ remains in the left-hand side of (4.6), yielding $H_i(z) \equiv 0$, which is also a contradiction.

Case 2 If $j \neq 1$, then a term of the form $q(z)\mathrm{e}^{Q_0(z)}H_1(z+c)\mathrm{e}^{(b_q+k_1 w)z^q}$, where $Q_0(z) = Q(z) - b_q z^q + k_1 w((z+c)^q - z^q)$, remains in the left-hand side of (4.6). Now $H_1(z+c) \equiv 0$ by Corollary 4.2, which is a contradiction. □

5 Proof of Theorem 1.1 (e)

We may write the exponential polynomial solution f in the form (4.2), where $q = \deg(Q)$ by (a). We may assume that $n = 2$ by (c).

The following lemma shows that exponential polynomial solutions of (1.4) are of the desired form in the case $m = 1$.

Lemma 5.1 *Let f be given by (4.2), where $m = 1$. If f is a solution of (1.4), then either $f \in \Gamma_1 \setminus \Gamma_0$ or $f \in \Gamma_0$. Moreover, if $f \in \Gamma_1 \setminus \Gamma_0$, then $\sigma(f) = 1$.*

Proof Substituting $f(z) = H_0(z) + H_1(z)e^{w_1 z^q}$ into (1.4) (with $n = 2$) yields

$$\begin{aligned} P(z) - H_0(z)^2 = {} & 2H_1(z)H_0(z)e^{w_1 z^q} + q(z)e^{Q_0(z)}H_0(z+c)e^{b_q z^q} \\ & + H_1(z)^2 e^{2w_1 z^q} + q(z)e^{Q_0(z)+P_1(z)}H_1(z+c)e^{(b_q+w_1)z^q}, \end{aligned} \tag{5.1}$$

where $Q_0(z) = Q(z) - b_q z^q$ and $P_1(z) = w_1(z+c)^q - w_1 z^q$ are polynomials of degree $\le q-1$. We break the rest of the proof into two cases.

Case 1 Suppose that $b_q \neq w_1$. Applying Corollary 4.2 to (5.1) in three different subcases $b_q = -w_1$, $b_q = 2w_1$ and $b_q \notin \{\pm w_1, 2w_1\}$ always results in $H_0(z) \equiv 0 \equiv H_1(z)$, which is a contradiction.

Case 2 Suppose that $b_q = w_1$. Then (5.1) reduces to the form

$$\begin{aligned} P(z) - H_0(z)^2 = {} & \big(2H_1(z)H_0(z) + q(z)e^{Q_0(z)}H_0(z+c)\big)e^{w_1 z^q} \\ & + \big(H_1(z)^2 + q(z)e^{Q_0(z)+P_1(z)}H_1(z+c)\big)e^{2w_1 z^q}, \end{aligned}$$

and an application of Corollary 4.2 yields the following equations:

$$H_1(z)^2 + q(z)e^{Q_0(z)+P_1(z)}H_1(z+c) = 0, \tag{5.2}$$

$$2H_1(z)H_0(z) + q(z)e^{Q_0(z)}H_0(z+c) = 0, \tag{5.3}$$

$$H_0(z)^2 = P(z). \tag{5.4}$$

Note that $H_0(z)$ is a polynomial by (5.4).

Suppose first that $\deg(Q_0 + P_1) = 0$. Now an exponential polynomial cannot satisfy (5.2), so we deduce that $H_1(z)$ must be a polynomial. Furthermore, an application of Lemma 2.4 to (5.2) shows that $H_1(z)$ reduces to a constant. Hence the polynomial $Q_0(z)$ in (5.3) must be a constant, and an application of Lemma 2.4 to (5.3) shows that $H_0(z)$ reduces to a constant also. This shows that $f \in \Gamma_1$.

Suppose then that $\deg(Q_0 + P_1) \ge 1$. Now $H_1 \in \Gamma_0$ by (5.2) and (c). Denote $H_1(z) = e^{\alpha(z)}$, where $\alpha(z)$ is a polynomial. Then (5.3) implies

$$2e^{\alpha(z)-Q_0(z)}H_0(z) = -q(z)H_0(z+c). \tag{5.5}$$

If $H_0(z) \equiv 0$, then $f \in \Gamma_0$, while if $H_0(z) \not\equiv 0$, then $\alpha(z) - Q_0(z)$ must be a constant. Hence, by Lemma 2.4 and (5.5), we deduce that $H_0(z)$ must reduce to a constant. This shows that $f \in \Gamma_1 \setminus \Gamma_0$ in the case $H_0(z) \not\equiv 0$.

Conclusion We have shown above that every solution f of (1.4) of the form $f(z) = H_0(z) + H_1(z)e^{w_1 z^q}$ must belong to either $\Gamma_1 \setminus \Gamma_0$ or Γ_0. If $f \in \Gamma_1 \setminus \Gamma_0$, then $H_0(z)$ is a constant $d \neq 0$ and $H_1 \in \Gamma_0$ with $\sigma(H_1) \le q-1$. Now (5.3) reads $2H_1(z) = -q(z)e^{Q_0(z)}$, and a substitution to (5.2) yields

$$H_1(z) = 2e^{P_1(z)}H_1(z+c). \tag{5.6}$$

We have $\sigma(H_1(z)) = \sigma(H_1(z+c))$ by Lemma 2.2, and hence $\sigma(H_1) = \deg(P_1)$ by (5.6) and the fact that $\sigma(H_1) \le q-1 = \deg(P_1)$. Write $H_1(z) = e^{\alpha(z)}$, where $\deg(\alpha) = \deg(P_1)$. We conclude by (5.6) that $\alpha(z) = P_1(z) + \alpha(z+c) + d_0$ for some $d_0 \in \mathbb{C}$. Since the polynomials $\alpha(z)$ and $\alpha(z+c)$ have the same leading coefficients, this leads to a contradiction, unless $q = 1$. In other words, $\sigma(f) = 1$. $\square$

Remark 5.2 We thank the referee for the following observations. Multiplying (5.6) by $\mathrm{e}^{w_1 z^q}$ and recalling that $P_1(z) + w_1 z^q = w_1(z+c)^q$, we obtain

$$f - d = 2(f(z+c) - d).$$

It thus follows that $f(z) = \Pi(z)2^{-z/c} + d$ for a c-periodic entire function $\Pi(z)$ of finite order. Since $\Pi(z)2^{-z/c}$ is of the form $\mathrm{e}^{\beta(z)}$ for some polynomial $\beta(z)$ of degree q, we must have $q = 1$ and $\Pi(z) = K\mathrm{e}^{2\pi \mathrm{i} z/c}$ for some $K \in \mathbb{C} \setminus \{0\}$. Therefore $f(z) = K\mathrm{e}^{(2\pi\mathrm{i} - \log 2)z/c} + d \in \Gamma_1 \setminus \Gamma_0$ with $\sigma(f) = 1$, as desired. In order that this f solves (1.4), we must have $K = -2d$, $q(z) \equiv -2K$ and $Q(z) = (2\pi\mathrm{i} - \log 2)z/c$.

The case $m = 1$ now being discussed, we complete the proof by dealing with the case $m \geq 2$.

Lemma 5.3 *If $m \geq 2$, then f of the form* (4.2) *is not a solution of* (1.4).

Proof Suppose on the contrary to the assertion that f solves (1.4). We write

$$G(z) = f(z)^2 - P(z) = M(z) + \sum_{\substack{i,j=0 \\ w_i + w_j \neq 0}}^{m} H_i(z)H_j(z)\mathrm{e}^{(w_i+w_j)z^q}, \tag{5.7}$$

where $M(z)$ is either an exponential polynomial of degree $< q$ or an ordinary polynomial in z (or a linear combination of both). Denote $Y = \{2\overline{w_1}, \ldots, 2\overline{w_m}\}$ and $X = \{\overline{w_i} + \overline{w_j} : w_i + w_j \neq 0\}$. It is clear by convexity and by the fact $Y \subset X$ that $\mathrm{co}(Y) = \mathrm{co}(X)$. We break the rest of the proof into two cases, and aim for a contradiction in both cases.

Case 1 Suppose that $H_0(z) \equiv 0$. By substituting $z + c$ in place of z in (4.2), we see that $f(z+c)$ has the same leading coefficients $w_1, \ldots, w_m$ as $f(z)$ does. Hence Theorem 4.3 yields

$$N\left(r, \frac{1}{f(z+c)}\right) = C(\mathrm{co}(W))\frac{r^q}{2\pi} + o(r^q) = N\left(r, \frac{1}{f}\right) + o(r^q), \tag{5.8}$$

and so, using (1.4) with $n = 2$, we have

$$N\left(r, \frac{1}{G}\right) = C(\mathrm{co}(W))\frac{r^q}{2\pi} + o(r^q). \tag{5.9}$$

If $M(z) \equiv 0$ in (5.7), then Theorem 4.3 yields

$$N\left(r, \frac{1}{G}\right) = C(\mathrm{co}(X))\frac{r^q}{2\pi} + o(r^q). \tag{5.10}$$

Comparing (5.9) and (5.10), we have $C(\mathrm{co}(X)) = C(\mathrm{co}(W))$. But this is a contradiction to the fact that $C(\mathrm{co}(X)) = C(\mathrm{co}(Y)) = 2C(\mathrm{co}(W))$. If $M(z) \not\equiv 0$, then $m\left(r, \frac{1}{G}\right) = o(r^q)$ by Theorem 4.3. Now (5.8) and (5.9) imply

$$\begin{aligned} 2T(r,f) &= T\left(r, \frac{1}{G}\right) + O(\log r) = N\left(r, \frac{1}{G}\right) + o(r^q) \\ &= N\left(r, \frac{1}{f}\right) + o(r^q) \leq T(r,f) + o(r^q), \end{aligned} \tag{5.11}$$

which is impossible by (a).

Case 2 Suppose that $H_0(z) \not\equiv 0$. Then

$$m\left(r, \frac{1}{f}\right) = o(r^q) = m\left(r, \frac{1}{f(z+c)}\right) + o(r^q)$$

by Theorem 4.3 and Lemma 2.1. Therefore, by (1.4) and Lemma 2.2,

$$\begin{aligned} N\left(r, \frac{1}{G}\right) &= N\left(r, \frac{1}{f(z+c)}\right) + O(\log r) \\ &= N\left(r, \frac{1}{f}\right) + o(r^q) = T(r, f) + o(r^q). \end{aligned} \tag{5.12}$$

If $M(z) \not\equiv 0$ in (5.7), then (5.11) holds by (5.12) and Theorem 4.3. This is again a contradiction, and hence $M(z) \equiv 0$. Applying Theorems 4.1 and 4.3 to (5.12), we deduce that

$$C(\mathrm{co}(Y)) = C(\mathrm{co}(W_0)). \tag{5.13}$$

The set $\mathrm{co}(W_0)$ cannot reduce to a line segment due to Lemma 4.5. Hence $\mathrm{co}(W_0)$ is a polygon with a non-empty interior. If the origin would be an interior point of $\mathrm{co}(W_0)$, then $\mathrm{co}(W_0) = \mathrm{co}(W)$, and (5.13) leads to a contradiction. Hence $0 \in \partial\,\mathrm{co}(W_0)$. The corner points of the polygon $\mathrm{co}(W_0)$ are among the points $0, \overline{w_1}, \dots, \overline{w_m}$. Noting that $\mathrm{co}(W_0)$ is also trigonometrically convex [10, p. 63], we may choose its non-zero corner points $u_1, \dots, u_t$, $t \le m$, such that $0 \le \arg(u_j) < \arg(u_{j+1}) \le 2\pi$ for $j \in \{1, \dots, t-1\}$. Now

$$C(\mathrm{co}(W_0)) = |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t|. \tag{5.14}$$

We see that the points $2u_1, \dots, 2u_t$ are corner points of $\mathrm{co}(Y)$. However, if $t < m$, then $\mathrm{co}(Y)$ may have more corner points. Such points, if any, are of the form $2\overline{w_j}$, where $\overline{w_j}$ is an interior point of $\mathrm{co}(W_0)$. Let L denote the length of the polygonal path on $\partial\,\mathrm{co}(Y)$ joining $2u_t$ and $2u_1$, measured in the counterclockwise direction. Hence $L \ge 2|u_t - u_1|$ with equality if and only if $2u_1, \dots, 2u_t$ are the only corner points of $\mathrm{co}(Y)$. Now

$$C(\mathrm{co}(Y)) = 2|u_2 - u_1| + \cdots + 2|u_t - u_{t-1}| + L. \tag{5.15}$$

By (5.13), we have

$$|u_1| + |u_t| = |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + L,$$

and hence, combining Corollary 4.4 with (5.13) and (5.15), it follows that

$$|b_q| = |u_1| + |u_t| - L/2.$$

Denote $W_{b,0} = \{0, \overline{b_q}, \overline{b_q} + \overline{w_1}, \dots, \overline{b_q} + \overline{w_m}\}$, $W_{b,0}^* = \{0, \overline{b_q} + \overline{w_1}, \dots, \overline{b_q} + \overline{w_m}\}$ and $g(z) = q(z)\mathrm{e}^{Q(z)} f(z+c)$. Then Theorem 4.1 yields

$$T(r, g) = C(\mathrm{co}(W_{b,0}))\frac{r^q}{2\pi} + o(r^q). \tag{5.16}$$

Note that the polygon $\mathrm{co}(W_{b,0})$ has a non-empty interior because the same is true for $\mathrm{co}(W_0)$.

Suppose that $\overline{b_q}$ is an interior point of $\mathrm{co}(W_{b,0})$. Then $\mathrm{co}(W_{b,0}) = \mathrm{co}(W_{b,0}^*)$. Similarly as in (5.14), we have

$$C(\mathrm{co}(W_{b,0}^*)) = |u_1 + \overline{b_q}| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t + \overline{b_q}|.$$

Equations (1.4) and (5.16) imply $2C(\mathrm{co}(W_0)) = C(\mathrm{co}(W^*_{b,0}))$. Recalling that $C(\mathrm{co}(W_0)) = 2|b_q|$, we now compute

$$\begin{aligned} 2|u_1| &+ |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t| \\ &= |u_1 + \overline{b_q}| + |u_t + \overline{b_q}| \leq |u_1| + |u_t| + 2|b_q| \\ &= |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + L + 2(|u_1| + |u_t| - L/2) \\ &= 2|u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t|, \end{aligned}$$

that is,

$$\begin{aligned} |u_1 + \overline{b_q}| + |u_t + \overline{b_q}| &= 2|u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t| \\ &= |u_1| + |u_t| + 2|b_q|. \end{aligned} \tag{5.17}$$

This shows that $\arg(u_1) = \arg(\overline{b_q}) = \arg(u_t)$, which is impossible.

Suppose then that $\overline{b_q}$ is not an interior point of $\mathrm{co}(W_{b,0})$. Then $\overline{b_q}$ must be a boundary point of $\mathrm{co}(W_{b,0})$. If $\overline{b_q} + u_j \neq 0$ for all $j \in \{1, \ldots, t\}$, then $\overline{b_q}, \overline{b_q} + u_1, \ldots, \overline{b_q} + u_t$ are the non-zero corner points of $\mathrm{co}(W_{b,0})$. Similarly as in (5.14), we have

$$C(\mathrm{co}(W_{b,0})) = |b_q| + |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + |u_t + \overline{b_q}|.$$

Following the reasoning above, we get $C(\mathrm{co}(W_{b,0})) = 2C(\mathrm{co}(W_0)) = 4|b_q|$, and hence

$$\begin{aligned} |u_1| &+ |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t| \\ &= |b_q| + |u_t + \overline{b_q}| \leq |u_t| + 2|b_q| \\ &= |u_2 - u_1| + \cdots + |u_t - u_{t-1}| - |u_1| + L + 2(|u_1| + |u_t| - L/2) \\ &= |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t|, \end{aligned}$$

that is,

$$|b_q| + |u_t + \overline{b_q}| = |u_1| + |u_2 - u_1| + \cdots + |u_t - u_{t-1}| + 2|u_t| = |u_t| + 2|b_q|.$$

We conclude that $\arg(u_t) = \arg(\overline{b_q})$, and hence one of $\overline{b_q}$ or $\overline{b_q} + u_t$ is not a non-zero corner point of $\mathrm{co}(W_{b,0})$. This is a contradiction. Finally, suppose that $\overline{b_q} + u_j = 0$ for some $j \in \{1, \ldots, t\}$. For this particular j, we have $C(\mathrm{co}(W_0)) = 2|b_q| = 2|u_j|$, which is a contradiction. □

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