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Probabilistic Normed Riesz Spaces

Celaleddin ŞENÇİMEN
Department of Mathematics, Faculty of Arts and Sciences, Mehmet Akif Ersoy University,
15030 *Burdur, Turkey*
E-mail: *sencimen@mehmetakif.edu.tr*

Serpil PEHLİVAN
Department of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University,
32260 *Isparta, Turkey*
E-mail: *serpilpehlivan@sdu.edu.tr*

Abstract In this paper, the concepts of probabilistic normed Riesz space and probabilistic Banach lattice are introduced, and their basic properties are studied. In this context, some continuity and convergence theorems are proved.

Keywords Probabilistic normed Riesz space, probabilistic Banach lattice, order convergence, strong convergence, probabilistic norm Cauchy system

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1 Introduction

The theories of probabilistic normed (PN) spaces and Riesz spaces are two independent and important areas of research in functional analysis. Much work has been done in both theories and they have many important applications in real world problems. PN spaces are the vector spaces in which the norms of the vectors are uncertain due to randomness. A Riesz space is a vector space endowed with the lattice structure. Both theories have many tools in common, therefore the aim of our work is to establish the first connections between these two large theories. In this context, we introduce the notions of probabilistic normed Riesz space and probabilistic Banach lattice, and investigate their certain properties.

First, let us cast a glance at the history of Riesz spaces. The concept of Riesz space, also called vector lattice or K-lineal, was first introduced by Riesz in [1]. The first contributions to the theory came from Freudenthal [2] and Kantorovich [3]. Since then many others have developed the subject (see, for instance, [4–7]). Most of the spaces encountered in analysis are Riesz spaces. They play an important role in optimization and analysis, especially in problems of Banach spaces, measure theory and operator theory. An important application of Riesz spaces can be found in [8].

A PN space is a generalization of an ordinary normed linear space. In a PN space, the norms of the vectors are represented by probability distribution functions instead of nonnegative real

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numbers. Such a generalization of normed spaces may well be adapted to the setting of physical quantities [9] and fixed point theory [10, 11]. PN spaces were first introduced by Šerstnev in [12] by means of a definition that was closely modelled on the theory of normed spaces. Since then the most deepest advances in this theory were obtained in [13–16]. In 1993, Alsina et al. [17] presented a new definition of a PN space which includes Šerstnev's definition in [12] as a special case. Here we will adopt this definition. Following [17], many papers (for instance, [18–22]) investigating the properties of PN spaces have appeared. A detailed history and the development of the subject up to 2006 can be found in [23].

Now, our work can be outlined as follows. In the second section, we recall some of the basic concepts related to PN spaces and Riesz spaces. In the third section, we introduce the concepts of probabilistic normed Riesz space and probabilistic Banach lattice, and study their basic properties. Next, we prove via two examples that the concepts of order convergence and strong convergence (that is, convergence with respect to the probabilistic norm) need not be equivalent in an arbitrary probabilistic normed Riesz space. Finally, we introduce the notion of probabilistic norm Cauchy system, and examine its certain properties.

2 Preliminaries

First, we recall some of the basic concepts related to the theory of PN spaces. We use the terminology of [17] and [24].

A *distance distribution function* is a non-decreasing function F that is left-continuous on $(-\infty,\infty)$, is equal to zero on $[-\infty,0]$ and $F(+\infty)=1$. The set of all distance distribution functions is denoted by Δ^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, and has both a maximal element ε_0 and a minimal element ε_∞, defined by

$$\varepsilon_0(x)=\begin{cases}0, & x\le 0,\\ 1, & x>0,\end{cases} \quad\text{and}\quad \varepsilon_\infty(x)=\begin{cases}0, & x<+\infty,\\ 1, & x=\infty,\end{cases}$$

respectively.

Now let $F,G\in\Delta^+$ and $h\in(0,1]$. If we denote the condition

$$G(x)\le F(x+h)+h \quad \text{for } x\in\left(0,\frac{1}{h}\right)$$

by $[F,G;h]$, then the function d_L defined on $\Delta^+\times\Delta^+$ by

$$d_L(F,G)=\inf\{h:\text{both } [F,G;h] \text{ and } [G,F;h] \text{ hold}\}$$

is called the *modified Lévy metric* on Δ^+. The metric space (Δ^+,d_L) is compact.

A *triangle function* is a binary operation τ on Δ^+, $\tau:\Delta^+\times\Delta^+\to\Delta^+$, that is associative, commutative, non-decreasing in each place, and has ε_0 as identity. A triangle function is said to be *Archimedean* provided that $\tau(F,F)=F$ implies that $F=\varepsilon_0$ or $F=\varepsilon_\infty$.

Definition 2.1 ([17, 21]) *A probabilistic normed space (briefly, a PN space) is a quadruple (V,ν,τ,τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions with $\tau\le\tau^*$, and ν is a mapping (the probabilistic norm) from V into the space of distribution functions Δ^+ such that, writing ν_p for $\nu(p)$, for all p,q in V, the following conditions hold:*

(N1) $\nu_p=\varepsilon_0$ *if and only if* $p=\theta$, *the null vector in* V, (N2) $\nu_{-p} = \nu_p$,

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$,

(N4) $\nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$, *for all* $\alpha \in [0,1]$.

A *Menger* PN *space* under T is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t-norm T and its t-conorm T^*; it is denoted by (V, ν, T).

For $p \in V$ and $t > 0$, the *strong t-neighborhood* of p is defined by the set

$$\mathcal{N}_p(t) = \{q \in V : d_L(\nu_{p-q}, \varepsilon_0) < t\}.$$

Since τ is continuous, the system of neighborhoods $\{\mathcal{N}_p(t) : p \in V \text{ and } t > 0\}$ determines a *Hausdorff* and *first countable* topology on V, called the *strong topology.*

A sequence (p_n) in (V, ν, τ, τ^*) is said to be *strongly convergent* (*convergent with respect to the probabilistic norm*) to a point p in V, and we will write $p_n \overset{\text{PN}}{\longrightarrow} p$, if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_p(t)$ whenever $n \geq N$. Thus, $p_n \overset{\text{PN}}{\longrightarrow} p$ if, and only if, $\lim_{n\to\infty} d_L(\nu_{p_n - p}, \varepsilon_0) = 0$. We will call p the *strong limit.*

A sequence (p_n) in (V, ν, τ, τ^*) is said to be *strong Cauchy*, if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_{p_m}(t)$ whenever $n, m \geq N$. If every strong Cauchy sequence is strongly convergent to a point p in V, then we say that (V, ν, τ, τ^*) is complete in the strong topology.

In the sequel, when we consider a PN space (V, ν, τ, τ^*), we will assume that it is endowed with the strong topology.

Now, we list some of the basic concepts and notations related to the theory of Riesz spaces, and we refer to [25] for more details.

Definition 2.2 *A real vector space E (with elements $f, g, \ldots$) with a partial order " $\leq$ " is called an ordered vector space if E is partially ordered in such a manner that the vector space structure and the order structure are compatible, that is to say,*

(i) *$f \leq g$ implies $f + h \leq g + h$ for every $h \in E$,*

(ii) *$f \geq \theta$ implies $\alpha f \geq \theta$ for every $\alpha \geq 0$ in $\mathbb{R}$, where θ is the null element with respect to vector addition. If, in addition, E is a lattice with respect to the partial ordering, then E is called a Riesz space or also a vector lattice. We will denote a Riesz space E by $(E, \leq)$.*

A Riesz space E is said to be *Archimedean* provided that given $f, g \geq \theta$ in E such that $\theta \leq nf \leq g$ for every $n \in \mathbb{N}$, it follows that $f = \theta$.

If E is a (real) Riesz space equipped with a norm $\|\cdot\|$ such that $|f| \leq |g|$ in E implies $\|f\| \leq \|g\|$, then the norm on E is called a *Riesz norm.* Any Riesz space equipped with a Riesz norm is called a *normed Riesz space.* We will denote a normed Riesz space E by $(E, \|\cdot\|, \leq)$.

A sequence (f_n) in a Riesz space E is said to *converge in order to* f if there exists a sequence $p_n \downarrow \theta$ such that $|f_n - f| \leq p_n$ holds for all $n \in \mathbb{N}$. In this case, we will write $f_n \overset{\text{ord}}{\longrightarrow} f$.

Now let E be a Riesz space and $\theta < u \in E$. A sequence (f_n) in E is said to *converge u-uniformly to* f, if for any number $t > 0$ there exists an index $N(t) \in \mathbb{N}$ such that $|f_n - f| \leq tu$ holds for all $n \geq N(t)$.

A sequence (f_n) in E is said to *converge relatively uniformly to* f, if there exists an element $u > \theta$ in E such that (f_n) converges u-uniformly to f.

3 Probabilistic Normed Riesz Spaces

First, we introduce the concepts of probabilistic normed Riesz space and probabilistic Banach lattice.

Definition 3.1 *Let $(E, \leq)$ be a (real) Riesz space equipped with a probabilistic norm ν, and continuous triangle functions τ and τ^*. The probabilistic norm on E is a probabilistic Riesz norm provided that $|f| \leq |g|$ in E implies $\nu_f \geq \nu_g$. Any Riesz space, equipped with a probabilistic Riesz norm is a probabilistic normed Riesz space* (PNR *space, briefly*). *If a* PNR *space E is complete with respect to the strong topology, then E is a probabilistic Banach lattice* (PBL, *in short*). *We will denote a* PNR *space by $(E, \nu, \tau, \tau^*, \leq)$ or just E, if the context is clear.*

Note that, for any $f \in E$, the elements f and $|f|$ have the same probabilistic Riesz norm.

Example 3.2 Let $(E, \|\cdot\|, \leq)$ be a normed Riesz space, and define $\nu : E \to \Delta^+$ via $\nu_f = \varepsilon_{\|f\|}$. Let τ and τ^* be triangle functions such that

$$\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b} \leq \tau^*(\varepsilon_a, \varepsilon_b)$$

for all $a, b \geq 0$. Then $(E, \nu, \tau, \tau^*, \leq)$ is a PNR space. Hence every ordinary normed Riesz space is a PNR space.

Example 3.3 Let $(E, \|\cdot\|, \leq)$ be the normed Riesz space of the real continuous functions f defined on some interval $[a, b]$, where "$\leq$" is the pointwise order. Let us consider the simple space $(E, \|\cdot\|, G, M)$, where

$$\|f\| = \max_{t \in [a,b]} |f(t)|; \quad G \in \Delta^+; \quad G \neq \varepsilon_0,\ \varepsilon_\infty,$$

and the probabilistic norm $\nu : E \longrightarrow \Delta^+$ is defined by $\nu_\theta = \varepsilon_0$ and

$$\nu_f(x) = G\left(\frac{x}{\|f\|}\right), \quad x > 0$$

if $f \neq \theta$, and M is the t-norm defined by $M(x, y) = \min\{x, y\}$. Hence $(E, \|\cdot\|, G, M, \leq)$ is a Menger PNR space under M.

Example 3.4 Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(B, \|\cdot\|, \leq)$ a Banach lattice and $L^0(\mathcal{F}, B)$ the random normed module of equivalence classes of B-valued, $\mathcal{F}$-random variables (see [26, 27]). Define $\nu : L^0(\mathcal{F}, B) \longrightarrow \Delta^+$ by

$$\nu_f(x) = P\{\omega \in \Omega : \|f(\omega)\| < x\}$$

for any $f \in L^0(\mathcal{F}, B)$ and $x \in \mathbb{R}$. Then $\left(L^0(\mathcal{F}, B), \nu, \tau_W, \tau_M\right)$ is an E-normed space, which is also a PN space (see [15]). Here the continuous triangle functions τ_W and τ_M are defined by

$$(\tau_W(F, G))(x) = \sup\{\max\{F(u) + G(v) - 1,\ 0\} : u + v = x\}$$

and

$$(\tau_M(F, G))(x) = \sup\{\min\{F(u), G(v)\} : u + v = x\},$$

where $F, G \in \Delta^+$ and $x \in \mathbb{R}$. If we define a partial order "$\leq$" on $L^0(\mathcal{F}, B)$ as

$$f \leq g \text{ if, and only if, } f^0(\omega) \leq g^0(\omega) \text{ a.s.},$$

where f^0 and g^0 are arbitrarily chosen representatives of f and g respectively, then $(L^0(\mathcal{F}, B), \nu, \tau_W, \tau_M, \leq)$ is a PBL.

In classical Riesz space theory, it is known that every normed Riesz space is Archimedean. However, as the following example shows, a PNR space need not be Archimedean.

Example 3.5 Let us consider the Riesz space $(\mathbb{R}^2, \leq)$ where "$\leq$" is the lexicographical order. Then this space is not Archimedean (see [25]). Now let us define a probabilistic norm on $\mathbb{R}^2$ by

$$\nu_f = \begin{cases} \varepsilon_0, & \text{if } f = \theta, \\ F, & \text{if } f \neq \theta, \end{cases}$$

where $F \in \Delta^+$ is fixed and $F \neq \varepsilon_0, \varepsilon_\infty$. If we take $\tau = \tau^* = \mathbf{M}$, where $\mathbf{M}$ is the maximal triangle function, then we get a PN space denoted by $(\mathbb{R}^2, F, \mathbf{M})$. Thus ν is a probabilistic Riesz norm, and hence $(\mathbb{R}^2, F, \mathbf{M}, \leq)$ is a PNR space.

However, if the triangle function τ^* of a PNR space $(E, \nu, \tau, \tau^*, \leq)$ is Archimedean and $\nu_f \neq \varepsilon_\infty$ for all $f \in E$, then the space is also Archimedean. To see this, we consider such a PNR space $(E, \nu, \tau, \tau^*, \leq)$. Let $f, g \geq \theta$ be given in E such that $\theta \leq nf \leq g$ for every $n \in \mathbb{N}$. Then $\nu_f \geq \nu_{\frac{1}{n}g}$ for every $n \in \mathbb{N}$. Since τ^* is Archimedean and $\nu_f \neq \varepsilon_\infty$ for all $f \in E$, the mapping $\mathcal{M} : \mathbb{R} \to E$ given by $\mathcal{M}(\alpha) = \alpha f$ for a fixed $f \in E$ is continuous (see [18]). Hence $\frac{1}{n}g \xrightarrow{\text{PN}} \theta$, and thus $\nu_{\frac{1}{n}g} \xrightarrow{d_L} \nu_\theta = \varepsilon_0$ since the probabilistic norm is continuous (see [18]). On the other hand, we have $d_L(\nu_f, \varepsilon_0) \leq d_L(\nu_{\frac{1}{n}g}, \varepsilon_0)$ for every $n \in \mathbb{N}$. This shows that $\nu_f = \varepsilon_0$, i.e., $f = \theta$. Hence E is Archimedean. Note that the triangle function $\mathbf{M}$ considered in Example 3.5 is not Archimedean.

Now we consider the continuity properties of certain mappings defined on a PNR space. Throughout the rest of the paper, E will denote a PNR space $(E, \nu, \tau, \tau^*, \leq)$.

Theorem 3.6 *Let E be a* PNR *space, (f_n) and (g_n) be sequences in E such that $f_n \xrightarrow{\text{PN}} f$ and $g_n \xrightarrow{\text{PN}} g$. Then we have*

$$f_n \vee g_n \xrightarrow{\text{PN}} f \vee g \quad \text{and} \quad f_n \wedge g_n \xrightarrow{\text{PN}} f \wedge g.$$

In particular, $f_n^+ \xrightarrow{\text{PN}} f^+$, $f_n^- \xrightarrow{\text{PN}} f^-$ and $|f_n| \xrightarrow{\text{PN}} |f|$. Thus the lattice operations $\vee$ and $\wedge$, and the mappings $f \longmapsto f^+$, $f \longmapsto f^-$ and $f \longmapsto |f|$ are continuous.

Proof Let $f_n \xrightarrow{\text{PN}} f$ and $g_n \xrightarrow{\text{PN}} g$. Then we have

$$|f_n \vee g_n - f \vee g| \leq |g - g_n| + |f - f_n|,$$

which means that

$$\nu_{f_n \vee g_n - f \vee g} \geq \nu_{|g-g_n|+|f-f_n|} \geq \tau(\nu_{g-g_n}, \nu_{f-f_n}),$$

and hence we get

$$d_L(\nu_{f_n \vee g_n - f \vee g}, \varepsilon_0) \leq d_L(\tau(\nu_{g-g_n}, \nu_{f-f_n}), \varepsilon_0)$$

for every $n \in \mathbb{N}$. Since the continuity of τ implies its uniform continuity, we can say that for any $t > 0$ there is a $\lambda > 0$ such that $d_L(\tau(F, G), \varepsilon_0) < t$ whenever $d_L(F, \varepsilon_0) < \lambda$ and $d_L(G, \varepsilon_0) < \lambda$, where $F, G \in \Delta^+$.

Now let $t > 0$. Then we can find a $\lambda > 0$ such that

$$d_L(\tau(\nu_{g-g_n}, \nu_{f-f_n}), \varepsilon_0) < t$$

(hence $d_L(\nu_{f_n \vee g_n - f \vee g}, \varepsilon_0) < t$) whenever $f_n \in \mathcal{N}_f(\lambda)$ (i.e., $d_L(\nu_{f_n - f}, \varepsilon_0) < \lambda$) and $g_n \in \mathcal{N}_g(\lambda)$ (i.e., $d_L(\nu_{g_n - g}, \varepsilon_0) < \lambda$). By hypothesis, for such a $\lambda > 0$ mentioned above, there exist

$N_1(\lambda) = N_1(t) \in \mathbb{N}$ and $N_2(\lambda) = N_2(t) \in \mathbb{N}$ such that $f_n \in \mathcal{N}_f(\lambda)$ for all $n \geq N_1$ and $g_n \in \mathcal{N}_g(\lambda)$ for all $n \geq N_2$. Now let $N = N(t) = \max\{N_1, N_2\}$. Then for every $n \geq N$ we have

$$d_L(\nu_{f_n \vee g_n - f \vee g}, \varepsilon_0) < t,$$

which shows that $f_n \vee g_n \xrightarrow{\text{PN}} f \vee g$. Similarly, we get $f_n \wedge g_n \xrightarrow{\text{PN}} f \wedge g$. Now by Definition 3.1 and the inequality $|f_n^+ - f^+| \leq |f_n - f|$, we get $f_n^+ \xrightarrow{\text{PN}} f^+$.

Similarly, we have $f_n^- \xrightarrow{\text{PN}} f^-$. Finally, the inequality $||f_n| - |f|| \leq |f_n - f|$ yields $|f_n| \xrightarrow{\text{PN}} |f|$. □

Theorem 3.7 *Let E be a* PNR *space and (f_n) be a sequence in E such that $f_n \xrightarrow{\text{PN}} f$ and $f_n \geq g$ for every $n \in \mathbb{N}$ and for some $g \in E$. Then we have $f \geq g$. Hence, if $f_n \xrightarrow{\text{PN}} f$ and $f_n \geq \theta$ for all $n \in \mathbb{N}$, then $f \geq \theta$. Thus the positive cone $E^+ = \{f \in E : f \geq \theta\}$ is closed with respect to the strong topology.*

The proof is obtained using Theorem 3.6, and we omit the details because, except for a change of language and notation, it is the same as in the classical Riesz space theory (see [25]).

In what follows, we compare the strong convergence with the relatively uniform convergence and order convergence. First we recall that, in an arbitrary normed Riesz space, relatively uniform convergence implies the convergence with respect to the Riesz norm. However, this may not hold in a PNR space. To hold this, we impose a sufficient condition as given in the following theorem.

Theorem 3.8 *Let E be a* PNR *space where the second triangle function τ^* is Archimedean and $\nu_f \neq \varepsilon_\infty$ for all $f \in E$. Then relatively uniform convergence implies the strong convergence.*

Proof Let (f_n) be a sequence in E such that (f_n) is convergent relatively uniformly to some $f \in E$. Then there exists an element $u > \theta$ in E such that (f_n) converges u-uniformly to f. To prove that (f_n) converges to f in the strong topology, we only need to prove that there exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ for every subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) such that $(f_{n_{k_j}})_{j\in\mathbb{N}}$ converges to f in the strong topology because by hypothesis, there always exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ for every subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of (f_n) such that $|f_{n_{k_j}} - f| \leq t_j u$ for every $j \in \mathbb{N}$, where (t_j) is a sequence of numbers such that $t_j \downarrow 0$. Thus we have $\nu_{f_{n_{k_j}} - f} \geq \nu_{t_j u}$ for all $j \in \mathbb{N}$. Since $t_j \to 0$, we have $t_j u \xrightarrow{\text{PN}} \theta$ as $j \to \infty$, by hypothesis. Thus $\nu_{t_j u} \xrightarrow{d_L} \nu_\theta = \varepsilon_0$, which yields $\nu_{f_{n_{k_j}} - f} \xrightarrow{d_L} \varepsilon_0$. Hence we may conclude that $f_n \xrightarrow{\text{PN}} f$, which completes the proof. □

If τ^* is not Archimedean, then relatively uniform convergence may not imply strong convergence. To see this, let us consider the following example.

Example 3.9 ([18]) Let $E = \mathbb{R}$, viewed as a one-dimensional linear space, $\tau = \tau_W$ and $\tau^* = \tau_M$. For $p \in E$, define ν by setting $\nu(0) = \varepsilon_0$, and

$$\nu(p) = \frac{1}{|p|+2}\,\varepsilon_0 + \frac{|p|+1}{|p|+2}\,\varepsilon_\infty \quad \text{for } p \neq 0.$$

It is easy to see that $(E, \nu, \tau_W, \tau_M, \leq)$ is a PNR space, where "$\leq$" is the usual order on $\mathbb{R}$. Note that τ_M is not Archimedean on all of Δ^+. Now consider the real sequence $(p_n) = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. Observe that (p_n) is convergent relatively uniformly to 0, however, it is not strongly convergent.

Now we shall prove via two examples that order convergence and strong convergence need not be equivalent in an arbitrary PNR space.

Example 3.10 Let us consider the PNR space $(E, \nu, \tau, \mathbf{M}, \leq)$, where $E = \mathbb{R}$, the probabilistic norm $\nu : \mathbb{R} \to \Delta^+$ is defined by

$$\nu_p = \begin{cases} \varepsilon_{\frac{a+|p|}{a}}, & \text{if } p \neq 0, \\ \varepsilon_0, & \text{if } p = 0, \end{cases}$$

$a > 0, \tau$ is a triangle function such that $\tau(\varepsilon_c,\ \varepsilon_d) \leq \varepsilon_{c+d}$ $(c, d > 0)$ and "$\leq$" is the usual order on $\mathbb{R}$. Now define $p_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ and take $a = 1$. Then we have $p_n \xrightarrow{\text{ord}} 1$, however, (p_n) is not strongly convergent.

Example 3.11 Let E be the Riesz space $L_1([0,1], \mu)$ of all Lebesgue integrable functions on $[0,1]$ with norm $\|f\| = \int_0^1 |f(x)|\ d\mu$. Let us define a probabilistic norm $\nu : E \to \Delta^+$ via

$$\nu_f = \begin{cases} \varepsilon_{\frac{\|f\|}{a+\|f\|}}, & \text{if } f \neq \theta, \\ \varepsilon_0, & \text{if } f = \theta, \end{cases}$$

where $a > 0$. Let τ be a triangle function such that

$$\tau(\varepsilon_c,\ \varepsilon_d) \leq \varepsilon_{c+d}, \quad c, d > 0.$$

Then $(E, \nu, \tau, \mathbf{M}, \leq)$ is a PNR space, where "$\leq$" is the pointwise order. Now let $(X_n)_{n \in \mathbb{N}}$ be the sequence of intervals

$$\left[0, \frac{1}{2}\right],\ \left[\frac{1}{2}, 1\right],\ \left[0, \frac{1}{3}\right],\ \left[\frac{1}{3}, \frac{2}{3}\right],\ \left[\frac{2}{3}, 1\right],\ \left[0, \frac{1}{4}\right], \ldots,$$

and f_n be the characteristic function of X_n for all $n \in \mathbb{N}$. Then (f_n) is not convergent in order (see [25]), but it is strongly convergent to the zero element of E.

Now we present the following relations between strong convergence and order convergence. Note that the monotone convergence considered in Theorem 3.12 is a particular case of order convergence (see [25]).

Theorem 3.12 *Let E be a* PNR *space and (f_n) be a sequence in E. If $f_n \uparrow$ and $f_n \xrightarrow{\text{PN}} f$, then $f_n \uparrow f$; similarly, for decreasing sequences.*

Proof It follows from Theorem 3.7. □

Theorem 3.13 *Let E be a* PNR *space and (f_n) be a sequence in E. If $f_n \xrightarrow{\text{PN}} f$ and $f_n \xrightarrow{\text{ord}} g$, then $f = g$.*

Proof It follows from Theorems 3.6 and 3.7. □

A result similar to Theorem 3.12 is also valid for directed subsets of a PNR space. Recall that a non-empty subset D of a Riesz space E is said to be *upwards directed* if for any $f, g \in D$ there exists an element $h \in D$ such that $h \geq f \vee g$. In this case, we write $D \uparrow$. If $D \uparrow$ and D has the supremum $f_0 \in E$, then we write $D \uparrow f_0$. A *downwards directed set* is defined similarly (see [25]).

Now we introduce the strong convergence of a directed set and prove a theorem related to this concept.

Definition 3.14 *Let D be an upwards directed set in a* PNR *space E. Then D is strongly convergent to some $f_0 \in E$ provided that for any $t > 0$ there is an $f(t) \in D$ such that $f \in \mathcal{N}_{f_0}(t)$ for all $f \in D$ satisfying $f \geq f(t)$.*

Theorem 3.15 *If D is an upwards directed set in a* PNR *space E such that D is strongly convergent to some f_0, then* $\sup D = f_0$.

Proof First, we will show that f_0 is an upper bound of D. Let $f^* \in D$ be given. We will prove that $f^* \leq f_0$. Since D is strongly convergent to f_0, there exists an element $f(1) \in D$ such that $f \in \mathcal{N}_{f_0}(1)$ for all $f \in D$ satisfying $f \geq f(1)$. Now choose $f_1 \in D$ such that $f_1 \geq f^* \vee f(1)$. Then $f_1 \geq f^*$ and $f_1 \in \mathcal{N}_{f_0}(1)$. Now there exists an element $f(1/2) \in D$ such that $f \in \mathcal{N}_{f_0}(1/2)$ for all $f \in D$ satisfying $f \geq f(1/2)$. Choose $f_2 \in D$ such that $f_2 \geq f_1 \vee f(1/2)$. Then $f_2 \geq f_1$ and $f_2 \in \mathcal{N}_{f_0}(1/2)$. Continuing in this way, we obtain a sequence $f^* \leq f_1 \leq f_2 \leq \cdots$ in D such that $f_n \in \mathcal{N}_{f_0}(1/n)$ for every $n \in \mathbb{N}$, i.e., $d_L(\nu_{f_n - f_0}, \varepsilon_0) < 1/n$. Thus $f_n \xrightarrow{\text{PN}} f_0$ as $n \to \infty$. Hence (f_n) is increasing and strongly convergent to f_0. Moreover, we have $f^* \leq f_n \uparrow f_0$ by Theorem 3.12. Since $f^* \in D$ is arbitrary, this shows that f_0 is an upper bound of D. For any other upper bound g of D, we have $f_n \leq g$ for all $n \in \mathbb{N}$, thus $\sup f_n = f_0 \leq g$ by Theorem 3.7. Hence $\sup D = f_0$. □

Now, for directed subsets of a PNR space, we introduce a notion similar to a strong Cauchy sequence.

Definition 3.16 *Let D be an upwards directed set in a* PNR *space E. Then D is a probabilistic norm Cauchy system provided that for any $t > 0$ there exists an element $f(t) \in D$ such that $f_1 \in \mathcal{N}_{f_2}(t)$ for all $f_1, f_2 \in D$ satisfying $f_1, f_2 \geq f(t)$. The definition for a downwards directed set is similar.*

It is clear that if the directed set D is strongly convergent, then it is a probabilistic norm Cauchy system. In the converse direction, we will prove that every probabilistic norm Cauchy system is strongly convergent in a PBL. For this, we first present the following theorem.

Theorem 3.17 *Let D be an upwards directed probabilistic norm Cauchy system in a* PNR *space E. Then there exists an increasing sequence in D such that the sequence and D have the same upper bounds (where it is possible that the set of these upper bounds is empty). A similar assertion holds for downwards directed probabilistic norm Cauchy system.*

Proof Let (t_n) be a sequence of positive numbers such that $t_n \downarrow 0$. By hypothesis, there is an element $f_1 \in D$ such that $f \in \mathcal{N}_{f_1}(t_1)$ for all $f \in D$ satisfying $f \geq f_1$. Also there exists $f_2^* \in D$ such that $f \in \mathcal{N}_{f_2^*}(t_2)$ for all $f \in D$ satisfying $f \geq f_2^*$. Now choose $f_2 \in D$ such that $f_2 \geq f_1 \vee f_2^*$. Then $f_2 \geq f_1$ and all $f \geq f_2$ in D satisfy the inequality $\theta \leq f - f_2 \leq f - f_2^*$, thus $\nu_{f-f_2} \geq \nu_{f-f_2^*}$, which implies that $d_L(\nu_{f-f_2}, \varepsilon_0) \leq d_L\left(\nu_{f-f_2^*}, \varepsilon_0\right) < t_2$, i.e., $f \in \mathcal{N}_{f_2}(t_2)$. Similarly, there exists an $f_3 \in D$ such that $f_3 \geq f_2$ and $f \in \mathcal{N}_{f_3}(t_3)$ for all $f \in D$ satisfying $f \geq f_3$. Continuing in this way, we obtain an increasing sequence (f_n) in D such that $f \in \mathcal{N}_{f_n}(t_n)$ for all $f \in D$ satisfying $f \geq f_n$. In particular, we have $f_m \in \mathcal{N}_{f_n}(t_n)$ for $m \geq n$. This shows that (f_n) is a strong Cauchy sequence. Now let us prove that if g is any upper bound of (f_n), then g is an upper bound of D, i.e., $g \vee f = g$ for every $f \in D$. To show this, let $f \in D$ and f_n in the sequence be given. Then there is an element $f' \in D$ such that $f' \geq f \vee f_n$, and thus $\nu_{f'-f_n} \leq \nu_{(f \vee f_n)-f_n}$, i.e., $d_L\left(\nu_{(f \vee f_n)-f_n}, \varepsilon_0\right) \leq d_L(\nu_{f'-f_n}, \varepsilon_0) < t_n$. On the other hand, we can write

$$\theta \leq (g \vee f) - g = g \vee (f \vee f_n) - g \vee f_n \leq (f \vee f_n) - f_n,$$

and so $\nu_{(g\vee f)-g} \geq \nu_{(f\vee f_n)-f_n}$, i.e.,

$$d_L(\nu_{(g\vee f)-g}, \varepsilon_0) \leq d_L(\nu_{(f\vee f_n)-f_n}, \varepsilon_0) < t_n.$$

This holds for all $n \in \mathbb{N}$. Since $t_n \to 0$, it follows that $g \vee f = g$. This completes the proof. □

Theorem 3.18 *Every probabilistic norm Cauchy system in a* PBL *is strongly convergent. If the system is upwards directed, the strong limit is the supremum of the system. In other words, if $D \uparrow$ and D is a probabilistic norm Cauchy system, then D is strongly convergent to some f_0, and $D \uparrow f_0$; similarly, if D is downwards directed.*

Proof Let D be an upwards directed probabilistic norm Cauchy system in a probabilistic Banach lattice E. Then by Theorem 3.17, D contains an increasing strong Cauchy sequence (f_n) such that $d_L(\nu_{f_n - f}, \varepsilon_0) < \lambda_n$ for all $f \in D$ satisfying $f \geq f_n$, where $\lambda_n \downarrow 0$. Since E is a PBL, the strong limit f_0 of the sequence exists, and since (f_n) is increasing, f_0 is also the supremum, i.e., $f_n \uparrow f_0$ (see Theorem 3.12).

Let us prove that D is strongly convergent to f_0. Since $f_n \xrightarrow{\text{PN}} f_0$, given $\lambda > 0$ there exists an $N(\lambda) \in \mathbb{N}$ such that $d_L(\nu_{f_n - f_0}, \varepsilon_0) < \lambda$ for all $n \geq N(\lambda)$. Now choose $n \geq N(\lambda)$ such that the corresponding λ_n satisfies $\lambda_n \leq \lambda$, hence $d_L(\nu_{f_n - f}, \varepsilon_0) < \lambda$ for all $f \in D$ satisfying $f \geq f_n$. Since τ is uniformly continuous, for each $t > 0$ we can find a $\lambda > 0$ such that $d_L(\nu_{f_n - f}, \varepsilon_0) < \lambda$ and $d_L(\nu_{f_n - f_0}, \varepsilon_0) < \lambda$ imply that $d_L(\tau(\nu_{f_n - f}, \nu_{f_n - f_0}), \varepsilon_0) < t$. Now let $t > 0$ and choose $\lambda > 0$ as mentioned just above. Then for any $f \in D$ satisfying $f \geq f_n$, we have $\nu_{f - f_0} \geq \tau(\nu_{f - f_n}, \nu_{f_n - f_0})$, i.e.,

$$d_L(\nu_{f - f_0}, \varepsilon_0) \leq d_L(\tau(\nu_{f - f_n}, \nu_{f_n - f_0}), \varepsilon_0) < t.$$

This shows that D is strongly convergent to f_0. Hence we have $D \uparrow f_0$ by Theorem 3.15. □

4 Conclusion

In the classical Riesz space theory, Banach lattices receive special attention. The current work serves as a brief introduction to the probabilistic analogues of Banach lattices. Thus we think that there are many open problems and applications in this new research area.

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References

[1] Riesz, F.: Sur la décomposition des opérations linéaires. *Proc. Internat. Congress of Math.* (*Bologna*), **3**, 143–148 (1928)

[2] Freudenthal, H.: Teilweise geordnete Modulen. *Proc. Acad. Amsterdam,* **39**, 641–651 (1936)

[3] Kantorovich, L. V.: Lineare halbgeordnete Raume. *Receueil Math.*, **2**, 121–168 (1937)

[4] Kakutani, S.: Concrete representation of abstract (L)-spaces and the mean ergodic theorem. *Ann. Math.,* **42**, 523–537 (1941)

[5] Luxemburg, W. A. J., Zaanen, A. C.: Notes on Banach function spaces I. *Indag. Math.,* **25**, 135–147 (1963)

[6] Maeda, F., Ogasawara, T.: Representation of vector lattices. *J. Sci. Hiroshima Univ. A,* **12**, 17–35 (1942)

[7] Nakano, H.: Linear topologies on semi-ordered linear spaces. *J. Fac. Sci. Hokkaido Univ.,* **12**, 87–104 (1953)

[8] Aliprantis, C. D., Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics, AMS Mathematical Surveys and Monographs (Vol. 105), USA, 2003

[9] Menger, K.: Probabilistic geometry. *Proc. Natl. Acad. Sci. USA,* **37**, 226–229 (1951)

[10] Shisheng, Z.: On the theory of probabilistic metric spaces with applications. *Acta Mathematica Sinica* (*New Ser.*), **1**, 366–377 (1985)

[11] Choudhury, B. S., Das, K.: A new contraction principle in Menger spaces. *Acta Mathematica Sinica, English Series*, **24**, 1379–1386 (2008)

[12] Šerstnev, A. N.: On the notion of a random normed space. *Dokl. Akad. Nauk,* **149**, 280–283 (1963)

[13] Guo, T. X., Ma, R. P.: Some reviews on various definitions of a random conjugate space together with various kinds of boundedness of a random linear functional. *Acta Anal. Funct. Appl.*, **6**, 16–38 (2004)

[14] Mustari, D. H.: On almost sure convergence in linear spaces of random variables. *Theory Probab. Appl.*, **15**, 337–342 (1970)

[15] Sherwood, H.: Isomorphically isometric probabilistic normed linear spaces. *Stochastica*, **3**, 71–77 (1979)

[16] Taylor, R. L.: Convergence of elements in random normed spaces. *Bull. Austral. Math. Soc.*, **12**, 156–183 (1975)

[17] Alsina, C., Schweizer, B., Sklar, A.: On the definition of a probabilistic normed space. *Aequationes Math.,* **46**, 91–98 (1993)

[18] Alsina, C., Schweizer, B., Sklar, A.: Continuity properties of probabilistic norms. *J. Math. Anal. Appl.,* **208**, 446–452 (1997)

[19] Lafuerza Guillén, B., Rodríguez Lallena, J. A., Sempi, C.: Some classes of probabilistic normed spaces. *Rend. Mat.*, **17**, 237–252 (1997)

[20] Lafuerza Guillén, B., Rodríguez Lallena, J. A., Sempi, C.: Probabilistic norms for linear operators. *J. Math. Anal. Appl.,* **220**, 462–476 (1998)

[21] Lafuerza Guillén, B., Rodríguez Lallena, J. A., Sempi, C.: A study of boundedness in probabilistic normed spaces. *J. Math. Anal. Appl.,* **232**, 183–196 (1999)

[22] Lafuerza Guillén, B.: D-bounded sets in probabilistic normed spaces and in their products. *Rend. Mat.,* **21**, 17–28 (2001)

[23] Sempi, C.: A short and partial history of probabilistic normed spaces. *Mediterr. J. Math.*, **3**, 283–300 (2006)

[24] Schweizer, B., Sklar, A.: Probabilistic Metric Spaces, Elsevier/North-Holland, New York, 1983 (Reissued with an errata list, notes and supplemental bibliography by Dover Publications, New York, 2005)

[25] Zaanen, A. C.: Introduction to Operator Theory in Riesz Spaces, Springer-Verlag, Berlin, 1997

[26] Guo, T. X.: Relations between some basic results derived from two kinds of topologies for a random locally convex module. *J. Funct. Anal.*, **258**, 3024–3047 (2010)

[27] Guo, T. X., Zeng, X. L.: Random strict convexity and random uniform convexity in random normed modules. *Nonlinear Anal.*, **73**, 1239–1263 (2010)