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Isotropic Affine Spheres

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Abstract In this paper, we study affine spheres which are isotropic and we obtain a complete classification. In particular, we show that all such affine spheres are hyperbolic affine spheres, isometric with $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$, $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ or $E_6(-26)/F_4$.

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1 Introduction

The notion of isotropic submanifolds of an arbitrary Riemannian manifold was first introduced by O'Neill [1], who studied the general properties of such class of submanifolds. These submanifolds, which can be considered as a generalization of the totally geodesic submanifolds, have been nearly always studied under the additional hypothesis of parallelism of the second fundamental form. When the ambient space is a sphere, this study was made by Sakamoto [2] and in the case of the complex projective space by Naitoh [3]. Montiel and Urbano [4] have studied n-dimensional, complete, totally real, isotropic submanifolds of a complex projective space without assumption about the parallelism of the second fundamental form and Vrancken [5] proved some local classification theorems for totally real isotropic submanifolds of a complex projective space.

In this paper, we study n-dimensional affine spheres in $\mathbb{R}^{n+1}$. Namely, let $M^n \to \mathbb{R}^{n+1}$ be an immersion. Then it is well known, since the publication of Blaschke's book in the early twenties, that on a non-degenerate affine hypersurface M there exists a canonical transversal vector field, called the affine normal. The second fundamental form h associated to the affine normal is called the affine metric. In the special case that M is locally strongly convex, this affine metric is a Riemannian metric. Also, using the affine normal, by the Gauss formula, one can introduce an affine connection on M, called the induced connection ∇. Therefore, on M we can consider two connections, namely the induced affine connection ∇ and the Levi–Civita connection $\hat{\nabla}$ of

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the affine metric h. The difference tensor K is defined by $K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$. Using the Weingarten formula, we can define the shape operator S and M is called an affine sphere if $S = c\,\mathrm{Id}$.

Moreover, we say that M is a λ-isotropic submanifold if at each point p of M, $h(K(v,v), K(v,v))$ is independent of the unit vector v, namely,

$$\lambda(p) = \|K(v,v)\|, \tag{1.1}$$

where λ is a function on M and $h(v,v) = 1$. If λ is a constant, we say that M is constant isotropic.

In this paper, we study n-dimensional affine spheres in $\mathbb{R}^{n+1}$ which are λ-isotropic and we obtain their complete classification.

Remark 1.1 Since M is strongly convex, if M is λ-isotropic, the case $\lambda = 0$ implies $K = 0$ and consequently, from (2.12) it follows $C = 0$. Furthermore, we remark that the theorem of Berwald states that C vanishes identically if and only if M is an open part of a non-degenerate locally convex quadric. Therefore, we assume $\lambda \neq 0$.

Remark 1.2 In Lemma 3.1 we prove that every surface in $\mathbb{R}^3$ is λ-isotropic.

For higher dimensions we prove the following

Main Theorem *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then M is a constant isotropic hyperbolic affine sphere and M is affine equivalent with a canonical immersion of one of the following symmetric spaces*:

- $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$;
- $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$;
- $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$;
- $E_6(-26)/F_4$.

In [6] the authors gave a complete classification of locally strongly convex affine hypersurfaces of $\mathbb{R}^{n+1}$ with parallel cubic form with respect to the Levi–Civita connection of the affine Berwald–Blaschke metric. It turns out that all such affine hypersurfaces can be obtained by applying repeatedly the Calabi product construction of hyperbolic affine hyperspheres, using as building blocks either the hyperboloid, or the standard immersion of one of the symmetric spaces $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$, $\mathrm{SL}(m,\mathbb{C})/\mathrm{SU}(m)$, $\mathrm{SU}^*(2m)/\mathrm{Sp}(m)$ or $E_6(-26)/F_4$.

2 Preliminaries

Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersion of a connected differentiable n-dimensional manifold into the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D and a parallel volume element ω and let ξ be an arbitrary local transversal vector field to $f(M^n)$. For any vector fields $X, Y, X_1, \ldots, X_n$, we write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$\theta(X_1, \ldots, X_n) = \omega(f_* X_1, \ldots, f_* X_n, \xi), \tag{2.2}$$

thus define an affine connection ∇, a symmetric $(0,2)$-type tensor h, as the second fundamental form and a volume element θ. We say that f is non-degenerate if h is non-degenerate (and this condition is independent of the choice of transversal vector field ξ). In this case, it is known (see [7]) that there is a unique choice (up to sign) of transversal vector field such that the induced connection ∇, the induced second fundamental form h and the induced volume element θ satisfy the following conditions:

$$\nabla\theta = 0, \tag{2.3}$$

$$\theta = \omega_h, \tag{2.4}$$

where ω_h is the metric volume element induced by h. We call ∇ the induced affine connection, ξ the affine normal and h the affine metric. By combining (2.3) and (2.4), we obtain the apolarity condition which states that $\nabla\omega_h = 0$. A non-degenerate immersion equipped with this special transversal vector field is called a Blaschke immersion. Throughout this paper, we will always assume that f is a Blaschke immersion. If h is positive (or negative) definite, the immersion is called locally strongly convex. Notice that if h is negative definite, we can always replace ξ by $-\xi$, thus making the new affine metric positive definite. Therefore, if we say that M is locally strongly convex, we will always assume that ξ is chosen so that h is positive definite.

Condition (2.3) implies that $D_X\xi$ is tangent to $f(M^n)$ for any tangent vector X to M. Hence, we can define a $(1,1)$-tensor field S, called the affine shape operator by

$$D_X\xi = -f_*(SX). \tag{2.5}$$

The following fundamental equations of Gauss, Codazzi and Ricci are given by

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY \quad \text{(Equation of Gauss)}, \tag{2.6}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z) \quad \text{(Equation of Codazzi for } h), \tag{2.7}$$

$$(\nabla_X S)Y = (\nabla_Y S)X \quad \text{(Equation of Codazzi for } S), \tag{2.8}$$

$$h(X,SY) = h(SX,Y) \quad \text{(Equation of Ricci)}. \tag{2.9}$$

If $\dim M \geq 2$ and M is an affine sphere, it follows from (2.8) that c is a constant. We call f a proper affine sphere if $c \neq 0$: if $c > 0$, the proper affine sphere is called elliptic, if $c < 0$, it is called hyperbolic. If $c = 0$, the affine sphere is called improper or parabolic. Hence, by applying a suitable homothetic transformation, we may assume that $c = -1$, $c = 0$ or $c = 1$. From (2.7), it follows that the cubic form

$$C(X,Y,Z) = (\nabla h)(X,Y,Z) \tag{2.10}$$

is symmetric in X, Y, Z.

Let $\hat{\nabla}$ denote the Levi–Civita connection of the affine metric h. The difference tensor K is defined by

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y$$

for vector fields X and Y on M. We also write $K_X Y = K(X,Y)$ and $K_X = \nabla_X - \hat{\nabla}_X$. Thus, for each X, it follows that K_X is a tensor of type $(1,1)$ that maps Y to $K(X,Y)$. Since both ∇ and $\hat{\nabla}$ have zero torsion, K is symmetric in X and Y,

$$K(X,Y) = K(Y,X). \tag{2.11}$$

We also have

$$C(X,Y,Z) = -2h(K(X,Y),Z) \tag{2.12}$$

and consequently

$$h(K(X,Y),Z) = h(K(X,Z),Y), \tag{2.13}$$

which says that the operator K_X is symmetric relative to h. Notice that the apolarity condition ($\nabla\omega_h = 0$) together with (2.4) (see [7, p. 51]) implies

$$\mathrm{trace} K_X = 0 \quad \text{for all vector fields } X. \tag{2.14}$$

Since

$$\hat{R}(X,Y)Z = \frac{1}{2}(h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y) - [K_X, K_Y]Z, \tag{2.15}$$

where $\hat{R}$ denotes the curvature tensor of $\hat{\nabla}$, in the special case that M is an affine sphere, equation (2.15) becomes

$$\hat{R}(X,Y)Z = c(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z. \tag{2.16}$$

Moreover, if M is an affine sphere, we have

$$(\hat{\nabla}_Y K)(X,Z) = (\hat{\nabla}_X K)(Y,Z), \tag{2.17}$$

where $(\hat{\nabla}_Y K)(X,Z) = \hat{\nabla}_Y(K(X,Z)) - K(\hat{\nabla}_Y X, Z) - K(X, \hat{\nabla}_Y Z)$.

3 The Construction of an Orthonormal Basis

In this section we consider an n-dimensional, locally strongly convex affine sphere M in $\mathbb{R}^{n+1}$ which is isotropic, namely, at each point p of M, $\|K(v,v)\|$ is independent of the unit vector v. Hence, there exists a function λ on M such that

$$\lambda^2(p) = h(K(v,v), K(v,v))$$

for $v \in U_pM$, where $U_pM = \{v \in T_pM | h(v,v) = 1\}$. We first investigate the algebraic properties of the difference tensor K at a point p where $\lambda(p) \neq 0$. In that case it is a straightforward computation to check the following conditions for orthonormal vectors x, y, z and w:

$$h(K(x,y), K(x,x)) = 0, \tag{3.1}$$

$$\lambda^2 h(x,x)h(y,y) - h(K(x,x), K(y,y)) - 2h(K(x,y), K(x,y)) = 0, \tag{3.2}$$

$$h(K(y,z), K(x,x)) + 2h(K(x,y), K(x,z)) = 0, \tag{3.3}$$

$$h(K(x,y), K(z,w)) + h(K(x,z), K(w,y)) + h(K(x,w), K(y,z)) = 0. \tag{3.4}$$

We now construct an orthonormal basis with respect to the affine metric h at the point $p \in M$, following the idea of Ejiri [8].

Since M is locally strongly convex and therefore U_pM is compact, we define a function f on U_pM by $f(v) = h(K(v,v), v)$. Let e_1 be an element of U_pM at which the function f attains an absolute maximum. After Remark 1.1, we conclude that $f(e_1) > 0$, since $f(e_1) = 0$ implies that f is identically 0 and consequently $K = 0$. Let $v \in U_pM$ such that $h(v, e_1) = 0$ and $\gamma(t) = \cos t e_1 + \sin t v$. Since $f(\gamma(t))$ attains an absolute maximum for $t = 0$, we have $\frac{d}{dt}(f(\gamma(t))|_0 = 0$. So $h(v, K(e_1,e_1)) + 2h(e_1, K(e_1, v)) = 0$, and using (2.13), we obtain $h(v, K(e_1, e_1)) = 0$. Therefore, e_1 is an eigenvector of K_{e_1} for the real eigenvalue $\lambda_1 = f(e_1)$.

Lemma 3.1 *Every surface in $\mathbb{R}^3$ is λ-isotropic.*

Proof Since K_{e_1} is a symmetric operator, there exists an orthonormal basis e_1, e_2 of U_pM composed of eigenvectors of K_{e_1}, with respective eigenvalues λ_1, λ_2, namely, satisfying

$$K_{e_1}e_1 = \lambda_1 e_1, \quad K_{e_1}e_2 = \lambda_2 e_2.$$

Since trace$K_{e_1} = 0$, we conclude $\lambda_2 = -\lambda_1$. Using (2.13), it follows $K(e_2, e_2) = -\lambda_1 e_1$.

It is clear that (1.1) holds for e_1 and e_2 and $\lambda = \lambda_1$. Moreover, for any unit vector $v = \alpha e_1 + \beta e_2$, using (2.11), we compute that (1.1) is satisfied, since $\alpha^2 + \beta^2 = 1$ and K is bilinear. □

From now on we suppose $n \geq 3$. Since K_{e_1} is a symmetric operator, there exists an orthonormal basis $e_1, e_2, \ldots, e_n$ of U_pM composed of the eigenvectors of K_{e_1}, with respective eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

This basis verifies the following relation, for all $i, j = 1, \ldots, n$,

$$h(e_1, K(e_i, e_j)) = h(K(e_1, e_i), e_j)) = \delta_{ij}\lambda_i. \tag{3.5}$$

Lemma 3.2 *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then*

$$h(e_1, K(e_1, e_1)) = \lambda; \tag{3.6}$$

$$h(e_1, K(e_1, x)) = 0, \quad x \in T_pM \cap \{e_1\}^{\perp}; \tag{3.7}$$

$$h(e_1, K(x, y)) = \frac{\lambda}{2} h(x, y), \quad x, y \in L_1; \tag{3.8}$$

$$h(e_1, K(x, y)) = -\lambda h(x, y), \quad x, y \in L_2; \tag{3.9}$$

$$h(e_1, K(x, y)) = 0, \quad x \in L_1, y \in L_2. \tag{3.10}$$

Proof Using (3.5) and (1.1), it follows $\lambda_1 = \lambda$, namely, we prove (3.6).

Furthermore, since h and K are bilinear, we get (3.7).

Now, applying (3.2) for $x = e_1$ and $y = e_i, i = 2, \ldots, n$, we obtain

$$(\lambda + \lambda_i)\,(\lambda - 2\lambda_i) = 0.$$

Let us first suppose that $\lambda_2 = \cdots = \lambda_n = \frac{\lambda}{2}$. Since trace$K_{e_1} = 0$, we compute $\lambda + (n-1)\frac{\lambda}{2} = 0$, which is a contradiction.

Let us now suppose that $\lambda_2 = \cdots = \lambda_n = -\lambda$. Since trace$K_{e_1} = 0$, it follows $\lambda(n-2) = 0$, which is again a contradiction.

Without loss of generality, we may assume $\lambda_2 = \cdots = \lambda_k = \frac{1}{2}\lambda$ and $\lambda_{k+1} = \cdots = \lambda_n = -\lambda$ with $2 \leq k < n$. Let us denote by L_1 and L_2 the linear subspaces of T_pM spanned by $e_2, \ldots, e_k$ and $e_{k+1}, \ldots, e_n$, respectively. Since h and K are bilinear, using (2.13) and the definition of L_i, we obtain (3.8)–(3.10). □

Lemma 3.3 *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then*

$$h(x, K(x, x)) = 0, \quad x \in L_i, i = 1, 2; \tag{3.11}$$

$$h(x, K(y, z)) = 0, \quad x, y, z \in L_i, i = 1, 2; \tag{3.12}$$

$$h(x, K(y, z)) = 0, \quad x \in L_1, y, z \in L_2. \tag{3.13}$$

Proof Using the definition of L_i, $i = 1, 2$ and relation (3.1) with $y = e_1$, we get

$$0 = h\left(\frac{\lambda}{2}\, x, K(x,x)\right), \quad x \in L_1,$$
$$0 = h(-\lambda\, x, K(x,x)), \quad x \in L_2,$$

which proves (3.11).

Let $x, y, z \in L_1$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Replacing x by $\alpha x + \beta y + \gamma z$ in (3.11), we obtain a polynomial in α, β and γ. Using (2.11) and (2.13), we compute that the coefficient of $\alpha\beta\gamma$ is $6h(x, K(y,z))$. Since all the coefficients of this polynomial vanish, we obtain (3.12). We have the same conclusion for $x, y, z \in L_2$.

Now, put $x \in L_1$, $y \in L_2$ and $z = e_1$ in (3.3). Using Lemma 3.2, we get $(\frac{\lambda}{2} - 2\lambda)h(x, K(y,y)) = 0$. Replacing y by $y + z$, with $z \in L_2$, we obtain (3.13). □

The next two lemmas will improve further our choice of orthonormal basis.

Lemma 3.4 *For linear subspaces L_1 and L_2 of T_pM spanned by $e_2, \ldots, e_k$ and $e_{k+1}, \ldots, e_n$, respectively, it follows*

$$\dim L_1 = 2l, \quad \dim L_2 = l + 1. \tag{3.14}$$

Proof Since $\mathrm{trace} K_{e_1} = 0$, using $\lambda_2 = \cdots = \lambda_k = \frac{1}{2}\lambda$ and $\lambda_{k+1} = \cdots = \lambda_n = -\lambda$, we compute $\lambda + (k-1)\frac{\lambda}{2} + (n - k - 1 + 1)(-\lambda) = 0$, namely, $1 + \frac{k-1}{2} - n + k = 0$.

Therefore, we conclude that $k - 1 = 2l$, $\dim L_1 = k - 1 = 2l$. Moreover, from the last relation we compute $n = 3l + 2$. Then, $\dim L_2 = n - k = l + 1$. □

Remark 3.5 As $n \geq 3$ and $n = 3l + 2$, we obtain $n \geq 5$.

Since $\dim L_2 \geq 1$, let $\zeta \in L_2$ be a fixed unit vector and let $x \in L_1$, $y \in L_2$. Then

- using Lemma 3.2, it follows $h(K(\zeta, x), e_1) = 0$;
- using (2.13) and (3.13), it follows $h(K(\zeta, x), y) = h(K(\zeta, y), x) = 0$.

Consequently, K_ζ maps L_1 into L_1 and therefore there exists an orthonormal basis $f_2, \ldots, f_k$ of L_1 such that

$$K_\zeta f_i = \mu_i\, f_i, \quad i = 2, \ldots, k. \tag{3.15}$$

We put $x = \zeta$, $y = f_i$, $i = 2, \ldots, k$ in (3.2) and compute

$$\lambda^2 - h(K(\zeta,\zeta), K(f_i, f_i)) - 2h(K(\zeta, f_i), K(\zeta, f_i)) = 0. \tag{3.16}$$

Using (3.8) and (3.15), we get $h(e_1, K(f_i, f_i)) = \frac{\lambda}{2}$ and $h(K(f_i, f_i), \zeta) = \mu_i$. Using (3.9) and (3.12), we find that there exist functions $\alpha_2, \ldots, \alpha_k$ such that $K(\zeta, \zeta) = -\lambda e_1 + \sum_{j=2}^k \alpha_j f_j$. By (3.12), we have $h(K(f_i, f_i), f_j) = 0$, $j = 2, \ldots, k$, and therefore $h(K(\zeta,\zeta), K(f_i, f_i)) = h(-\lambda e_1, \frac{\lambda}{2} e_1)$. Then it follows from (3.16) and (3.15) that $\mu_i^2 = \frac{3}{4}\lambda^2$.

Let L_{11} and L_{12} denote the subspaces of L_1 corresponding to the eigenvalues $\frac{\sqrt{3}}{2}\lambda$ and $-\frac{\sqrt{3}}{2}\lambda$, respectively.

Lemma 3.6 *Under the above notations, there holds*

$$\dim L_{11} = \dim L_{12} = l. \tag{3.17}$$

Proof Let us suppose that $\dim L_{11} = a$ and $\dim L_{12} = b$. Since $\mathrm{trace} K_\zeta = 0$, we compute $0 = \frac{\sqrt{3}}{2}\lambda a + (-\frac{\sqrt{3}}{2})\lambda b$. Moreover, since $a + b = 2l$, we conclude $a = b = l$, namely, $\dim L_{11} = \dim L_{12} = l$. Then the orthogonal complement of ζ in L_2 also has dimension l. □

Combining the results from this section, we have the following proposition.

Proposition 3.7 *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then there exists an orthonormal basis $\{e_1, g_1, \ldots, g_l, g_{l+1}, \ldots g_{2l}, g_{2l+1}, \ldots, g_{3l}, \zeta\}$ of T_pM^n satisfying $g_i \in L_{11}$ for $i = 1, \ldots, l$, $g_i \in L_{12}$ for $i = l+1, \ldots, 2l$, $g_i \in L_2$ for $i = 2l+1, \ldots, 3l$, $\zeta \in L_2$ and*

$$
\begin{aligned}
&K(\zeta, g_i) = (-1)^{j+1}\frac{\sqrt{3}}{2}\lambda g_i, \quad \text{for } g_i \in L_{1j};\\
&K(\zeta, g_i) = 0, \quad \text{for } g_i \in L_2 \cap \zeta^{\perp}; \quad K(\zeta, \zeta) = -\lambda e_1; \quad K(\zeta, e_1) = -\lambda\zeta;\\
&K(e_1, e_1) = \lambda e_1; \quad K(e_1, g_i) = \frac{\lambda}{2} g_i, \quad \text{for } g_i \in L_{11} \cup L_{12};\\
&K(e_1, g_i) = -\lambda g_i, \quad \text{for } g_i \in L_2 \cap \zeta^{\perp};\\
&K(g_i, g_i) = \frac{\lambda}{2} e_1 + (-1)^{j+1}\frac{\sqrt{3}}{2}\lambda\zeta, \quad \text{for } g_i \in L_{1j};\\
&K(g_i, g_i) = -\lambda e_1, \quad \text{for } g_i \in L_2 \cap \{\zeta\}^{\perp};\\
&K(g_j, g_k) = 0, \quad \text{if } g_j, g_k \in L_{1i} \text{ or if } g_j, g_k \in L_2 \cap \{\zeta\}^{\perp};\\
&K(g_j, g_k) \in L_2 \cap \{\zeta\}^{\perp}, \quad \text{for } g_j \in L_{11} \text{ and } g_k \in L_{12};\\
&K(g_k, g_m) \in L_{1i}, \quad \text{for } g_k \in L_{1j} \text{ with } j \neq i \text{ and } g_m \in L_2 \cap \{\zeta\}^{\perp}.
\end{aligned}
$$

Proof Using (3.8), (3.9) and (3.11), we find

$$h(K(x,x), K(y,y)) = -\frac{\lambda^2}{2} h(x,x)h(y,y) \tag{3.18}$$

for $x \in L_1$ and $y \in L_2$. We put (3.18) in (3.2) and obtain

$$h(K(x,y), K(x,y)) = \frac{3\lambda^2}{4} \tag{3.19}$$

for orthonormal vectors.

Replacing y by $y+z$ with $z \in L_2$ orthogonal to y, we get

$$h(K(x,y), K(x,z)) = 0. \tag{3.20}$$

Applying (3.20) to $x \in L_{1i}$, $y \in L_2 \cap \{\zeta\}^{\perp}$ and $z = \zeta$, we have $h(y, K(x,x)) = 0$.

Replacing x by $x+w$ with $w \in L_{1i}$, we get

$$h(y, K(x,w)) = 0, \quad y \in L_2 \cap \{\zeta\}^{\perp}, \quad x, w \in L_{1i}. \tag{3.21}$$

Combining (3.21) with Lemma 3.2 and Lemma 3.3, we conclude the proof of this proposition. □

4 Some Results on Isotropic Affine Spheres

Proposition 4.1 *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then M is constant isotropic.*

Proof Under the previous notation, we set $e'_1 = e_1$, $e'_n = \zeta$ and $e'_i = g_{i-1}$ for all $i = 2, \ldots, n-1$ and we denote by $\widehat{\mathrm{Ric}}_{jk}$ the Ricci tensor, namely, $\widehat{\mathrm{Ric}}_{jk} = \sum_{i=1}^{n} h(\widehat{R}(e'_i, e'_j)e'_k, e'_i)$.

Using (2.13), (2.16) and (2.14), we get

$$\begin{aligned}\widehat{\mathrm{Ric}}_{jk} &= (n-1)c\delta_{jk} + \sum_{i=1}^{n} h(K(e'_j, e'_i), K(e'_i, e'_k)) \\ &= (n-1)c\delta_{jk} + \sum_{i=1}^{n} h(K(e'_j, e'_i), K(e'_i, e'_k)) + \frac{1}{2}\sum_{i=1}^{n} h(K(e'_i, e'_i), K(e'_j, e'_k)).\end{aligned}$$

Then, from (3.1) and (3.3), we obtain

$$\widehat{\mathrm{Ric}}_{jk} = (n-1)c\delta_{jk} + \left[\sum_{i=1}^{n} h(K(e'_i, e'_j), K(e'_i, e'_j)) + \frac{1}{2}\sum_{i=1}^{n} h(K(e'_i, e'_i), K(e'_j, e'_j))\right]\delta_{jk}.$$

Now we use (3.2) and compute

$$\widehat{\mathrm{Ric}}_{jk} = (n-1)c\delta_{jk} + \left(\frac{3}{2}\lambda^2 + \sum_{i\neq j}\frac{\lambda^2}{2}\right)\delta_{jk} = \left[(n-1)c + \frac{n+2}{2}\lambda^2\right]\delta_{jk}.$$

Since $n \geq 3$, by Schur's lemma, it follows that λ is a constant. □

Proposition 4.2 *Let $n \geq 3$ and M be an n-dimensional affine sphere in $\mathbb{R}^{n+1}$ which is λ-isotropic. Then the difference tensor K is parallel with respect to the Levi–Civita connection of the affine metric h.*

Proof The isotropy condition (1.1) implies

$$h(K(x,x), K(x,x)) = \lambda^2\, h^2(x,x) \tag{4.1}$$

for each vector $x \in T_pM$. From Proposition 4.1, it follows that λ is a constant. Then, since $\hat{\nabla}$ is a Levi–Civita connection, covariant differentiation of (4.1) gives

$$h((\hat{\nabla}_y K)(x,x), K(x,x)) = 0. \tag{4.2}$$

Let $x = \alpha v + \beta w$, where v, w are unit orthogonal tangent vectors to M and $\alpha, \beta \in \mathbb{R}$. Using (2.11) and (2.17), the linearization argument implies

$$h((\hat{\nabla}_v K)(v,v), K(v,w)) = 0. \tag{4.3}$$

From (2.13), we get

$$\lambda^2 = h(K(v,v), K(v,v)) = h(K(v, K(v,v)), v) \tag{4.4}$$

and using (3.1), it follows

$$0 = h(K(v,v), K(v,w)) = h(K(v, K(v,v)), w), \tag{4.5}$$

for all w orthogonal to v. Then (4.4) and (4.5) imply

$$K(v, K(v,v)) = \lambda^2 v. \tag{4.6}$$

Putting $K(v,v)$ instead of w in (4.3), we obtain

$$0 = h((\hat{\nabla}_v K)(v,v), K(v, K(v,v))) = \lambda^2\, h((\hat{\nabla}_v K)(v,v), v). \tag{4.7}$$

Since $\lambda \neq 0$, we conclude

$$h((\hat{\nabla}_v K)(v,v), v) = 0. \tag{4.8}$$

Let $v = \alpha x + \beta y + \gamma z + \delta w$, where x, y, z, w are tangent vectors to M, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Using (2.13), (2.17) and (4.8), the linearization argument implies

$$h((\hat{\nabla}_x K)(y,z), w) = 0 \tag{4.9}$$

for all $x, y, z, w \in T_pM$. □

Proposition 4.3 *Let $n \geq 3$ and M be an n-dimensional λ-isotropic affine sphere in $\mathbb{R}^{n+1}$. Then M is a hyperbolic affine sphere and $\lambda = \frac{1}{\sqrt{2}}$.*

Proof From Proposition 4.2, we conclude $\hat{\nabla} K = 0$. Hence we get $\hat{R} \cdot K = 0$ and we obtain for $x, y, z, w \in T_pM$ that

$$\hat{R}(x,y)K(z,w) = K(\hat{R}(x,y)z, w) + K(z, \hat{R}(x,y)w). \tag{4.10}$$

Applying this formula for $z = w = e_1$, $x = e_1$, $y = e_i$, we obtain

$$\hat{R}(e_1, e_i)K(e_1, e_1) = K(\hat{R}(e_1, e_i)e_1, e_1) + K(e_1, \hat{R}(e_1, e_i)e_1). \tag{4.11}$$

Using (2.16), it follows

$$\hat{R}(e_1, e_i)e_1 = c(h(e_i, e_1) - h(e_1, e_1)e_i) - [K_{e_1}, K_{e_i}]e_1 \tag{4.12}$$

and from Lemma 3.2 and (4.12), for $e_i \in L_2$, we obtain

$$\hat{R}(e_1, e_i)e_1 = (-c - 2\lambda^2)e_i. \tag{4.13}$$

Since, for $e_i \in L_2$, using (4.13) and (3.9), we compute

$$K(e_1, \hat{R}(e_1, e_i)e_1) = K((-c - 2\lambda^2)e_i, e_1) = (c + 2\lambda^2)\lambda e_i. \tag{4.14}$$

Using (4.11) and (3.6) and (4.14), for $e_i \in L_2$, we conclude

$$\lambda(2\lambda^2 + c) = 0. \tag{4.15}$$

Since $\lambda \neq 0$, it follows $c = -2\lambda^2$, i.e., M is a hyperbolic affine sphere. Consequently, we have $\lambda^2 = \frac{1}{2}$. □

Lemma 4.4 *Let $n \geq 3$ and M be an n-dimensional λ-isotropic affine sphere in $\mathbb{R}^{n+1}$. Then M is a symmetric space.*

Proof Since $\hat{\nabla}$ is a Levi–Civita connection with respect to h, using Proposition 4.2 and the Gauss equation (2.16), we conclude $\hat{\nabla}\hat{R} = 0$. □

Proposition 4.5 *Let $n \geq 3$ and M be an n-dimensional λ-isotropic affine sphere in $\mathbb{R}^{n+1}$. Then $n = 5, 8, 14$ or 26.*

Proof In order to prove Proposition 4.5, we are going to define a map

$$\alpha : L_{1\,1} \times L_{1\,2} \to L_2 \setminus \{\zeta\},$$

which is bilinear and satisfies the following condition of multiplicativity

$$h(\alpha(x,y), \alpha(x,y)) = h(x,x) \cdot h(y,y) \tag{4.16}$$

for any $x \in L_{11}$ and $y \in L_{12}$. We remember that Lemma 3.4 and Lemma 3.6 imply $\dim L_{11} = \dim L_{12} = \dim L_2 \setminus \{\zeta\}$.

Then, using the Hurwitz's theorem it follows $l = 1, 2, 4$ or 8, and therefore $n = 5, 8, 14$ or 26, which completes the proof of Proposition 4.5.

Since K is bilinear, let us define α as

$$\alpha(x,y)=\frac{2}{\sqrt{3}\lambda}\,K(x,y)$$

and prove that the condition (4.16) is satisfied.

For $x\in L_{11}$, using Lemma 3.2 and the basis of Proposition 3.7, we compute

$$\begin{aligned}K(x,x)&=h(K(x,x),e_1)e_1+h(K(x,x),\zeta)\zeta+\sum_{g_i\in L_1}h(K(x,x),g_i)g_i\\&\quad+\sum_{g_j\in L_2\setminus\zeta}h(K(x,x),g_j)g_j\\&=h(K_{e_1}x,x)e_1+h(K_\zeta x,x)\zeta=\frac{\lambda}{2}h(x,x)e_1+\frac{\sqrt{3}}{2}\lambda h(x,x)\zeta\end{aligned}$$

since for $g_i\in L_1$, using (3.12), we have $h(K(x,x),g_i)=0$ and for $g_j\in L_2\cap\{\zeta\}^\perp$, using (3.21), we have $h(K(x,x),g_j)=0$.

Similarly, for $y\in L_{12}$, we compute

$$K(y,y)=\frac{\lambda}{2}h(y,y)e_1-\frac{\sqrt{3}}{2}\lambda h(y,y)\zeta.$$

Using (3.2) and the previous formulas, we compute

$$\begin{aligned}&h(K(x,y),K(x,y))\\&\quad=\frac{1}{2}\left[\lambda^2h(x,x)h(y,y)-h\left(\frac{\lambda}{2}h(x,x)e_1+\frac{\sqrt{3}}{2}\lambda h(x,x)\zeta,\frac{\lambda}{2}h(y,y)e_1-\frac{\sqrt{3}}{2}\lambda h(y,y)\zeta\right)\right]\\&\quad=\frac{3}{4}\lambda^2h(x,x)\,h(y,y)\end{aligned}$$

for any $x\in L_{11}$ and $y\in L_{12}$ and therefore, the condition (4.16) is satisfied. $\square$

5 Four Examples and Proof of Main Theorem

In order to finish the proof of our Main Theorem (in Subsection 5.5), we first consider separately each of the four dimensions determined in Proposition 4.3. Namely, in Subsections 5.1–5.4, we construct the natural imbeddings, as hypersurfaces, of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$, $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ and $E_6(-26)/F_4$, into the affine spaces $\mathbb{R}^6$, $\mathbb{R}^9$, $\mathbb{R}^{15}$, $\mathbb{R}^{27}$, respectively and we prove that they are $\frac{1}{\sqrt{2}}$-isotropic.

Note that these examples already appear in the study of homogeneous hyperbolic affine hyperspheres by Sasaki [9], who has shown that these immersions are indeed homogeneous hyperbolic affine hyperspheres.

5.1 For $n=5$, we will construct an imbedding $M^5=\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)\to s(3)\simeq\mathbb{R}^6$, where we denote by $s(3)$ the vector space of all real symmetric matrices of degree 3. The mapping

$$f:\,\mathrm{SL}(3,\mathbb{R})\to s(3)\quad\text{given by } f(a)={}^t a\,a$$

induces an imbedding

$$f:\,\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)\to s(3),$$

which is a Blaschke imbedding as a centro-affine hypersurface. We consider the decomposition of the Lie algebra $\mathrm{sl}(3,\mathbb{R})$: $\mathrm{sl}(3,\mathbb{R})=s_0\oplus o_3$ where $s_0=\{M\in s(3)/\operatorname{trace}(M)=0\}$ and $o_3=\{M\in \mathrm{sl}(3,\mathbb{R})/\operatorname{trace}(M)=0 \text{ and } {}^tM=M^{-1}\}$.

Proposition 5.1 ([7, p. 112]) *The Blaschke structure of the imbedding*

$$f : M^5 = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) \to s(3) \simeq \mathbb{R}^6$$

can be expressed algebraically in terms of the Lie algebra, as follows:

$$\nabla_X Y = XY + YX - \frac{2}{3}\mathrm{trace}(XY)I, \tag{5.1}$$

$$h(X,Y) = \frac{4}{3}\mathrm{trace}(XY), \tag{5.2}$$

$$S = -I \tag{5.3}$$

for $X, Y \in s_0 \simeq T_I M^5$.

Then it follows $h(X,X) = \frac{4}{3}\mathrm{trace}(X^2)$, which shows that h is positive-definite.

The Levi–Civita connection $\hat{\nabla}$ for h coincides with the canonical invariant connection ∇^0 on the symmetric homogeneous space given by $\nabla^0_X Y = 0$, for $X, Y \in s_0$. Therefore, $\hat{\nabla}_X Y = 0$, and using (5.1), we obtain

$$K(X,X) = 2X^2 - \frac{2}{3}\mathrm{trace}(X^2)I$$

and consequently it follows

$$h(K(X,X),K(X,X)) = \frac{16}{3}\mathrm{trace}(X^4) - \frac{16}{9}(\mathrm{trace}(X^2))^2. \tag{5.4}$$

Using the Cayley–Hamilton theorem, for each X in the Lie algebra of M^5, we get $X^4 - \frac{1}{2}\left(\mathrm{trace}(X^2)\right)X^2 - \det(X)X = 0$ and therefore

$$\mathrm{trace}(X^4) = \frac{1}{2}(\mathrm{trace}(X^2))^2. \tag{5.5}$$

Now, using (5.4) and (5.5), we compute

$$h(K(X,X),K(X,X)) = \frac{1}{2}h^2(X,X),$$

i.e., $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is $\frac{1}{\sqrt{2}}$-isotropic.

5.2 $n = 8$

Denoting by $s'(3)$ the vector space of all Hermitian matrices of degree 3 on $\mathbb{C}$, we conclude that the mapping

$$f : \mathrm{SL}(3,\mathbb{C}) \to s'(3) \quad \text{given by } f(a) = {}^t a\,\overline{a}$$

induces an imbedding

$$f : \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3) \to s'(3) \simeq \mathbb{R}^9.$$

We consider the decomposition of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$: $\mathrm{sl}(3,\mathbb{C}) = s'_0 \oplus u_3$ where $s'_0 = \{M \in s'(3)/\,\mathrm{trace}(M) = 0\}$ and $u_3 = \{M \in \mathrm{sl}(3,\mathbb{C})/\,\mathrm{trace}(M) = 0 \text{ and } {}^tM = \overline{M}^{-1}\}$.

Proposition 5.2 *The Blaschke structure of the imbedding* $f : M^8 = \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3) \to s'(3) \simeq \mathbb{R}^9$ *can be expressed algebraically in terms of the Lie algebra, as follows:*

$$\nabla_X Y = XY + YX - \frac{2}{3}\mathrm{trace}(XY)I, \tag{5.6}$$

$$h(X,Y) = \frac{4}{3}\mathrm{trace}(XY) \tag{5.7}$$

for $X, Y \in s'_0 \simeq T_I M^8$ *and* $S = -I$.

Proof Let $X \in s_0'$, $a_s = \exp(sX)$, $\pi(a_s) = x_s \in \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$, where $\pi : \mathrm{SL}(3,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$ is the natural projection. We have $f(x_s) = {}^t a_s\, \overline{a}_s$ and hence

$$f_*(X) = \left(\frac{d}{ds}\right)_{s=0} \left({}^t\exp(sX)\, \overline{\exp(sX)}\right) = 2\,\overline{X}. \tag{5.8}$$

Let ϕ be the representation $\phi : \mathrm{SL}(3,\mathbb{C}) \to \mathbf{SA}(9)$, where $\mathbf{SA}(9)$ is the group of unimodular affine transformations of $\mathbb{R}^9$, such that

$$f(M_1\, M_2) = \phi(M_2)\, f(M_1).$$

It gives a representation ϕ of $\mathrm{SL}(3,\mathbb{C})$ on s_0' by

$$\phi(M_2)X = {}^tM_2\, X\, \overline{M}_2,$$

for each $X \in s_0'$ and that (f,ϕ) is an equivariant immersion of $(\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3), \mathrm{SL}(3,\mathbb{C}))$ into $(R^9, \mathbf{SA}(9))$. For more details we refer to [7].

Then for the tangent vector $\vec{x}_s \in T_IM \simeq s_0'$ to the curve x, using (5.8), we have

$$f_*(\vec{x}_s) = f_*(a_sX) = \phi(a_s)f_*(X) = {}^t(\exp(sX))\,(2\overline{X})\,\overline{\exp(sX)}. \tag{5.9}$$

Since $\left(\frac{d}{ds}\right)_{s=0} f_*(\vec{x}_s) = f_*(\nabla_X X) + h(X,X)I$, using (5.9) and (5.8), we obtain

$$4\overline{X}^2 = 2\overline{(\nabla_X X)} + h(X,X)I. \tag{5.10}$$

Taking the trace of (5.10), it follows $4\mathrm{trace}(\overline{X}^2) = 3h(X,X)$. Since $\mathrm{trace}(\overline{X})^2 = \mathrm{trace}(X^2)$ for $X \in s_0'$, we compute

$$\nabla_X X = 2\,X^2 - \frac{2}{3}\mathrm{trace}(X^2)I, \tag{5.11}$$

$$h(X,X) = \frac{4}{3}\mathrm{trace}(X^2). \tag{5.12}$$

Polarization of (5.11) and (5.12) gives (5.6) and (5.7). Then we compute the curvature tensor and therefore, using the Gauss equation (2.6), we get $S = -I$. $\square$

Using the same argument as for $n = 5$, we obtain

$$\mathrm{trace}(X^4) = \frac{1}{2}\left(\mathrm{trace}(X^2)\right)^2. \tag{5.13}$$

We note that h coincides, up to a scalar, with the Killing form B of the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. It is a natural Riemannian metric on the symmetric space M^8 and therefore its Levi–Civita connection $\hat{\nabla}$ is given by $\hat{\nabla}_X Y = \frac{1}{2}[X,Y]$.

Consequently $K(X,X) = 2X^2 - \frac{2}{3}\mathrm{trace}(X^2)I$ and using (5.13), we compute

$$h(K(X,X), K(X,X)) = \frac{1}{2}h^2(X,X).$$

Therefore, $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$ is $\frac{1}{\sqrt{2}}$-isotropic.

5.3 $n = 14$

The mapping $f : \mathrm{SU}^*(6) \to \mathrm{a}$ given by $f(N) = {}^tN\,\overline{N}$, where

$$\mathrm{a} = \left\{ \begin{pmatrix} E & F \\ -\overline{F} & \overline{E} \end{pmatrix} \,/\, {}^tE = \overline{E},\ {}^tF = -F \right\}$$

induces an imbedding $f : \mathrm{SU}^*(6)/\mathrm{Sp}(3) \to \mathrm{a} \simeq \mathbb{R}^{15}$.

We consider the decomposition $su^*(6) = sp(3) \oplus p_0$ of the Lie algebra $su^*(6)$ where $p_0 = \{N \in a/\operatorname{trace}(N) = 0\}$ in order to represent any invariant structure on the space $M^{14} = \mathrm{SU}^*(6)/\mathrm{Sp}(3)$ (see [10, Chapter XI]). In particular, p_0 represents $T_I M^{14}$.

Proposition 5.3 *The Blaschke structure of the imbedding $f : M^{14} = \mathrm{SU}^*(6)/\mathrm{Sp}(3) \to a \simeq \mathbb{R}^{15}$ can be expressed algebraically in terms of the Lie algebra, as follows:*

$$\nabla_X Y = XY + YX - \frac{1}{3}\operatorname{trace}(XY)I, \tag{5.14}$$

$$h(X,Y) = \frac{2}{3}(XY) \tag{5.15}$$

for $X, Y \in p_0 \simeq T_I M^{14}$ and $S = -I$.

Proof Let $X \in p_0$, $a_s = \exp(sX)$, $\pi(a_s) = x_s \in \mathrm{SU}^*(6)/\mathrm{Sp}(3)$ where $\pi : \mathrm{SU}^*(6) \to \mathrm{SU}^*(6)/\mathrm{Sp}(3)$ is the natural projection. Since after a straightforward computation, we get ${}^tX = \overline{X}$ for $X \in p_0$, we compute $f_*(X) = 2\,\overline{X}$.

Let ϕ be the representation $\phi : \mathrm{SU}^*(6) \to \mathbf{SA}(15)$, where $\mathbf{SA}(15)$ is the group of unimodular affine transformations of $\mathbb{R}^{15}$, such that $f(M_1 M_2) = \phi(M_2)\, f(M_1)$. A representation ϕ of $\mathrm{SU}^*(6)$ on p_0 is given by $\phi(M_2)X = {}^t M_2\, X\, \overline{M}_2$ and (f, ϕ) is an equivariant immersion of $(\mathrm{SU}^*(6)/\mathrm{Sp}(3), \mathrm{SU}^*(6))$ into $(R^{15}, \mathbf{SA}(15))$.

Then for the tangent vector $\vec{x}_s \in T_I M^{14} \simeq p_0$ to the curve x, we obtain the same relation as (5.9). Consequently, the relation (5.10) follows and therefore we compute $4\operatorname{trace}(\overline{X}^2) = 6h(X,X)$.

Since in a fairly straightforward way, we obtain $\operatorname{trace}(\overline{X})^2 = \operatorname{trace}(X^2)$ for $X \in p_0$. We get

$$\nabla_X X = 2\,X^2 - \frac{1}{3}\operatorname{trace}(X^2)I, \quad h(X,X) = \frac{2}{3}\operatorname{trace}(X^2).$$

Hence, by polarization, (5.15) and (5.14) follow. Like in the previous case, $S = -I$. □

After a long but straightforward computation, we obtain

$$\operatorname{trace}(X^4) = \frac{1}{4}(\operatorname{trace}(X^2))^2. \tag{5.16}$$

As it is easy to verify that $B(X,Y) = \operatorname{trace}(XY)$ is a Killing form on $su^*(6)$, the metric h is a natural Riemannian metric on the symmetric space $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$. Therefore the difference tensor is given by

$$K(X,X) = 2X^2 - \frac{1}{3}\operatorname{trace}(X^2)I,$$

and using (5.16), we conclude that $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ is $\frac{1}{\sqrt{2}}$-isotropic since

$$\begin{aligned} h(K(X,X),K(X,X)) &= \frac{2}{3}\operatorname{trace}\left(\left((2X^2) - \frac{1}{3}\operatorname{trace}(X^2)I\right)^2\right) \\ &= \frac{2}{3}\operatorname{trace}(4X^4) - \frac{8}{9}(\operatorname{trace}(X^2))^2 + \frac{2}{27}(\operatorname{trace}(X^2))^2 \cdot 6 \\ &= \frac{1}{2}h^2(X,X). \end{aligned}$$

5.4 $n = 26$

$M^{26} = E_6(-26)/F_4$. We denote by $\mathcal{M}_3(\mathbb{O})$ the vector space of all 3×3 matrices with entries in the space of octonions $\mathbb{O}$. Let $\mathfrak{h}_3(\mathbb{O})$ be the set of Hermitian matrices with entries in $\mathbb{O}$, i.e.,

$\mathfrak{h}_3(\mathbb{O}) = \{N \in \mathcal{M}_3(\mathbb{O}) \ / \ {}^t\overline{N} = N\}$. Any element $N \in \mathfrak{h}_3(\mathbb{O})$ is of the form

$$N = N(\xi, x) = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} ; \xi_i \in \mathbb{R} \text{ and } x_i \in \mathbb{O} \right\}.$$

In the 27-dimensional real vector space $\mathfrak{h}_3(\mathbb{O})$, the multiplication $X \circ Y$, called the Jordan multiplication, is defined by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

$\mathfrak{h}_3(\mathbb{O})$, equipped with the product $\circ$, is a real Jordan algebra.

Despite noncommutativity and nonassociativity, the determinant of a matrix N in $\mathfrak{h}_3(\mathbb{O})$, defined by

$$\det N = \frac{1}{3}\text{trace}(N \circ N \circ N) - \frac{1}{2}\text{trace}(N)\text{trace}(N \circ N) + \frac{1}{6}(\text{trace}(N))^3, \tag{5.17}$$

where $\text{trace}(N) = \xi_1 + \xi_2 + \xi_3$, $N = N(\xi, x)$ is a well-defined and useful concept. The group of determinant-preserving linear transformations of $\mathfrak{h}_3(\mathbb{O})$ is a noncompact real form of E_6 which is sometimes called $E_6(-26)$, because its Killing form has signature -26. For more explanation we refer to [11].

We consider the decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus s\mathfrak{h}_3(\mathbb{O})$ of the Lie algebra $\mathfrak{e}_6$ of $E_6(-26)$ where $\mathfrak{f}_4 = \mathfrak{D}er(\mathbb{O}) \oplus \left\{N \in \mathcal{M}_3(\mathbb{O})/{}^tN = -\overline{N}, \ \text{tr}(N) = 0\right\}$, where the derivations of the octonions $\mathfrak{D}er(\mathbb{O})$ is the Lie algebra of $G_2 = \text{Aut}(\mathbb{O})$, the automorphism group of the octonion algebra. So we have

$$T_I M^{26} \simeq \left\{N \in \mathcal{M}_3(\mathbb{O})/{}^tN = \overline{N}, \ \text{tr}(N) = 0\right\}.$$

Using Lemma 2.2.4 of [12], we conclude

$$F_4 = \{\text{iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))/\det(\alpha N) = \det N, \alpha I = I\},$$
$$E_6(-26) = \{\text{iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))/\det(\alpha N) = \det N\},$$

where I denotes the identity matrix and $\text{iso}_{\mathbb{R}}(\mathfrak{h}_3(\mathbb{O}))$ denotes all $\mathbb{R}$-linear isomorphisms of $\mathfrak{h}_3(\mathbb{O})$. We deduce that the stabilizer of I in $\mathfrak{h}_3(\mathbb{O})$ is F_4. Then M^{26} is locally isomorphic to $\mathcal{A}$ which is the connected component of I in $\left\{N \in \mathcal{M}_3(\mathbb{O})/{}^tN = \overline{N}, \ \det(N) = 1\right\}$. Therefore, $\mathcal{A}$ is the homogeneous space and we have a natural immersion of this space in $\mathfrak{h}_3(\mathbb{O})$.

Since we are interested to construct local coordinates around the identity matrix, we choose local coordinate system such that $\xi_1 = 1$.

For $N = \begin{pmatrix} 1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \in \mathfrak{h}_3(\mathbb{O})$, there exist 26 real numbers $y_1, \ldots, y_{26}$ such that

$$\xi_2 = y_1, \quad \xi_3 = y_2, \quad x_1 = \sum_{i=0}^{i=7} y_{3+i}e_i, \quad x_2 = \sum_{i=0}^{i=7} y_{11+i}e_i, \quad x_3 = \sum_{i=0}^{i=7} y_{19+i}e_i, \tag{5.18}$$

where $(e_0, \ldots, e_7)$ is a basis of the real vector space $\mathbb{O}$.

Since for $N \in \mathfrak{h}_3(\mathbb{O})$, the matrix of the form $\det^{-\frac{1}{3}}(N)N$ has a determinant equal to 1, we put $g(p) = \det \begin{pmatrix} 1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}$ at $p = (y_1, \ldots, y_{26})$.

Using (5.17) we compute

$$\det(N(\xi, x)) = \xi_1\xi_2\xi_3 + 2Re(x_1x_2x_3) - \xi_1 x_1\overline{x}_1 - \xi_2 x_2\overline{x}_2 - \xi_3 x_3\overline{x}_3. \tag{5.19}$$

Now, using the octonion multiplication table, notation (5.18) and (5.19), we compute

$$\begin{aligned}
g(p) = y_1y_2 + 2\,\{&y_{19}(y_3y_{11} - y_4y_{12} - y_5y_{13} - y_6y_{14} - y_7y_{15} - y_8y_{16} - y_9y_{17} - y_{10}y_{18})\\
&- y_{20}(y_3y_{12} + y_4y_{11} + y_5y_{14} - y_6y_{13} + y_7y_{16} - y_8y_{15} + y_9y_{18} - y_{10}y_{17})\\
&- y_{21}(y_3y_{13} - y_4y_{14} + y_5y_{11} + y_6y_{12} - y_7y_{17} + y_8y_{18} + y_9y_{15} - y_{10}y_{16})\\
&- y_{22}(y_3y_{14} + y_4y_{13} - y_5y_{12} + y_6y_{11} + y_7y_{18} + y_8y_{17} - y_9y_{16} - y_{10}y_{15})\\
&- y_{23}(y_3y_{15} - y_4y_{16} + y_5y_{17} - y_6y_{18} + y_7y_{11} + y_8y_{12} - y_9y_{13} + y_{10}y_{14})\\
&- y_{24}(y_3y_{16} + y_4y_{15} - y_5y_{18} - y_6y_{17} - y_7y_{12} + y_8y_{11} + y_9y_{14} + y_{10}y_{13})\\
&- y_{25}(y_3y_{17} - y_4y_{18} - y_5y_{15} + y_6y_{16} + y_7y_{13} - y_8y_{14} + y_9y_{11} + y_{10}y_{12})\\
&- y_{26}(y_3y_{18} + y_4y_{17} + y_5y_{16} + y_6y_{15} - y_7y_{14} - y_8y_{13} - y_9y_{12} + y_{10}y_{11})\}\\
&- (y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 + y_9^2 + y_{10}^2)\\
&- y_1(y_{11}^2 + y_{12}^2 + y_{13}^2 + y_{14}^2 + y_{15}^2 + y_{16}^2 + y_{17}^2 + y_{18}^2)\\
&- y_2(y_{19}^2 + y_{20}^2 + y_{21}^2 + y_{22}^2 + y_{23}^2 + y_{24}^2 + y_{25}^2 + y_{26}^2).
\end{aligned}$$

With these notations we obtain local coordinates which define the hypersurface by

$$F : \mathbb{R}^{26} \longrightarrow \mathbb{R}^{27}, \quad p \mapsto g^{-\frac{1}{3}}(p)\,(1, p). \tag{5.20}$$

We set the notation: $\frac{\partial F}{\partial y_i} = F_i$, $\frac{\partial g}{\partial y_i} = g_i$ and let f_i be a point where only the i-th coordinate is 1 and the others are 0.

Now, let us choose the local transversal vector field to be the position vector field. Following (2.1), we decompose $D_{F_j}F_i$ as

$$F_{ji}(p) = \nabla_{F_j}F_i\,(p) + h(F_j, F_i)(p)F(p), \tag{5.21}$$

where ∇ is the induced connection and h is the affine metric.

We compute

$$F_i(p) = -\frac{1}{3}g_i(p)g^{-\frac{4}{3}}(p)(1, p) + g^{-\frac{1}{3}}(p)f_i \tag{5.22}$$

and

$$F_{ji}(p) = \frac{4}{9}\,g_i(p)\,g_j(p)\,g^{-\frac{7}{3}}(p)\,(1, p) - \frac{1}{3}\,g_{ji}(p)\,g^{-\frac{4}{3}}(p)\,(1, p) + g^{-\frac{1}{3}}(p)\,f_i. \tag{5.23}$$

Expressing $(1, p)$ from (5.20) and f_i from (5.22), in terms of g and F, and using (5.23), we obtain

$$\begin{aligned}
F_{ji}(p) = &\left(\frac{2}{9}\,g^{-2}(p)\,g_i(p)\,g_j(p) - \frac{1}{3}\,g^{-1}(p)\,g_{ji}(p)\right)\,F(p)\\
&- \frac{1}{3}\,g^{-1}(p)\,g_j(p)\,F_i(p) - \frac{1}{3}\,g^{-1}(p)\,g_i(p)\,F_j(p).
\end{aligned}$$

From (5.21), it follows

$$h(F_j, F_i)(p) = \frac{2}{9}\,g^{-2}(p)\,g_i(p)\,g_j(p) - \frac{1}{3}\,g^{-1}(p)\,g_{ji}(p), \tag{5.24}$$

$$\nabla_{F_j}F_i\,(p) = -\frac{1}{3}\,g^{-1}(p)\,g_j(p)\,F_i(p) - \frac{1}{3}\,g^{-1}(p)\,g_i(p)\,F_j(p). \tag{5.25}$$

Using (5.24) and (5.25), we get

$$(\nabla h)(F_k, F_i, F_j)(p) = \frac{\partial}{\partial y_k}(h(F_i, F_j))(p) - h(\nabla_{F_k}F_i, F_j)(p) - h(F_i, \nabla_{F_k}F_j)(p)$$

$$
\begin{aligned}
&= -\frac{4}{27}\, g^{-3}(p)\, g_i(p)\, g_j(p)\, g_k(p) \\
&\quad + \frac{1}{9}\, g^{-2}(p)(g_j(p)\, g_{ki}(p) + g_j(p)\, g_{kj}(p) + g_k(p)\, g_{ij}(p)) \\
&\quad - \frac{1}{3}\, g^{-1}(p)\, g_{ijk}(p).
\end{aligned} \tag{5.26}
$$

We compute

$$
K(F_i, F_j)(p) = -\frac{1}{2} \sum_{l,k} (\nabla h)(F_l, F_i, F_j)(p) h^{-1}_{lk}(p) F_k(p) \tag{5.27}
$$

since $(\nabla h)(F_k, F_i, F_j)(p) = -2h(K(F_i, F_j), F_k)(p)$ follows from (2.10) and (2.12). Here we denote by $(h_{ij})(p)$ the matrix $h(F_i, F_j)(p)$, namely the matrix of the metric at p and we denote by $(h^{-1}_{ij})(p)$ its inverse matrix. Using (5.24), we obtain, at $p_0 = (1, 1, 0, \ldots, 0)$,

$$
h(F_i, F_j)(p_0) = \begin{pmatrix}
\frac{2}{9} & -\frac{1}{9} & 0 & \cdots & 0 \\
-\frac{1}{9} & \frac{2}{9} & 0 & \cdots & 0 \\
0 & 0 & \frac{2}{3} & \cdots & 0 \\
0 & & & \ddots & \\
0 & & & & \frac{2}{3}
\end{pmatrix} \tag{5.28}
$$

and

$$
h^{-1}(F_i, F_j)(p_0) = \begin{pmatrix}
6 & 3 & 0 & \cdots & 0 \\
3 & 6 & 0 & \cdots & 0 \\
0 & 0 & \frac{3}{2} & \cdots & 0 \\
0 & & & \ddots & \\
0 & & & & \frac{3}{2}
\end{pmatrix}. \tag{5.29}
$$

Using (5.27), (5.29) and (5.26), we compute, at p_0,

$$
\begin{aligned}
K(F_i, F_j)(p_0) = -\frac{1}{54}\, &(-24 g_i g_j - 54 g_{1ij} + 18 g_i g_{1j} + 18 g_j g_{1i} + 18 g_{ij} \\
&- 12 g_i g_j - 27 g_{2ij} + 9 g_i g_{2j} + 9 g_j g_{2i} + 9 g_{ij})\, F_1 \\
&+ (-24 g_i g_j - 54 g_{2ij} + 18 g_i g_{2j} + 18 g_j g_{2i} + 18 g_{ij} \\
&- 12 g_i g_j - 27 g_{1ij} + 9 g_i g_{1j} + 9 g_j g_{1i} + 9 g_{ij})\, F_2 \\
&+ \sum_{k=3}^{26} \left(-\frac{27}{2} g_{ijk} + \frac{9}{2} g_i g_{jk} + \frac{9}{2} g_j g_{ik} \right) F_k.
\end{aligned} \tag{5.30}
$$

For example,

$$
\begin{aligned}
&K_{11} = \frac{1}{3} F_1, \quad K_{22} = \frac{1}{3} F_2, \\
&K_{ii} = F_1 + F_2, \quad i = 3, \ldots, 10, \\
&K_{ii} = -F_1, \quad i = 13, \ldots, 18, \\
&K_{ii} = -F_2, \quad i = 19, \ldots, 26,
\end{aligned}
$$

$$K_{12} = -\frac{1}{3}F_1 - \frac{1}{3}F_2.$$

The proof of $\mathrm{trace}_h K = 0$ at p_0 (i.e., $\sum_{j,k} h^{jk} K^i_{jk} = 0$ for all i) is a matter of straightforward computation. Consequently, $F(p_0)$ is the affine normal at p_0 and we conclude that M is an affine hypersphere. Since h and K are bilinear, for $V = \sum_{i=1}^{26} v_i F_i$, we have

$$h(K(V,V), K(V,V)) = \sum_{i,j,k,l=1}^{26} v_i v_j v_k v_l \, h(K(F_i, F_j), K(F_k, F_l)), \tag{5.31}$$

$$h^2(V,V) = \sum_{i,j,k,l=1}^{26} v_i v_j v_k v_l \, h_{ij} h_{kl}. \tag{5.32}$$

Using (5.30), (5.28), (5.31) and (5.31), we compute $h(K(v,v), K(v,v))$ and $h^2(v,v)$, where v is a tangent vector at p_0. These are polynomial functions of fourth order on 26 variables. After a long but straightforward computation, we conclude $h(K(v,v), K(v,v)) = \frac{1}{2}h^2(v,v)$.

5.5 We can now conclude the proof of the main theorem.

Lemma 5.4 *Let $n \geq 3$ and M be an n-dimensional λ-isotropic affine sphere in $\mathbb{R}^{n+1}$. There exists an orthonormal basis $\{e'_1, \dots, e'_n\}$ of T_pM^n, defined by Proposition* 3.7, *such that $K(e'_j, e'_k) = \sum_{i=1}^n \alpha^i_{j,k} e'_i$ where $\alpha^i_{j,k}$ are constants which depend on λ.*

Proof Let $\{e'_1, \dots, e'_n\} = \{e_1, g_1, \dots, g_l, g_{l+1}, \dots g_{2l}, g_{2l+1}, \dots, g_{3l}, \zeta\}$ be an orthonormal basis of T_pM^n defined by Proposition 3.7. We recall that $L_{11} = \mathrm{Vect}\{g_1, \dots, g_l\}$, $L_{12} = \mathrm{Vect}\{g_{l+1}, \dots, g_{2l}\}$ and $L_2 \cap \{\zeta\}^\perp = \mathrm{Vect}\{g_{2l+1}, \dots, g_{3l}\}$. It is clear from the results of Section 4 that these normed vectors spaces are isomorphic. We denote by π_1 and π_2 the isomorphisms between L_{11} and L_{12}, and L_{11} and $L_2 \cap \{\zeta\}^\perp$, respectively. Following the proof of Proposition 4.5, the mapping $\beta : L_{11} \times L_{11} \longrightarrow L_{11}$ defined by

$$\beta(v,w) = \alpha(v, \pi_1(w)) = \frac{2}{\sqrt{3}\lambda} K(v, \pi_1(w))$$

satisfies $h(\beta(v,w), \beta(v,w)) = h(v,v) \cdot h(w,w)$. So by using the Hurwitz's theorem, we can choose an orthonormal basis of L_{11} such that, with respect to this basis, β is given by the traditional multiplication table for the real (resp. complex, quaternionic, Cayley) numbers. Using Proposition 3.7, we conclude the proof. $\square$

We have shown that our examples are $\frac{1}{\sqrt{2}}$-isotropic. Since K is parallel, parallel transports along geodesics preserve all components of K. Using Lemma 5.4 and Theorem 5.5, which is known as a theorem of Cartan on determination of the metric, we conclude that there exist local isometries between each example and a $\frac{1}{\sqrt{2}}$-isotropic affine sphere with the same dimension, respectively.

Theorem 5.5 ([13, p. 157, Theorem 2.1]) *Let M and $\widetilde{M}$ be two Riemannian manifolds of dimension n. Chosen a linear isometry i between the two tangent spaces and f a mapping from a normal neighborhood V of a point of M to $\widetilde{M}$, we define $\phi_t : T_qM \to T_{f(q)}\widetilde{M}$, $v \mapsto \phi_t(v) = \widetilde{P}_t \circ i \circ P_t^{-1}$, where $P_t, \widetilde{P}_t$ are parallel transports along good geodesics. If for all q in V and all $x, y, u, v \in T_qM$ we have*

$$\langle R(x,y)u, v\rangle = \langle \widetilde{R}(\phi_t(x), \phi_t(y))\phi_t(u), \phi_t(v)\rangle,$$

then $f : V \to f(V) \subset M$ is a local isometry.

Finally, using the following fundamental uniqueness theorem of affine geometry, we prove that the λ-isotropic affine spheres, namely hyperbolic affine spheres M^n for $n = 5, 8, 14, 28$, and the imbeddings of of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$, $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ and $E_6(-26)/F_4$, into the affine spaces $\mathbb{R}^6$, $\mathbb{R}^9$, $\mathbb{R}^{15}$, $\mathbb{R}^{27}$, respectively, constructed in Subsections 5.1–5.4 are affine equivalent, which completes the proof of Main Theorem.

Theorem 5.6 ([7, p. 73, Theorem 8.1]) *Let M be a simply connected differentiable manifold with a torsion-free affine connection ∇, a symmetric $(0,2)$-tensor filed h, and a $(1,1)$-tensor field S that satisfy the equation of Gauss* (2.6)*, the equations of Codazzi,* (2.7), (2.8), *and the equation of Ricci* (2.9)*. Then there exists a ∇-parallel volume element θ on M and a global equiaffine immersion $f : (M, \nabla, \theta) \to \mathbb{R}^{n+1}$ with h and S as affine fundamental form and shape operator. Such an immersion is uniquely determined up to affine transformation of $\mathbb{R}^{n+1}$. If, moreover, h is non-degenerate and the given ∇ and h satisfy the apolarity condition, then there is a parallel volume element ω in $\mathbb{R}^{n+1}$ such that f is a Blaschke immersion.*

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