Acta Mathematica Sinica, English Series
Aug., 2012, Vol. 28, No. 8, pp. 1615–1622
Published online: February 1, 2012
DOI: 10.1007/s10114-012-0172-6
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Derivations on the Algebra of Operators in Hilbert C^*-Modules

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Abstract Let $\mathcal{M}$ be a full Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and let $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ be the algebra of adjointable operators on $\mathcal{M}$. We show that if $\mathcal{A}$ is unital and commutative, then every derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is an inner derivation, and that if $\mathcal{A}$ is σ-unital and commutative, then innerness of derivations on "compact" operators completely decides innerness of derivations on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. If $\mathcal{A}$ is unital (no commutativity is assumed) such that every derivation of $\mathcal{A}$ is inner, then it is proved that every derivation of $\mathrm{End}^*_{\mathcal{A}}(L_n(\mathcal{A}))$ is also inner, where $L_n(\mathcal{A})$ denotes the direct sum of n copies of $\mathcal{A}$. In addition, in case $\mathcal{A}$ is unital, commutative and there exist $x_0, y_0 \in \mathcal{M}$ such that $\langle x_0, y_0\rangle = 1$, we characterize the linear $\mathcal{A}$-module homomorphisms on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ which behave like derivations when acting on zero products.

Keywords Derivations, inner derivations, C^*-algebras, Hilbert C^*-modules

MR(2000) Subject Classification 47B47, 47B49

1 Introduction and Preliminaries

Throughout this paper, we only consider derivations from an algebra into itself. Recall that a *derivation* of an algebra $\mathcal{A}$ is a linear mapping Δ from $\mathcal{A}$ into itself, such that $\Delta(ab) = \Delta(a)b + a\Delta(b)$ holds for all $a, b \in \mathcal{A}$. For a fixed $b \in \mathcal{A}$, the mapping $a \to ba - ab$ is clearly a derivation, which is usually called an *inner derivation* (implemented by b).

One of the interesting problems in the theory of derivations is to identify those algebras on which all the derivations are inner, i.e., the first cohomology group is trivial. The first result of this kind is probably due to Kaplansky who proved in [1] that every derivation of a type I

Received March 25, 2010, accepted April 6, 2011
The first author is supported by National Natural Science Foundation of China (Grant No. 11171151) and Natural Science Foundation of Jiangsu Province of China (Grant No. BK2011720); the third author is supported by Singapore Ministry of Education Academic Research Fund Tier 1 (Grant No. R-146-000-136-112)

W^*-algebra is inner. The complete solution for the W^*-algebra case was settled by Sakai in [2] who proved that every derivation of a W^*-algebra is inner. Kadison showed in [3] that every derivation of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is spatial (i.e., it has the form $a \to ta - at$ for some bounded linear operator t on H, where t need not be in $\mathcal{A}$), and more importantly every derivation of a von Neumann algebra is inner. Similar problems have been investigated for non-self-adjoint operator algebras, and we refer to [4–11] for more information about this topic. We particularly point out that Christensen in [4] proved that every derivation of a nest algebra is inner, and later this result was generalized by the second named author of this paper to nest algebras on Banach spaces (see [6]). More general cases were also studied, for example, Gilfeather and Moore in [5, 9] investigated the spatiality and quasi-spatiality of derivations of certain CSL algebras which is a generalization of nest algebras.

This note is devoted to the study for innerness of derivations on the algebra of operators in Hilbert C^*-modules. Hilbert C^*-modules first appeared in the work of Kaplansky (see [1]), which play a significant role in theory of operator algebras, operator K-theory, group representation theory (via strong Morita equivalence), theory of operator spaces and so on (see [12]).

For the convenience of the reader, let us review some basics of Hilbert C^*-modules. Let $\mathcal{A}$ be a C^*-algebra. A *pre-Hilbert $\mathcal{A}$-module* is a complex linear space $\mathcal{M}$ which is a left $\mathcal{A}$-module (with compatible scalar multiplication: $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for $\lambda \in \mathbb{C}$ where $\mathbb{C}$ denotes the complex field, $a \in \mathcal{A}$ and $x \in \mathcal{M}$), and equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ satisfying: for $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$, $x, y, z \in \mathcal{M}$,

(1) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(2) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$;

(3) $\langle ax, y \rangle = a \langle x, y \rangle$;

(4) $\langle x, y \rangle = \langle y, x \rangle^*$.

A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is called a *Hilbert C^*-module over the C^*-algebra $\mathcal{A}$*, or briefly, *a Hilbert $\mathcal{A}$-module*, if it is complete with respect to the norm: $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, $x \in \mathcal{M}$.

Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Denote by $\langle \mathcal{M}, \mathcal{M} \rangle$ the closure of the linear span of all $\langle x, y \rangle, x, y \in \mathcal{M}$. We call $\mathcal{M}$ *full* if $\langle \mathcal{M}, \mathcal{M} \rangle = \mathcal{A}$. Since the set $\langle \mathcal{M}, \mathcal{M} \rangle$ is obviously a closed two-sided involutive ideal in the C^*-algebra $\mathcal{A}$, one can always consider any Hilbert C^*-module as a full Hilbert C^*-module over the C^*-algebra $\langle \mathcal{M}, \mathcal{M} \rangle$. A bounded linear $\mathcal{A}$-module homomorphism from $\mathcal{M}$ into itself is called an *operator* of $\mathcal{M}$. Following [12], denote by $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ the Banach algebra of all operators of $\mathcal{M}$. It is well known that there is no natural involution on this algebra. For $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we say that T is *adjointable*, if there exists an operator $T^* \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{M}$; usually, call T^* the *adjoint* of T. Denote by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ the set of all adjointable operator in $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$. Then it becomes a C^*-algebra. For $x, y \in \mathcal{M}$, define the operator $\theta_{x,y}$ of $\mathcal{M}$ by $\theta_{x,y}(\xi) = \langle \xi, y \rangle x$, $\xi \in \mathcal{M}$. Note that $\theta_{x,y}$ is quite different from rank one operators, in the usual sense, on a Hilbert space. For instance, we cannot infer that $x = 0$ or $y = 0$ from $\theta_{x,y} = 0$. Denote by $\mathcal{K}(\mathcal{M})$ the closed linear span of $\{\theta_{x,y} : x, y \in \mathcal{M}\}$. Then $\mathcal{K}(\mathcal{M})$ is a closed two-side ideal in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Elements of $\mathcal{K}(\mathcal{M})$ are often referred to as "compact" operators. But considered as operators on the Banach space $\mathcal{M}$ they need not be compact. We collect some properties of $\theta_{x,y}$ in the following lemma.

Lemma 1.1 *Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, $x, y \in \mathcal{M}$. Then*

(1) $\theta_{x,y} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ *and* $\theta^*_{x,y} = \theta_{y,x}$;

(2) $T\theta_{x,y} = \theta_{Tx,y}$ *for* $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$, *and* $\theta_{x,y}S = \theta_{x,S^*y}$ *for* $S \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$;

(3) *if in addition, $\mathcal{A}$ is commutative then* $a\theta_{x,y} = \theta_{ax,y} = \theta_{x,a^*y}$ *for* $a \in \mathcal{A}$.

A C^*-algebra $\mathcal{A}$ is said to be *σ-unital* if it possesses a countable approximate unit. The following lemma is essentially due to Brown [13]. For a more direct proof, see [12, Lemma 2.4.3].

Lemma 1.2 *Let $\mathcal{A}$ be a σ-unital C^*-algebra, and let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then there exists a sequence $\{x_i\}$ in $\mathcal{M}$, such that the sequence of partial sums of the series $\sum_i \langle x_i, x_i\rangle$ is a countable approximate unit of the algebra $\mathcal{A}$. In particular, if $\mathcal{A}$ is unital with unit 1, then there exist a positive integer k and elements $x_1, x_2, \dots, x_k \in \mathcal{M}$ such that $\sum_{i=1}^k \langle x_i, x_i\rangle = 1$.*

To prove our results, we also need the following three lemmas. The latter two are famous, which were established by Kadison and Sakai, respectively; see also [14].

Lemma 1.3 ([12, Lemma 2.1.1]) *Let $\mathcal{M}$ be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, and T, S be two mappings from $\mathcal{M}$ into itself such that $\langle Tx, y\rangle = \langle x, Sy\rangle$. Then T, S are both bounded linear $\mathcal{A}$-module homomorphism of $\mathcal{M}$ and so, belong to $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.*

Lemma 1.4 ([3]) *Every derivation of a C^*-algebra annihilates its center.*

Lemma 1.5 ([15]) *Every derivation of a C^*-algebra is bounded.*

2 Main Results

Our first main result reads as follows.

Theorem 2.1 *Let $\mathcal{A}$ be a unital commutative C^*-algebra and let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. Then every derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is an inner derivation.*

Proof Let Δ be an arbitrary derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. By Lemma 1.2, there exist a positive integer k and elements $x_1, x_2, \dots, x_k \in \mathcal{M}$ such that $\sum_{i=1}^k \langle x_i, x_i\rangle = 1$. Define two mappings $T, S : \mathcal{M} \to \mathcal{M}$ by

$$Tx = \sum_{i=1}^k \Delta(\theta_{x,x_i})x_i, \quad x \in \mathcal{M}, \quad Sy = -\sum_{i=1}^k \Delta(\theta_{x_i,y})^* x_i, \quad y \in \mathcal{M}.$$

Clearly, T and S are both linear. For any $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and any $x \in \mathcal{M}$, we have $\Delta(A\theta_{x,x_i}) = \Delta(A)\theta_{x,x_i} + A\Delta(\theta_{x,x_i})$, and then $\Delta(A\theta_{x,x_i})x_i = \langle x_i, x_i\rangle\Delta(A)x + A\Delta(\theta_{x,x_i})x_i$, $1 \le i \le k$. Hence

$$TAx = \sum_{i=1}^k \Delta(A\theta_{x,x_i})x_i = \sum_{i=1}^k \langle x_i, x_i\rangle\Delta(A)x + A\sum_{i=1}^k \Delta(\theta_{x,x_i})x_i = \Delta(A)x + ATx.$$

Consequently, $\Delta(A) = TA - AT$ holds for every $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Thus, it suffices to show that $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

Denote by $\mathcal{Z}$ the center of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Since $\mathcal{A}$ is commutative, for every $a \in \mathcal{A}$, we can define an operator T_a on $\mathcal{M}$ by $T_a x = ax, x \in \mathcal{M}$. It is worthwhile to notice that if $\mathcal{A}$ is not commutative then T_a will not be an $\mathcal{A}$-module homomorphism. We then have $T_a \in \mathcal{Z}$ and $(T_a)^* = T_{a^*}$. We claim that $\mathcal{Z} = \{T_a : a \in \mathcal{A}\}$. In fact, let $A \in \mathcal{Z}$. Then for each $x \in \mathcal{M}$, we have $A\theta_{x,x_i}x_i = \theta_{x,x_i}Ax_i$, $1 \le i \le k$. It follows that $Ax = \sum_{i=1}^k \langle Ax_i, x_i\rangle x$ and so $A = T_{\sum_{i=1}^k \langle Ax_i, x_i\rangle}$, as desired. By Lemma 1.4, for every $a \in \mathcal{A}$, $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, we have

$\Delta(T_a) = 0$ and $\Delta(aA) = \Delta(T_a A) = \Delta(T_a)A + T_a\Delta(A) = a\Delta(A)$. This implies that Δ is an $\mathcal{A}$-module homomorphism.

Suppose $x, y \in \mathcal{M}$ are arbitrary elements. Then

$$\begin{aligned}\Delta(\theta_{x,y}) &= \sum_{i=1}^{k} \Delta(\theta_{x,x_i}\theta_{x_i,y}) \\ &= \sum_{i=1}^{k} \Delta(\theta_{x,x_i})\theta_{x_i,y} + \sum_{i=1}^{k} \theta_{x,x_i}\Delta(\theta_{x_i,y}) \\ &= \sum_{i=1}^{k} \theta_{\Delta(\theta_{x,x_i})x_i,y} + \sum_{i=1}^{k} \theta_{x,\Delta(\theta_{x_i,y})^* x_i} \\ &= \theta_{Tx,y} - \theta_{x,Sy}.\end{aligned}$$

Let $x, y, u, v \in \mathcal{M}$. On the one hand,

$$\Delta(\theta_{u,y}\theta_{x,v}) = \langle x, y\rangle \Delta(\theta_{u,v}) = \langle x, y\rangle\theta_{Tu,v} - \langle x, y\rangle\theta_{u,Sv}.$$

On the other hand,

$$\begin{aligned}\Delta(\theta_{u,y}\theta_{x,v}) &= \Delta(\theta_{u,y})\theta_{x,v} + \theta_{u,y}\Delta(\theta_{x,v}) \\ &= (\theta_{Tu,y} - \theta_{u,Sy})\theta_{x,v} + \theta_{u,y}(\theta_{Tx,v} - \theta_{x,Sv}) \\ &= \langle x, y\rangle\theta_{Tu,v} - \langle x, Sy\rangle\theta_{u,v} + \langle Tx, y\rangle\theta_{u,v} - \langle x, y\rangle\theta_{u,Sv}.\end{aligned}$$

Consequently, we obtain $\langle Tx, y\rangle\theta_{u,v} = \langle x, Sy\rangle\theta_{u,v}$. Then

$$\langle Tx, y\rangle\langle v, v\rangle\langle u, u\rangle = \langle Tx, y\rangle\langle\theta_{u,v}v, u\rangle = \langle x, Sy\rangle\langle\theta_{u,v}v, u\rangle = \langle x, Sy\rangle\langle v, v\rangle\langle u, u\rangle.$$

In particular, taking $u = x_i, v = x_j$, we obtain

$$\langle Tx, y\rangle\langle x_i, x_i\rangle\langle x_j, x_j\rangle = \langle x, Sy\rangle\langle x_i, x_i\rangle\langle x_j, x_j\rangle$$

where $1 \le i, j \le k$. Since $\sum_{i=1}^{k}\langle x_i, x_i\rangle = 1$, it follows that $\langle Tx, y\rangle = \langle x, Sy\rangle$ holds for all $x, y \in \mathcal{M}$. By Lemma 1.3, we get $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. So Δ is an inner derivation. The proof is complete. □

Remark 2.2 In Theorem 2.1, suppose that the condition "$\mathcal{A}$ is unital" is substituted with "$\mathcal{A}$ is σ-unital". Then by Lemma 1.2, there exists a sequence $\{x_i\}$ in $\mathcal{M}$, such that the sequence of partial sums of the series $\sum_i\langle x_i, x_i\rangle$ is a countable approximate unit of the algebra $\mathcal{A}$. Similar to the proof of Theorem 2.1, for every positive integer k, we define two mappings $T_k, S_k : \mathcal{M} \to \mathcal{M}$ by

$$T_k x = \sum_{i=1}^{k} \Delta(\theta_{x,x_i})x_i, \quad x \in \mathcal{M}, \quad S_k y = -\sum_{i=1}^{k} \Delta(\theta_{x_i,y})^* x_i, \quad y \in \mathcal{M}.$$

Then we can similarly obtain $\sum_{i=1}^{k}\langle x_i, x_i\rangle\Delta(A) = T_k A - AT_k$ for all $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, and $\langle T_k x, y\rangle = \langle x, S_k y\rangle$ for all $x, y \in \mathcal{M}$ which means that $T_k \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. If we define $\Delta_k = \sum_{i=1}^{k}\langle x_i, x_i\rangle\Delta$, then Δ_k is an inner derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. For every $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $x \in \mathcal{M}$, recalling that $\{\sum_{i=1}^{k}\langle x_i, x_i\rangle\}_{k=1}^{\infty}$ is an approximate unit of the algebra $\mathcal{A}$, it is easily seen that $\lim_{k\to\infty}\Delta_k(A)x = \Delta(A)x$. Similar to [16], in this case, Δ is called an (weakly) *approximately inner derivation* of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

The following result states that innerness of derivations on $\mathcal{K}(\mathcal{M})$ completely decide innerness of derivations on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$.

Theorem 2.3 *Let $\mathcal{A}$ be a σ-unital commutative C^*-algebra and let $\mathcal{M}$ be a full Hilbert $\mathcal{A}$-module. If every derivation of $\mathcal{K}(\mathcal{M})$ is inner, then any derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is also inner.*

Proof Let Δ be a derivation of $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and let $\{x_i\}$ in $\mathcal{M}$ such that the sequence of partial sums of the series $\sum_i \langle x_i, x_i\rangle$ is a countable approximate unit of the algebra $\mathcal{A}$. For any $x, y \in \mathcal{M}$, we have

$$\Delta(\theta_{\sum_{i=1}^k \langle x_i,x_i\rangle x,y}) = \sum_{i=1}^k \Delta(\theta_{x,x_i}\theta_{x_i,y}) = \sum_{i=1}^k \Delta(\theta_{x,x_i})\theta_{x_i,y} + \sum_{i=1}^k \theta_{x,x_i}\Delta(\theta_{x_i,y}) \in \mathcal{K}(\mathcal{M}).$$

Noticing that $\|\theta_{\sum_{i=1}^k \langle x_i,x_i\rangle x,y} - \theta_{x,y}\| \le \|\sum_{i=1}^k \langle x_i, x_i\rangle x - x\|\|y\|$, we get $\theta_{\sum_{i=1}^k \langle x_i,x_i\rangle x,y}$ converges to $\theta_{x,y}$ in norm topology, as k tends to ∞. It follows from Lemma 1.5 that Δ maps $\mathcal{K}(\mathcal{M})$ into itself. By the assumption that every derivation of $\mathcal{K}(\mathcal{M})$ is inner, there exists an operator $T \in \mathcal{K}(\mathcal{M})$ such that $\Delta(K) = KT - TK$ for all $K \in \mathcal{K}(\mathcal{M})$. Suppose that $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Then for every $x \in \mathcal{M}$ and every i,

$$\begin{aligned} A\theta_{x,x_i}T - TA\theta_{x,x_i} &= \Delta(A\theta_{x,x_i}) = \Delta(A)\theta_{x,x_i} + A\Delta(\theta_{x,x_i}) \\ &= \Delta(A)\theta_{x,x_i} + A\theta_{x,x_i}T - AT\theta_{x,x_i}. \end{aligned}$$

This implies that $\Delta(A)\theta_{x,x_i} = (AT - TA)\theta_{x,x_i}$. Moreover, one has

$$\sum_{i=1}^k \langle x_i, x_i\rangle \Delta(A)x = \sum_{i=1}^k \langle x_i, x_i\rangle (AT - TA)x$$

for any positive integer k. Hence $\Delta(A) = AT - TA$ for all $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, i.e., Δ is an inner derivation. This completes the proof. □

The proof of Theorem 2.1 depends heavily on the commutativity of the C^*-algebra $\mathcal{A}$. In the next theorem, this commutativity condition will be removed. However, in this case we only infer that innerness of derivations for a concrete class of Hilbert C^*-module. Let $\mathcal{A}$ be a C^*-algebra. With the inner product of elements $x, y \in \mathcal{A}$ defined by $\langle x, y\rangle = xy^*$, the algebra $\mathcal{A}$ becomes a Hilbert $\mathcal{A}$-module. Denote by $L_n(\mathcal{A})$ the direct sum of n copies of the Hilbert module $\mathcal{A}$. Then $L_n(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module, whose inner product is given by the formula $\langle x, y\rangle = \sum_{i=1}^n \langle x_i, y_i\rangle$, where $x = x_1 \oplus \cdots \oplus x_n, y = y_1 \oplus \cdots \oplus y_n \in L_n(\mathcal{A})$.

Theorem 2.4 *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose that every derivation of $\mathcal{A}$ is inner. Then for any positive integer n, every derivation of $\mathrm{End}^*_{\mathcal{A}}(L_n(\mathcal{A}))$ is an inner derivation.*

Proof Since $\mathcal{A}$ is unital, $\theta_{1,1}$ is the identity operator on the Hilbert $\mathcal{A}$-module $\mathcal{A}$. Thus $\mathrm{End}^*_{\mathcal{A}}(\mathcal{A}) = \mathcal{K}(\mathcal{A})$. For a positive integer n, we can identify $\mathrm{End}^*_{\mathcal{A}}(L_n(\mathcal{A}))$ with the set of $n \times n$ matrices over $\mathrm{End}^*_{\mathcal{A}}(\mathcal{A})$. Hence $\mathrm{End}^*_{\mathcal{A}}(L_n(\mathcal{A})) = \mathcal{K}(L_n(\mathcal{A}))$. We write $M_n(\mathcal{A})$ for the C^*-algebra of $n \times n$ matrices over $\mathcal{A}$. From [12, Proposition 2.2.2], we know that there exists an isometric algebra *-isomorphism ϕ_n from $\mathcal{K}(L_n(\mathcal{A}))$ into $M_n(\mathcal{A})$ such that

$$\phi_n : \theta_{a_1\oplus\cdots\oplus a_n, b_1\oplus\cdots\oplus b_n} \longrightarrow \begin{bmatrix} a_1 b_1^* & \cdots & a_1 b_n^* \\ \vdots & \ddots & \vdots \\ a_n b_1^* & \cdots & a_n b_n^* \end{bmatrix}.$$

So it suffices to prove that every derivation of $M_n(\mathcal{A})$ is inner. To do this, we will proceed by induction. It should be mentioned that the techniques used here are similar to those in the proof of [17, Theorem 2], which characterizes the derivations of matrix algebras.

The conclusion holds clearly for $n = 1$, since every derivation of $\mathcal{A}$ is inner. Suppose that the conclusion is true for $n-1$ and that Δ is a derivation of $M_n(\mathcal{A})$. Let E_{ij} $(1 \le i, j \le n)$ be the $n \times n$ matrix units and E be the $n \times n$ identity matrix. We write $P_1 = E_{11}$, $P_2 = E - E_{11}$ and $\mathfrak{B}_{ij} = P_i M_n(\mathcal{A}) P_j$, where $1 \le i, j \le 2$. Then P_1 and P_2 are idempotents with $P_1P_2 = 0$, and $\mathfrak{B}_{11} = a_{11}E_{11}$, $\mathfrak{B}_{12} = \sum_{j=2}^n a_{1j}E_{1j}$, $\mathfrak{B}_{21} = \sum_{i=2}^n a_{i1}E_{i1}$, $\mathfrak{B}_{22} = \sum_{i,j=2}^n a_{ij}E_{ij}$, where all a_{ij} $(1 \le i, j \le n)$ are in $\mathcal{A}$. Obviously,

$$M_n(\mathcal{A}) = \mathfrak{B}_{11} \oplus \mathfrak{B}_{12} \oplus \mathfrak{B}_{21} \oplus \mathfrak{B}_{22}.$$

Since $\Delta(P_1) = \Delta(P_1)P_1 + P_1\Delta(P_1)$, we see that $P_1\Delta(P_1)P_1 = P_2\Delta(P_1)P_2 = 0$. Put $S_0 = P_1\Delta(P_1) - \Delta(P_1)P_1$. Define a mapping $\Delta_1 : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by

$$\Delta_1(A) = \Delta(A) - (AS_0 - S_0A), \quad A \in M_n(\mathcal{A}).$$

Then clearly, Δ_1 is a derivation such that $\Delta_1(P_1) = 0$. Because of $\Delta_1(E) = 0$, we have $\Delta_1(P_2) = 0$. Moreover, for any $A \in M_n(\mathcal{A})$, we have

$$\Delta_1(P_2AP_2) = \Delta_1(P_2)AP_2 + P_2\Delta_1(A)P_2 + P_2A\Delta_1(P_2) = P_2\Delta_1(A)P_2.$$

Accordingly, $\Delta_1|_{\mathfrak{B}_{22}}$, the restriction of Δ_1 to $\mathfrak{B}_{22}$, is a derivation of $\mathfrak{B}_{22}$. Since $\mathfrak{B}_{22}$ is isomorphic with $M_{n-1}(\mathcal{A})$, by the induction hypothesis, there exists an operator $T_0 \in \mathfrak{B}_{22}$ such that $\Delta_1(A_{22}) = A_{22}T_0 - T_0A_{22}$ for all $A_{22} \in \mathfrak{B}_{22}$. Define a new derivation $\Delta_2 : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by

$$\Delta_2(A) = \Delta_1(A) - (AT_0 - T_0A), \quad A \in M_n(\mathcal{A}).$$

Then clearly, $\Delta_2(E_{11}) = 0$ and $\Delta_2(A_{22}) = 0$ for every $A_{22} \in \mathfrak{B}_{22}$, and in particular, $\Delta_2(E_{22}) = 0$. Thus $\Delta_2(E_{12}) = \Delta_2(E_{11}E_{12}E_{22}) = E_{11}\Delta_2(E_{12})E_{22} = a_0E_{12}$ for some element $a_0 \in \mathcal{A}$. Define another new derivation $\Delta_3 : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by

$$\Delta_3(A) = \Delta_2(A) + A(a_0E_{11}) - (a_0E_{11})A, \quad A \in M_n(\mathcal{A}).$$

It is easily seen that $\Delta_3(E_{11}) = \Delta_3(E_{12}) = 0$, and $\Delta_3(A_{22}) = 0$ for all $A_{22} \in \mathfrak{B}_{22}$. Noting that $aE_{2j} \in \mathfrak{B}_{22}$ for $2 \le j \le n$ and $a \in \mathcal{A}$, we obtain

$$\Delta_3(aE_{1j}) = \Delta_3(E_{12}(aE_{2j})) = \Delta_3(E_{12})(aE_{2j}) + E_{12}\Delta_3(aE_{2j}) = 0.$$

Hence $\Delta_3(A_{12}) = 0$ for all $A_{12} \in \mathfrak{B}_{12}$. Thus for $2 \le i \le n$ and $b \in \mathcal{A}$, it follows that from $\Delta_3(E_{11}) = \Delta_3(E_{1i}) = \Delta_3(bE_{ii}) = 0$ that

$$\begin{aligned}\Delta_3(bE_{i1}) &= \Delta_3((bE_{i1})E_{11}) = \Delta_3(bE_{i1})E_{11} = \Delta_3(bE_{i1})E_{1i}E_{i1} \\ &= (\Delta_3(bE_{i1})E_{1i} + bE_{i1}\Delta_3(E_{1i}))E_{i1} \\ &= \Delta_3(bE_{ii})E_{i1} = 0,\end{aligned}$$

which implies that $\Delta_3(A_{21}) = 0$ for all $A_{21} \in \mathfrak{B}_{21}$. Also, for every $c \in \mathcal{A}$, from $cE_{12} \in \mathfrak{B}_{12}$ and $E_{21} \in \mathfrak{B}_{21}$ we see that $\Delta_3(cE_{11}) = \Delta_3((cE_{12})E_{21}) = 0$, i.e., $\Delta_3(A_{11}) = 0$ for all $A_{11} \in \mathfrak{B}_{11}$. Consequently, for any $A \in M_n(\mathcal{A})$, we have

$$\Delta_3(A) = \Delta_3(P_1AP_1) + \Delta_3(P_1AP_2) + \Delta_3(P_2AP_1) + \Delta_3(P_2AP_2) = 0.$$

We now can conclude that Δ is an inner derivation, which is implemented by $T_0 - a_0E_{11} + E_{11}\Delta(E_{11}) - \Delta(E_{11})E_{11}$. The proof is complete. □

In [18], Jing et al. proved that an additive mapping on a standard operator algebra is almost a derivation if it satisfies the expansion formula of derivations on pairs of elements with zero product. The authors in [19, 20] continued the investigation in this direction, who extended the result in [18] to prime rings, and obtained a similar result for generalized skew derivations of prime rings.

We will conclude by characterizing the linear $\mathcal{A}$-module homomorphisms on $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ which behave like derivations when acting on zero products. It should be mentioned that a C^*-algebra (in particular, $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$) is semiprime, but not prime.

Theorem 2.5 *Let $\mathcal{A}$ be a unital commutative C^*-algebra and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module with the property that there exist $x_0, y_0 \in \mathcal{M}$ such that $\langle x_0, y_0\rangle = 1$. Suppose that $\Delta : \mathrm{End}^*_{\mathcal{A}}(\mathcal{M}) \to \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is a linear $\mathcal{A}$-module homomorphism, such that $\Delta(AB) = \Delta(A)B + A\Delta(B)$ for every pair $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ with $AB = 0$. Then $\Delta(AB) = \Delta(A)B + A\Delta(B) - A\Delta(I)B$ for all $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. Particularly, if in addition $\Delta(I) = 0$, then Δ is a derivation. Here, 1 and I denote the unit of $\mathcal{A}$ and the identity operator on $\mathcal{M}$, respectively.*

Proof Let $P \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ be an arbitrary idempotent. For every $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$, it follows from $AP(I-P) = A(I-P)P = 0$ that

$$\Delta(AP)(I-P) + AP\Delta(I-P) = \Delta(AP(I-P)) = 0$$

and

$$\Delta(A-AP)P + (A-AP)\Delta(P) = \Delta(A(I-P)P) = 0.$$

Comparing these equalities, we obtain $\Delta(AP) = \Delta(A)P + A\Delta(P) - AP\Delta(I)$. If we take $A = I$, then $P\Delta(I) = \Delta(I)P$. Hence

$$\Delta(AP) = \Delta(A)P + A\Delta(P) - A\Delta(I)P. \tag{2.1}$$

Let $A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ and $x \in \mathcal{M}$. We claim that

$$\Delta(A\theta_{x,y_0}) = \Delta(A)\theta_{x,y_0} + A\Delta(\theta_{x,y_0}) - A\Delta(I)\theta_{x,y_0}. \tag{2.2}$$

In fact, we can take a positive number λ small enough, such that $\langle \lambda x + x_0, y_0\rangle$ is invertible in $\mathcal{A}$. Denote $x' = \lambda x + x_0$ and $a = \langle x', y_0\rangle^{-1}$. Then θ_{ax',y_0} is an idempotent. By the equality (2.1), it follows that $\Delta(A\theta_{ax',y_0}) = \Delta(A)\theta_{ax',y_0} + A\Delta(\theta_{ax',y_0}) - A\Delta(I)\theta_{ax',y_0}$. Since Δ is assumed to be an $\mathcal{A}$-module homomorphism, we get

$$\Delta(A\theta_{x',y_0}) = \Delta(A)\theta_{x',y_0} + A\Delta(\theta_{x',y_0}) - A\Delta(I)\theta_{x',y_0}.$$

This equality minus the following one

$$\Delta(A\theta_{x_0,y_0}) = \Delta(A)\theta_{x_0,y_0} + A\Delta(\theta_{x_0,y_0}) - A\Delta(I)\theta_{x_0,y_0}$$

will imply that (2.2) holds.

Let $A, B \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$. For any $x \in \mathcal{M}$, by the equality (2.2), we have

$$\Delta(AB\theta_{x,y_0}) = \Delta(AB)\theta_{x,y_0} + AB\Delta(\theta_{x,y_0}) - AB\Delta(I)\theta_{x,y_0}.$$

On the other hand,

$$\begin{aligned}\Delta(AB\theta_{x,y_0}) &= \Delta(A)\theta_{Bx,y_0} + A\Delta(\theta_{Bx,y_0}) - A\Delta(I)\theta_{Bx,y_0} \\ &= \Delta(A)B\theta_{x,y_0} + A\Delta(B)\theta_{x,y_0} + AB\Delta(\theta_{x,y_0}) \\ &\quad - AB\Delta(I)\theta_{x,y_0} - A\Delta(I)B\theta_{x,y_0}.\end{aligned}$$

Hence

$$\Delta(AB)\theta_{x,y_0} = \Delta(A)B\theta_{x,y_0} + A\Delta(B)\theta_{x,y_0} - A\Delta(I)B\theta_{x,y_0}.$$

By letting the two sides of this equation act on x_0, we obtain

$$\Delta(AB)x = (\Delta(A)B + A\Delta(B) - A\Delta(I)B)x$$

for all $x \in \mathcal{M}$, which implies $\Delta(AB) = \Delta(A)B + A\Delta(B) - A\Delta(I)B$. Obviously, if $\Delta(I) = 0$ then Δ is a derivation. We are done. □

Acknowledgements This work started while the first and the third named authors were visiting the Department of Mathematics, University of Central Florida in 2008 and 2009. They wish to thank the hospitality of the Mathematics Department during their visits. The authors also thank Professor Wu Jing for his helpful comments on this paper.

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