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# A Vertex Cover with Chorded 4-cycles

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**Abstract** Let $k$ be an integer with $k \geq 2$ and $G$ a graph with order $n > 4k$. We prove that if the minimum degree sum of any two nonadjacent vertices is at least $n + k$, then $G$ contains a vertex cover with exactly $k$ components such that $k - 1$ of them are chorded 4-cycles. The degree condition is sharp in general.

**Keywords** Degree condition, vertex-disjoint, chorded quadrilateral

**MR(2000) Subject Classification** 05C35, 05C38

## 1 Terminology and Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges and we use Bondy and Murty [1] for terminology and notation not defined here. Let $G = (V, E)$ be a graph; the order of $G$ be $|G| = |V|$ and its size be $e(G) = |E|$. A set of subgraphs is said to be vertex-disjoint or independent if no two of them have common vertex in $G$, and we use disjoint or independent to stand for vertex-disjoint throughout this paper. Let $G_1$ and $G_2$ be two subgraphs of $G$ or subsets of $V(G)$. If $G_1$ and $G_2$ have no common vertex in $G$, we define $E(G_1, G_2)$ to be the set of edges of $G$ between $G_1$ and $G_2$, and let $e(G_1, G_2) = |E(G_1, G_2)|$. Let $H$ be a subgraph of $G$ and $u \in V(G)$ a vertex. $N(u, H)$ is the set of neighbors of $u$ contained in $H$. We write $d(u, H) = d_H(u) = |N(u, H)|$. Clearly, $d(u, G)$ is the degree of $u$ in $G$, and we write $d(x)$ to replace $d(x, G)$. If there is no fear of confusion, we often identify a subgraph $H$ of $G$ with its vertex set $V(H)$. For a subset $U$ of $V(G)$, we denote by $G[U]$ the subgraph of $G$ induced by $U$ and write $d_H(U) = \sum_{x \in U} d_H(x)$ for a subgraph $H$ of $G$. Let $C$ be a cycle, we

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use $l(C)$ to denote the length of $C$. That is, $l(C) = |C|$. A Hamiltonian cycle of $G$ is a cycle which contains all vertices of $G$, and a Hamiltonian path of $G$ is a path of $G$ which contains every vertex in $G$. A cycle of length 4 is called a quadrilateral. If $S$ is a set of subgraphs of $G$, we write $G \supseteq S$. For a noncomplete graph $G$, let $\sigma_2(G) = \min\{d(x) + d(y) | xy \notin E(G)\}$; if $G$ is a complete graph, let $\sigma_2(G) := \infty$. In this manuscript, we always write $D$ to be the graph obtained from $K_4$ by removing exactly one edge. For a cycle $C$ of $G$, a chord of $C$ is an edge of $G - E(C)$ which joins two vertices of $C$. Throughout this paper, we call a cycle $C$ in $G$ a *chorded* 4-*cycle* if $C$ is a quadrilateral with at least one chord.

In his very excellent paper [2], Enomoto proposed the following interesting conjecture.

**Conjecture 1.1** ([2]) *Let $s$ and $k$ be two positive integers with $1 \le s \le k$ and $G$ be a graph with order $n \ge 3s + 4(k - s) + 3$. Suppose $\sigma_2(G) \ge n + s$. Then $G$ can be partitioned into $k + 1$ disjoint cycles $H_1, \dots, H_{k+1}$ satisfying $|H_i| = 3$ for $1 \le i \le s$ and $|H_i| \le 4$ for $s < i \le k$.*

It is probably the first step to specify the length of $|H_i|$ for $s < i \le k$ to solve Enomoto's conjecture. The following result obtained by Yan stated that the length of these cycles is four.

**Theorem 1.2** ([3]) *Let $s$ and $k$ be two positive integers with $1 \le s \le k$ and $G$ be a graph with order $n \ge 3s + 4(k - s) + 3$. Suppose $\sigma_2(G) \ge n + s$. Then $G$ contains $k$ disjoint cycles $H_1, \dots, H_k$ satisfying $|H_i| = 3$ for $1 \le i \le s$ for $1 \le i \le s$, and $|H_i| = 4$ for $s < i \le k$.*

In fact, by a tedious proof, we can improve the condition $n \ge 3s+4(k-s)+3$ of Theorem 1.2 to $n \ge 3s + 4(k - s) + 1$. With respect to Conjecture 1.1, Enomoto [2] verified the case when $s = k$. To our knowledge, the general case for Conjecture 1.1 is still open. It is worthy to cite the paper of Fujita [4], which solved the packing problem for disjoint $D$ and verified a conjecture proposed by Kawarabayashi [5].

**Theorem 1.3** ([4]) *Let $k$ be an integer with $k \ge 2$, and $G$ a graph of order $n \ge 4k + 1$. If $\sigma_2(G) \ge n + k$, then $G$ contains $k$ disjoint $D$.*

In this article, we consider the following problem: Given a graph $G$, when does $G$ have a vertex cover with exactly $k$ components which satisfy the specified condition? Many studies have been conducted regarding partitions of graphs into vertex-disjoint cycles, which contains specified elements, see the survey paper [6]. In particular, as a basic result, we cite a classical result of Brandt et al. [7].

**Theorem 1.4** ([7]) *Let $k$ be a positive integer and let $G$ be a graph of order $n \ge 4k$. If $\sigma_2(G) \ge n$, then $G$ has a 2-factor with exactly $k$ disjoint cycles.*

The goal of this article is to prove a more specific version of Theorem 1.4. We specify the structure of most components of a vertex cover with exactly $k$ components in a given graph $G$. The main purpose of this paper is to prove the following theorem.

**Theorem 1.5** *Let $k$ be an integer with $k \ge 2$. Suppose $G$ is a graph of order $n > 4k$ with $\sigma_2(G) \ge n + k$. Then $G$ contains a vertex cover with exactly $k$ disjoint cycles such that $k - 1$ of them are chorded 4-cycles.*

**Remark 1.6** The condition on $\sigma_2(G)$ is the best possible. Consider the graph $G = \overline{K_{k-1}} + (\overline{K_{\frac{n-k+1}{2}}} + \overline{K_{\frac{n-k+1}{2}}})$. Then $\sigma_2(G) = n + k - 1$, but we can not find a vertex cover with exactly $k$ components such that $k - 1$ of them are chorded 4-cycles. Note also the conclusion of Theorem 1.5 does not hold when $n = 4k$; this can be shown by $G = K_1 + (\overline{K_{\frac{3k}{2}}} + K_{\frac{5k}{2}-1})$.

Very recently, we obtain a result stating that except at most one cycle, the other $k-1$ cycles are chorded 4-cycles.

**Theorem 1.7** ([8]) *Suppose $G$ is a graph of order $n \geq 4k+3$ with $\sigma_2(G) \geq n+k$. Then $G$ contains a vertex cover with exactly $k+1$ disjoint cycles $C_1, \ldots, C_k, C_{k+1}$ such that $C_i$ are chorded quadrilateral for $1 \leq i \leq k-1$ and the length of $C_k$ is at most four.*

## 2 Lemmas

In the following, $G$ is a graph of order $n \geq 3$.

**Lemma 2.1** ([9]) *Let $P = x_1x_2\cdots x_m$ be a path of $G$ with $m \geq 2$ and $y \in V(G) - V(P)$. If $d(y, P) + d(x_m, P) \geq m+1$, then $G$ has a path $P'$ from $x_1$ to $y$ such that $V(P') = V(P) \cup \{y\}$. Furthermore, if $yx_1 \notin E(G)$ and $d(y, P) + d(x_m, P) \geq m$, then $G$ has a path $P'$ with vertex set $V(P') = V(P) \cup \{y\}$ whose end vertices are $y$ and $x_1$.*

**Lemma 2.2** *Let $P = x_1 \cdots x_p$ be a path with $p \geq 2$, $M = y_1z_1$ be an edge and $S$ be a subgraph in $G$ such that all of them are disjoint, where $S$ is isomorphic to $D$ or $K_4$. Suppose $e(\{x_1, x_p\} \cup M, S) \geq 11$; then $G[V(M \cup P \cup S)]$ contains two disjoint subgraphs $S'$ and $P'$ such that $P'$ is a path of order $p+1$, where $S'$ is isomorphic to $D$ or $K_4$.*

*Proof* For convenience, we write $V(S) = \{a, b, c, d\}$ so that $d_S(a) \geq d_S(b) \geq d_S(c) \geq d_S(d)$. Note that $d_S(a) = d_S(b) = 3$ and $d_S(c) = d_S(d) \geq 2$. Therefore,

$$11 \leq e(\{x_1, x_p\} \cup M, S) = e(\{x_p, y_1\}, S) + e(\{x_1, z_1\}, S)$$

and $e(\{x_p, y_1\}, S) \leq 8$, we may assume that $e(\{x_p, y_1\}, S) \geq 6$ and then $e(\{x_1, z_1\}, S) \geq 3$. Furthermore, we observe that $e(\{x_p, y_1\}, S) \leq 6$. Otherwise, it is easy to see that $G[V(M \cup S \cup P)]$ contains two required disjoint subgraphs. Then it follows that $e(\{x_p, y_1\}, S) = 6$ and $e(\{x_1, z_1\}, S) \geq 5$. If $d(y_1, S) = 4$, then we have nothing to prove as $d(x_p, S) \geq 2$. So, we assume $2 \leq d(y_1, S) \leq 3$ and then $d(x_p, S) \geq 3$.

**Case 1** $d(y_1, S) = 3$. Then $d(x_p, S) = 3$.

Suppose that $N(y_1, S) = \{a, b, c\}$. If $x_pd \in E(G)$, then $G[V(M \cup S \cup P)]$ contains two required subgraphs $S' = G[\{y_1, a, b, c\}]$ and $P' = P + d$. Therefore, $x_pd \notin E(G)$ and then $\{a, b, c\} = N(x_p, S)$. However, we observe $N(z_1, S) \cap N(x_1, S) = \emptyset$, which contradicts the fact that $e(\{x_1, z_1\}, S) \geq 5$. Hence, by symmetry, we may assume that $N(y_1, S) = \{d, b, c\}$. As $G[\{y_1, d, b, c\}] \supseteq D$, $x_1a \notin E(G)$ and $x_pa \notin E(G)$. Then $N(x_p, S) = \{d, b, c\}$. Note that $N(x_1, S) \cap N(z_1, S) \subseteq \{b\}$, which follows from $e(\{x_1, z_1\}, S) \geq 5$ that $\{z_1b, x_1b\} \subseteq E(G)$. Hence, $G[V(M \cup S \cup P)]$ contains two required disjoint subgraphs $S' = G[\{y_1, z_1, b, d\}]$ and $P' = P + c$.

**Case 2** $d(y_1, S) = 2$. Then $d(x_p, S) = 4$.

Suppose $N(y_1, S) = \{c, d\}$. If $d(z_1, S) \geq 3$, then we have nothing to prove. Hence, we may assume that $d(z_1, S) \leq 2$ and so $d(x_1, S) \geq 3$. By symmetry, say $cx_1 \in E(G)$. Then $G[V(M \cup S \cup P)]$ contains two required subgraphs $S' = G[\{x_p, a, b, d\}$ and $P' = x_{p-1} \cdots x_1cy_1$. Hence, by symmetry, we may assume that $N(y_1, S) = \{a, b\}$ or $N(y_1, S) = \{c, a\}$. In both cases, since $G[\{y_1, a, c, b\}] \supseteq D$, then we can choose $P' = P + d$. The proof is complete. $\square$

**Lemma 2.3** ([10]) *Let $P = x_1 \cdots x_k$ be a path of $G$ with $k \geq 3$. If $d(x_1, P) + d(x_k, P) \geq k$, then $G[V(P)]$ contains a cycle $C$ such that $V(C) = V(P)$.*

**Lemma 2.4** *Let $S$ and $P = x_1x_2 \cdots x_p$ be two disjoint subgraphs such that $S$ is isomorphic to $D$ or $K_4$ and $P$ be a path with order at least 2. Suppose $d(x_1, S) + d(x_p, S) \geq 6$, then $G[V(S \cup P)]$ contains a spanning cycle.*

*Proof* By contradiction. Label $V(S) = \{a, b, c, d\}$ such that $d_S(a) \geq d_S(b) \geq d_S(c) \geq d_S(d)$. If $d(x_1, S) = 4$ or $d(x_p, S) = 4$ holds, it is easy to check that $G[V(S \cup P)]$ contains a spanning cycle. A contradiction. Hence, by symmetry, we may assume that $d(x_1, S) = 3$ and so $d(x_p, S) = 3$. Furthermore, by the symmetric role of $c$ and $d$, we divide the proof into two cases:

Suppose $N(x_1, S) = \{a, b, c\}$. In this case, $x_pd \notin E(G)$, otherwise, $G[V(S \cup P)]$ contains a spanning cycle $C' = x_1 \cdots x_pdabcx_1$. A contradiction. Hence, $N(x_p, S) = \{a, b, c\}$. Then $G[V(S \cup P)]$ contains a spanning cycle $C' = x_1 \cdots x_pbdacx_1$. A contradiction again.

Suppose $N(x_1, S) = \{a, d, c\}$. Then, $x_pb \notin E(G)$, otherwise, $G[V(S \cup P)]$ contains a spanning cycle $C' = x_1 \cdots x_pbdacx_1$, a contradiction. Hence, $N(x_p, S) = \{a, d, c\}$ and so $G[V(S \cup P)]$ contains a spanning cycle $C' = x_1 \cdots x_pdbacx_1$, a final contradiction, which completes the proof. □

Since it is easy to check that the following lemma holds, we omit the proof.

**Lemma 2.5** *Let $S$ and $P = x_1x_2 \cdots x_p$ be two disjoint subgraphs such that $S$ is isomorphic to $D$ or $K_4$, and $P$ be a path with order at least 2. Suppose $d(x_1, S) + d(x_p, S) \geq 7$, then for each pair of distinct vertices $w_1, w_2 \in V(S)$, there is a hamiltonian path $P'$ of $G[V(S \cup P)]$ with two end-vertices $w_1$ and $w_2$.*

**Lemma 2.6** *Let $S$ and $P = x_1x_2 \cdots x_p$ be two disjoint subgraphs such that $S$ is isomorphic to $D$ or $K_4$, and $P$ be a path with order at least 2. Suppose $d(x_1, S) + d(x_p, S) \geq 5$, then for each $Q \subseteq V(S)$ with $|Q| = 3$, there exist two pairs of $w_1, w_2 \in Q$ such that $G[V(S \cup P)]$ contains a hamiltonian path $P'$ with two end-vertices $w_1$ and $w_2$.*

*Proof* Label $V(S) = \{a, b, c, d\}$ such that $d_S(a) \geq d_S(b) \geq d_S(c) \geq d_S(d)$. As $d(x_1, S) + d(x_p, S) \geq 5$, by symmetry, say $d(x_1, S) \geq 3$ and so $d(x_p, S) \geq 1$. Let us assume for the moment, that $d(x_1, S) = 4$. We conclude that $x_pc \notin E(G)$. Otherwise, we define

$$P' = \begin{cases} adx_1 \cdots x_pcb, & \text{if } Q \in \{\{a, b, c\}, \{a, b, d\}\}, \\ cx_p \cdots x_1abd, & \text{if } Q \in \{\{a, c, d\}, \{b, c, d\}\}. \end{cases}$$

Therefore, by symmetry, $x_pd \notin E(G)$, Then we may assume that $x_pb \in E(G)$, and we define $P'$ as follows:

$$P' = \begin{cases} bx_p \cdots x_1dac, & \text{if } Q \in \{\{a, b, c\}, \{b, c, d\}\}, \\ dx_1 \cdots x_pbca, & \text{if } Q \in \{\{a, b, d\}, \{a, c, d\}\}. \end{cases}$$

Hence, $d(x_1, S) = 3$ and so $d(x_p, S) \geq 2$. By the symmetric role of $c$ and $d$, we consider two cases.

**Case 1** $N(x_1, S) = \{a, b, c\}$. In this cases, we claim $x_pd \notin E(G)$, for otherwise, define

$$P' = \begin{cases} bdx_p \cdots x_1ac, & \text{if } Q \in \{\{a, b, c\}, \{b, c, d\}\}, \\ dx_p \cdots x_pcba, & \text{if } Q \in \{\{a, b, d\}, \{a, c, d\}\}. \end{cases}$$

Consequently, we may assume $x_pb \in E(G)$ as $d(x_p, S) \geq 2$, then define

$$P' = \begin{cases} dacx_1 \cdots x_pb, & \text{if } Q \in \{\{a,b,d\},\{b,c,d\}\}, \\ adbx_p \cdots x_1c, & \text{if } Q \in \{\{a,b,c\},\{a,c,d\}\}. \end{cases}$$

**Case 2** $N(x_1, S) = \{a, d, c\}$. We claim that $x_pb \notin E(G)$, for otherwise, define

$$P' = \begin{cases} bx_p \cdots x_1cad, & \text{if } Q \in \{\{a,b,d\},\{b,c,d\}\}, \\ adbx_p \cdots x_1c, & \text{if } Q \in \{\{a,b,c\},\{a,c,d\}\}. \end{cases}$$

Consequently, we may assume $x_pc \in E(G)$ as $d(x_p, S) \geq 2$. Then define

$$P' = \begin{cases} abdx_1 \cdots x_pc, & \text{if } Q \in \{\{a,b,c\},\{a,c,d\}\}, \\ bacx_p \cdots x_1d, & \text{if } Q \in \{\{a,b,d\},\{b,c,d\}\}. \end{cases}$$ □

## 3 Proof of Theorem 1.5

Let $G$ be a graph of order $n > 4k$ with $\sigma_2(G) \geq n + k$ and $k \geq 2$. Suppose that Theorem 1.5 is false. According to Theorem 1.3, $G$ contains $k$ vertex-disjoint subgraphs $S_1, \ldots, S_k$ such that $S_i$ is isomorphic $D$ or $K_4$ for each $i \in \{1, \ldots, k\}$. We choose $k$ disjoint $S_1, \ldots, S_k$ in $G$ so that

$$\text{the length of a longest path in } G - V\left(\bigcup_{i=1}^{k} S_i\right) \text{ is maximum.} \tag{3.1}$$

Let $P = x_1 \cdots x_p$ be a longest path of $G - V(\bigcup_{i=1}^{k} S_i)$. Subject to (3.1), we choose $k$ disjoint subgraphs $S_1, \ldots, S_k$ and $P$ in $G$ such that

$$\text{size of the maximum matching in } G - \left(V\left(\bigcup_{i=1}^{k} S_i\right) \cup V(P)\right) \text{ is maximum.} \tag{3.2}$$

Let $H = \bigcup_{i=1}^{k} S_i$, $F = G - V(H)$ and $|F| = f$. Furthermore, let $M = \{y_1z_1, \ldots, y_rz_r\}$ be a maximum matching of $F - V(P)$. Define $\mathcal{H}_1 = \{S_i : 1 \leq i \leq k\}$. By assumption, we suppose that $G[V(F \cup S_i)]$ contains no Hamiltonian cycle for each $S_i \in \mathcal{H}_1$ and $1 \leq i \leq k$. Our proof includes several claims.

For convenience, for $i = 1, \ldots, k$, we write $V(S_i) = \{a_i, b_i, c_i, d_i\}$ such that $d_{S_i}(a_i) \geq d_{S_i}(b_i) \geq d_{S_i}(c_i) \geq d_{S_i}(d_i)$. Note that $d_{S_i}(a_i) = d_{S_i}(b_i) = 3$ and $d_{S_i}(c_i) = d_{S_i}(d_i) \geq 2$.

**Claim 3.1** $p + 2r \geq f - 1$.

*Proof* To the contrary, suppose that $p+2r \leq f-2$. Let $w_1$ and $w_2$ be two nonadjacent vertices in $F - V(P) \cup V(M)$ subject to (3.2). Then $e(\{w_1, w_2\}, y_iz_i) \leq 2$ for each $i \in \{1, 2, \ldots, r\}$ by the maximality of $M$. We prove that $e(\{w_1, w_2\}, P) \leq p$. If $p = 1$, then by (3.1), we see that $e(\{w_1, w_2\}, P) = 0 < 1$. Thus, there remains the case that $p \geq 2$, by the maximality of $P$ and Lemma 2.1, we see that $e(\{w_1, w_2\}, P) \leq p$. Thus, $e(\{w_1, w_2\}, F) \leq p + 2r \leq f - 2$. It follows that

$$e(\{w_1, w_2\}, H) \geq n + k - (f - 2) = 5k + 2.$$

By pigeonhole principle, there exists $S_i \in \mathcal{H}_1$ such that $e(\{w_1, w_2\}, S_i) \geq 6$. Without loss of generality, say $d(w_1, S_i) \geq d(w_2, S_i)$. Then $d(w_1, S_i) \geq 3$ and $d(w_2, S_i) \geq 2$. We will show that

$G[V(S_i) \cup \{w_1, w_2\}]$ contains a subgraph $S_i'$ and an edge $e$ such that they are disjoint, where $S_i^{'}$ is isomorphic $D$ or $K_4$.

If $d(w_1, S_i) = 4$, then it is obvious as $d(w_2, S_i) \geq 2$. So we may assume that $N(w_1, S_i) = N(w_2, S_i)$ and $d(w_1, S_i) = d(w_2, S) = 3$. Without loss of generality, we may assume that $\{d_i, b_i, c_i\} = N(w_1, S_i)$ or $\{a_i, b_i, c_i\} = N(w_1, S_i)$. In both cases, we choose $S_i' = G[\{w_1, c_i, w_2, b_i\}]$ and $e = a_i d_i$.

In both cases, replace $S_i$ with $S_i'$ resulting in a contradiction to the maximality of $M$ while (3.1) still holds. Thus, $p + 2r \geq f - 1$. □

**Claim 3.2** $p \geq f - 1$.

*Proof* By contradiction, suppose that $p \leq f - 2$. According to Claim 3.1, we see that $M \neq \emptyset$. Since $P$ is a longest path in $F$, let $R = \{x_1, x_p, y_1, z_1\}$. By the maximality of $P$ and Lemma 2.1, we obtain $e(\{x_1, y_1\}, P) \leq p$ and $e(\{x_p, z_1\}, P) \leq p$. Note that $e(\{x_1, x_p\}, F - V(P)) = 0$. Thus, $e(R, F) \leq 2p + 2(f - p - 1) = 2f - 2$. As $x_1 y_1 \notin E(G)$ and $x_p z_1 \notin E(G)$, we obtain

$$e(R, H) \geq 2(n + k) - (2f - 2) = 10k + 2.$$

This implies that there exists some $S_i \in \mathcal{H}_1$ such that $e(R, S_i) \geq 11$. By Lemma 2.2, $G[V(S_i \cup P) \cup \{y_1, z_1\}]$ contains a subgraph $S_i' \supseteq D$ and a path $P'$ of order $p+1$ such that $S_i'$ and $P'$ are disjoint. Replacing $S_i$ with $S_i'$, we obtain a contradiction to (3.1). Thus, $p \geq f - 1$. □

**Claim 3.3** We can properly choose $S_1, \ldots, S_k$ such that $P$ is a Hamiltonian path of $F$.

*Proof* Otherwise, suppose $p < f$. By Claim 3.2, $p = f - 1$. Take $y \in V(F - P)$. By Lemma 2.1, $d(x_i, P) + d(y, P) \leq p$ for each $i \in \{1, p\}$. So, $d(x_i, F) + d(y, F) \leq p + d(y, F - P) \leq p + f - p - 1 = f - 1$ for each $i \in \{1, p\}$. It follows that $d(x_1, H) + d(x_p, H) + 2d(y, H) \geq 2(n + k) - 2(f - 1) = 10k + 2$. This implies that there exists $S_i \in \mathcal{H}_1$ such that $d(x_1, S_i) + d(x_p, S_i) + 2d(y, S_i) \geq 11$.

Now we will show that $G[V(S_i \cup F)]$ can be partitioned into a subgraph $S_i' \supseteq D$ and $P'$ of order $f$ such that they are disjoint, a contradiction. Clearly, $d(y, S_i) \geq 2$. If $d(y, S_i) = 4$, as $d(x_1, S_i) + d(x_p, S_i) \geq 3$, we may assume that $zx_1 \in E(G)$ with $z \in V(S_i)$. Then $G[V(S_i - z) \cup \{y\}] \supseteq S_i' \supseteq D$, which disjoints the path $P' = P + z$. Hence, we have $d(y, S_i) \leq 3$ and so $d(x_1, S_i) + d(x_p, S_i) \geq 5$. Without loss of generality, assume $d(x_1, S_i) \geq d(x_p, S_i)$. Then $d(x_1, S_i) \geq 3$ and $d(x_p, S_i) \geq 1$.

We claim that $d(y, S_i) \leq 2$. Otherwise, suppose $d(y, S_i) = 3$. We observe that $G[N(y, S_i) \cup \{y\}] \supseteq D$, thus, it follows that $N(y, S_i) = N(x_1, S_i)$ and so $d(x_p, S_i) \geq 2$. If $N(y, S_i) = \{a_i, b_i, c_i\}$, then $x_p d_i \notin E(G)$. If $a_i x_p \in E(G)$, then we can choose $S_i' = G[\{y, b_i, x_1, c_i\}]$ and $P' = P - x_1 + a_i d_i$. Hence, $a_i x_p \notin E(G)$ and so $b_i x_p \notin E(G)$ by symmetry. It follows that $d(x_p, S_i) \leq 1$. A contradiction. Therefore, by symmetry, we assume $N(y, S_i) = \{d_i, b_i, c_i\}$. Clearly, $x_p c_i \notin E(G)$ and $x_p d_i \notin E(G)$. Consequently, $N(x_p, S_i) = \{a_i, b_i\}$. Then we can choose $S_i^{'} = G[\{x_1, b_i, y, d_i\}]$ and $P' = P - x_1 + a_i c_i$ such that they are disjoint.

From the above arguments, we obtain $d(y, S_i) = 2$ and so $d(x_1, S_i) = 4$ and $d(x_p, S_i) \geq 3$. Furthermore, we observe that $N(y, S_i) = \{c_i, d_i\}$. As $d(x_p, S_i) \geq 3$, by symmetry, we may assume that $\{a_i, b_i, c_i\} \subseteq N(x_p, S_i)$ or $\{c_i, b_i, d_i\} \subseteq N(x_p, S_i)$. In both cases, we can choose $S_i' = G[\{x_p, a_i, b_i, c_i\}]$ and $P' = P - x_p + d_i y$ such that $S_i'$ and $P'$ are disjoint. This completes the proof of Claim 3.3. □

**Claim 3.4** $p \geq 2$.

*Proof* Otherwise, suppose $p = 1$. Label $P = z$. Then $n = 4k + 1$ by Claim 3.3. For each $S_i \in \mathcal{H}_1$, there exists at most one edge between $z$ and $\{c_i, d_i\}$, for otherwise, $G[V(S_i) \cup \{z\}]$ induces a spanning cycle, a contradiction. By symmetry, say $zc_i \notin E(G)$. Clearly, we may assume that $N(z, S_i) = \{a_i, b_i\}$ and $S_i \cong D$ if $d(z, S_i) = 2$. Recalling $\sigma_2(G) \geq n + k$, therefore, there exists $S_i \in \mathcal{H}_1$, say $S_1$ without loss of generality, such that $d(z, S_1) \geq 2$. This implies $d(z, S_1) = \{a_1, b_1\}$ and $S_1 \cong D$. Since $\{c_1, d_1, z\}$ is an independent set, we obtain

$$d(c_1, H - S_1) + d(d_1, H - S_1) + 2d(z, H - S_1) \geq 2\sigma_2(G) - 8 > 10(k - 1).$$

This implies that there exists $S_j \in \mathcal{H}_1 - S_1$, say $j = 2$, such that $d(c_1, S_2) + d(d_1, S_2) + 2d(z, S_2)) \geq 11$. We claim $d(c_1, S_2) \leq 2$. Otherwise, we can insert $c_1$ into $S_2$ and obtain a spanning cycle $S_2'$. Note that $G[\{z\} \cup S_1 - c_1] \supseteq D$, denoted by $S_1'$, then $G$ contains a vertex cover with exactly $k$ disjoint cycles $S_1', S_2', S_3, \ldots, S_k$, a contradiction. By the symmetry role of $c_1$, $d_1$ and $z$, $d(d_1, S_2) \leq 2$ and $d(z, S_2) \leq 2$. This gives $d(c_1, S_2) + d(d_1, S_2) + 2d(z, S_2) \leq 8$. A contradiction. The claim is complete. □

**Claim 3.5** If $p \geq 3$, then $G[V(P)]$ is Hamiltonian.

*Proof* On the contrary, suppose that $G[V(P)]$ is not Hamiltonian. Clearly, $x_1x_f \notin E(G)$ and $d(x_1, P) + d(x_f, P) \leq f - 1$ by applying Lemma 2.3 to $P$. Then $d(x_1, H) + d(x_f, H) \geq n + k - (f - 1) = 5k + 1$. This implies that there exists $S_i \in \mathcal{H}_1$, say $S_1$, such that $d(x_1, S_1) + d(x_f, S_1) \geq 6$. By Lemma 2.4, $G[V(S_1 \cup P)]$ contains a spanning cycle, denoted by $S_1'$. Then $G$ contains a vertex cover with exactly $k$ disjoint cycles $S_1', S_2, \ldots, S_k$, a contradiction. This completes the proof for Claim 3.5. □

From Claim 3.4 and Claim 3.5, we may assume that $x_1x_f \in E(G)$ throughout the rest of this paper.

**Claim 3.6** There exist $x_i \in V(P)$ and $S_t \in \mathcal{H}_1$ such that $d(x_i, S_t) \geq 2$.

*Proof* By contradiction. Suppose that Claim 3.6 is false. Since $\sigma_2(G) \geq n + k$, then by Ore's classical theorem [11], $G$ contains a hamiltonian cycle. Hence, there exist $x_i \in V(P)$ and $S_j \in \mathcal{H}_1$ such that $d(x_i, S_j) \geq 1$, $1 \leq i \leq f$ and $1 \leq j \leq k$. By Claim 3.4 and Claim 3.5, without loss of generality, say $i = j = 1$.

Suppose for the moment, $x_1c_1 \in E(G)$. As $G[V(S_1 \cup P)]$ does not contain a Hamiltonian cycle, then $e(x_f, S_1 - c_1) = 0$. By Lemma 2.1, if $d(d_1, P) + d(x_f, P) \geq f + 1$, then there exists a Hamiltonian path $d_1Px_1$ on $V(P) \cup \{d_1\}$ such that two end-vertices are $x_1$ and $d_1$. Consequently, $G[V(S_1 \cup P)]$ contains a Hamiltonian cycle $C' = d_1Px_1c_1b_1a_1d_1$, and so $G$ contains a vertex cover with $k$ cycles $C', S_2, \ldots, S_k$, a contradiction. Hence, $d(d_1, P) + d(x_f, P) \leq f$. Then $d(d_1, S_1) + d(x_f, S_1) = d(d_1) + d(x_f) - (d(d_1, P) + d(x_f, P)) - d(d_1, H - S_1) - d(x_f, H - S_1) \geq n + k - f - \sum_{i=2}^{k} |S_i| - (k - 1) = 5$. So, $d(x_f, S_1) \geq 5 - 3 = 2$ as $d(d_1, S_1) \leq 3$, a contradiction. Thus, $x_1c_1 \notin E(G)$ and $x_1d_1 \notin E(G)$ by symmetry. Then we can assume that $x_1a_1 \in E(G)$ by the symmetry role of $a_1$ and $b_1$, it follows that $x_fc_1 \notin E(G)$ and $x_fd_1 \notin E(G)$, namely, $d(x_f, S_1) \leq 2$. By Lemma 2.1 and our assumption that Theorem 1.5 is false, $d(x_f, P) + d(c_1, P) \leq f - 1$. However, we have $d(x_f, S_1) + d(c_1, S_1) \geq n + k - (f - 1) - \sum_{i=2}^{k} |S_i| - (k - 1) = 6$, then $d(x_f, S_1) \geq 6 - 3 = 3$, a contradiction. □

Since $G[V(P)]$ contains a Hamiltonian cycle by Claim 3.5, therefore, we may assume that $d(x_1, S_1) \geq 2$ by Claim 3.6. Next, we shall show that there exist some $x_i \in V(P)$ and $S_j \in \mathcal{H}_1$, such that $d(x_i, S_j) \geq 3$, $1 \leq i \leq f$ and $1 \leq j \leq k$. Otherwise, $d(x_1, S_1) = 2$, $d(x_2, S_j) \leq 2$ and $d(x_f, S_j) \leq 2$ for each $1 \leq j \leq k$. By symmetry, we must be in one of the following two cases:

**Case 1** $a_1 \in N(x_1, S_1)$. In this case, by our assumption, Theorem 1.5 is false, neither $c_1$ nor $d_1$ belongs to $N(x_2, S_1) \cup N(x_f, S_1)$. By Lemma 2.1, $d(c_1, P) + d(x_2, P) \leq f - 1$ and $d(d_1, P) + d(x_f, P) \leq f - 1$. Let $W = \{c_1, x_2, d_1, x_f\}$. Then

$$\sum_{x \in W} d(x, H - S_1) \geq 2(n + k) - 2(f - 1) - 6 - 4 = 10(k - 1) + 2,$$

thus, we can assume that there exists $S_2 \in \mathcal{H}_1 - S_1$, such that $\sum_{x \in W} d(x, S_2) \geq 11$. Since $d(x_i, S_2) \leq 2$ for each $i \in \{2, f\}$, we obtain $8 \geq d(c_1, S_2) + d(d_1, S_2) \geq 7$. By symmetry, say $d(c_1, S_2) = 4$, $d(d_1, S_2) \geq 3$ and $d(x_f, S_2) = 2$. Take $y' \in N(x_f, S_2) \cap N(d_1, S_2)$, clearly, $G[V(S_2 - y') \cup \{c_1\}] \supseteq S_1' \supseteq D$, consequently, we obtain a desired vertex cover of $G$: $S_1', x_f y' d_1 b_1 a_1 x_1 \cdots x_f, S_3, \ldots, S_k$, a contradiction.

**Case 2** $c_1 \in N(x_1, S_1)$. From the above case, we see that $S_1 \cong D$ and $N(x_1, S_1) = \{c_1, d_1\}$. Furthermore, by our assumption, Theorem 1.5 is false, neither $c_1$ nor $d_1$ belongs to $N(x_2, S_1) \cup N(x_f, S_1)$. By Lemma 2.1, $d(c_1, P) + d(x_2, P) \leq f$ and $d(d_1, P) + d(x_f, P) \leq f$. Let $W' = \{c_1, x_2, d_1, x_f\}$. Then

$$\sum_{x \in W'} d(x, H - S_1) \geq 2(n + k) - 2f - 4 - 4 = 10(k - 1) + 2,$$

thus, we can assume that there exists $S_2 \in \mathcal{H}_1 - S_1$, such that $\sum_{x \in W'} d(x, S_2) \geq 11$. Since $d(x_i, S_2) \leq 2$ for each $i \in \{2, f\}$, we obtain $8 \geq d(c_1, S_2) + d(d_1, S_2) \geq 7$. By symmetry, say $d(c_1, S_2) = 4$, $d(d_1, S_2) \geq 3$ and $d(x_f, S_2) = 2$. Take $y' \in N(x_f, S_2) \cap N(d_1, S_2)$, we observe $G[V(S_2 - y') \cup \{c_1\}] \supseteq S_1' \supseteq D$, then, we obtain a desired vertex cover of $G$: $S_1', x_f y' d_1 b_1 a_1 x_1 \cdots x_f, S_3, \ldots, S_k$, a contradiction once again.

From the above arguments, we may assume that $d(x_1, S_1) \geq 3$. Without loss of generality, we assume $\{a_1, b_1, c_1\} \subseteq N(x_1, S_1)$ (otherwise, $\{c_1, a_1, d_1\} \subseteq N(x_1, S_1)$, then we choose $b_1$ to replace the role of $d_1$ in the following proof and the proof is similar). It is obvious that $d(x_2, S_1) = 0$ and $d(x_f, S_1) = 0$ as $G[V(S_1 \cup P)]$ contains no spanning cycle. Furthermore, by Lemma 2.1, $d(x_2, P) + d(d_1, P) \leq f$ and $d(x_f, P) + d(d_1, P) \leq f$. Hence, $2d(d_1, H - S_1) + d(x_2, H - S_1) + d(x_f, H - S_1) \geq 2(n + k) - 2f - 6 > 10(k - 1)$. This implies that there exists $S_2 \in \mathcal{H}_1 - S_1$ such that $2d(d_1, S_2) + d(x_2, S_2) + d(x_f, S_2) \geq 11$. Note $d(x_2, S_2) + d(x_f, S_2) \leq 8$, thus, $d(d_1, S_2) \geq 2$. Without loss of generality, say $d(x_2, S_2) \geq d(x_f, S_2)$. Then $d(x_2, S_2) \geq 2$ as a consequence. Let $G_1 = G[V(S_1 \cup S_2 \cup P)]$ and define $S_1' = S_1 - d_1 + x_1$. Clearly, $S_1' \supseteq D$.

**Claim 3.7** $d_1$ has at most three neighbors in $V(S_2)$.

*Proof* By contradiction, suppose $d(d_1, S_2) = 4$. Then $G[V(S_2) \cup \{d_1\} - \{u\}]$ contains $D$ for each $u \in V(S_2)$. Define $P' = P - x_1$.

We first consider the case $c_2 x_2 \in E(G)$. We observe $f \geq 3$. Otherwise, $f = 2$. Then $d(x_2, S_2) \geq 2$ and we show that $G_1$ can be partitioned into $S_1'$ and $S_2'$, and so $G$ contains a desired vertex cover $S_1', S_2', S_3, \ldots, S_k$, where $S_2'$ are defined as follows: If $x_2 a_2 \in E(G)$, then choose $S_2' = x_2 c_2 b_2 d_1 d_2 a_2 x_2$. By symmetry, there remains the case $x_2 d_2 \in E(G)$, then choose

$S_2' = x_2c_2a_2d_1b_2d_2x_2$. Since $S_1' \supseteq D$ and Theorem 1.5 is false by our assumption, thus, $G[V(P' \cup S_2) \cup \{d_1\}]$ does not contains a spanning cycle. If $x_fa_2 \in E(G)$, then $G[V(P' \cup S_2) \cup \{d_1\}] \supseteq S_2' = x_fa_2d_1d_2b_2c_2x_2P'x_f$, a contradiction. So, $x_fa_2 \notin E(G)$ and then $x_fb_2 \notin E(G)$ by symmetry. If $x_fd_2 \in E(G)$, then $G[V(P' \cup S_2) \cup \{d_1\}] \supseteq S_2' = x_fd_2b_2a_2d_1c_2x_2P'x_f$, a contradiction again. Hence, $x_fd_2 \notin E(G)$. Note that $d(x_2, S_2) \geq 2$, we can also prove that $x_fc_2 \notin E(G)$ and so $d(x_f, S_2) = 0$. If $d(x_f, P') + d(c_2, P') \geq f$, then by Lemma 2.1, $G[V(P') \cup \{c_2\}]$ contains a spanning path $P''$ such that the end-vertices are $c_2$ and $x_2$. Recalling $d(x_2, S_2) \geq 2$, we may assume that $x_2a_2 \in E(G)$ or $x_2d_2 \in E(G)$ by symmetry. If the former holds, then $G_1$ can be partitioned into $S_1'$ and $S_2'' = c_2P''x_2a_2d_1d_2b_2c_2$, a contradiction. If the latter holds, then $G_1$ can be partitioned into $S_1'$ and $S_2'' = c_2P''x_2d_2d_1a_2b_2c_2$, a contradiction again. In both cases, $G$ contains a vertex cover with $k$ disjoint cycles $S_1', S_2'', S_3, \ldots, S_k$, a contradiction. Hence, $d(x_f, P') + d(c_2, P') \leq f - 1$ and so $d(c_2, G_1) + d(x_f, G_1) \leq f - 1 + 2 + 3 + 4 = f + 8$. Then $d(c_2, H - S_1 - S_2) + d(x_f, H - S_1 - S_2) \geq n + k - (f + 8) = 5(k - 2) + 2$, which implies that there exists $S_3 \in \mathcal{H}_1 - S_1 - S_2$, such that $d(c_2, S_3) + d(x_f, S_3) \geq 6$. By Lemma 2.4, $G[V(S_3 \cup P') \cup \{c_2\}]$ contains a spanning cycle, denoted by $S_3'$. As $S_2 - c_2 + d_1$ is a chorded 4-cycle, then $G$ contains a vertex cover with exactly $k$ disjoint cycles $S_1', S_2 - c_2 + d_1, S_3', S_4, \ldots, S_k$, a contradiction.

Hence, we may assume that $x_2c_2 \notin E(G)$ and $x_2d_2 \notin E(G)$ by symmetry. Then it follows that $\{a_2, b_2\} = N(x_2, S_2)$ and $d(x_f, S_2) \geq 1$. If $x_fc_2 \in E(G)$, then $G_1$ can be partitioned into $S_1'$ and $S_2'' = x_fP'x_2b_2a_2d_2d_1c_2x_f$, a contradiction. Hence, $x_fc_2 \notin E(G)$ and so $x_fd_2 \notin E(G)$. Therefore, by the symmetry role of $a_2$ and $b_2$, we may assume that $x_fa_2 \in E(G)$, then $G_1$ can be partitioned into $S_1'$ and $S_2'' = x_fP'x_2b_2c_2d_1d_2a_2x_f$, a contradiction once again. $\square$

By Claim 3.7, we obtain $d(x_2, S_2) + d(x_f, S_2) \geq 5$. Note that $d(d_1, S_2) \geq 2$. If $d(d_1, S_2) = 2$, take $w_1, w_2 \in N(d_1, S_2)$ with $w_1 \neq w_2$, then $d(x_2, S_2) + d(x_f, S_2) \geq 7$. By Lemma 2.5, $G[V(S_2) \cup P - x_1]$ contains a hamiltonian path $P'$ connecting $w_1$ and $w_2$, consequently, we can merge $d_1$ into $P'$ and obtain a spanning cycle $S_2'$. Then $G$ can be partitioned into $S_1', S_2', S_3, \ldots, S_k$, a contradiction. Therefore, we may assume that $d(d_1, S_2) = 3$ by Claim 3.7. As $d(x_2, S_2) + d(x_f, S_2) \geq 5$, by Lemma 2.6, for each $Q \subseteq V(S_2)$ with $|Q| = 3$, there exist two pairs $w_1, w_2 \in Q$ such that $G[V(S_2 \cup P) - \{x_1\}]$ contains a hamiltonian path $P'$ with two end-vertices $w_1$ and $w_2$.

Now, we are in a position to complete the proof. By symmetry, we consider two cases: Either $\{a_2, c_2, d_2\} = N(d_1, S_2)$ or $\{a_2, b_2, c_2\} = N(d_1, S_2)$. No matter how the neighbors of $d_1$ in $S_2$ are, by Lemma 2.6 and the symmetric role of $a_2c_2$ and $b_2c_2$, or $a_2c_2$ and $a_2d_2$, we can always assume that there exists a Hamiltonian path $P'$ of $G[V(S_2 \cup P) - \{x_1\}]$ with two end-vertices $c_2$ and $a_2$. Then $G_1$ can be partitioned into $S_1'$ and $S_2' = d_1c_2P'a_2d_1$, and so $G$ contains a desired vertex cover with exact components $S_1', S_2', S_3, \ldots, S_k$, a final contradiction completes the proof.

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