Acta Mathematica Sinica, English Series
Jun., 2011, Vol. 27, No. 6, pp. 1071–1078
Published online: May 15, 2011
DOI: 10.1007/s10114-011-8474-7
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Hamilton-type Gradient Estimates for a Nonlinear Parabolic Equation on Riemannian Manifolds

Bin QIAN
Department of Mathematics, Changshu Institute of Technology,
Changshu 215500*, P. R. China*
and
Institut de Mathématiques de Toulouse, Université de Toulouse, CNRS 5219*, France*
E-mail: *binqiancn@yahoo.com.cn*

Abstract Let (M, g) be a complete noncompact Riemannian manifold. In this note, we derive a local Hamilton-type gradient estimate for positive solution to a simple nonlinear parabolic equation

$$\partial_t u = \Delta u + au \log u + qu$$

on $M \times (0, \infty)$, where a is a constant and q is a C^2 function. This result can be compared with the ones of Ma (*JFA*, **241**, 374–382 (2006)) and Yang (*PAMS*, **136**, 4095–4102 (2008)). Also, we obtain Hamilton's gradient estimate for the Schödinger equation. This can be compared with the result of Ruan (*JGP*, **58**, 962–966 (2008)).

Keywords Nonlinear parabolic equations, Li–Yau inequalities, Harnack differential inequalities, gradient estimates

MR(2000) Subject Classification 58J05, 58J35

1 Introduction

Let (M, g) be a complete noncompact Riemannian manifold. Let us consider the nonlinear parabolic equation

$$\partial_t u = \Delta u + au \log u + qu, \tag{1.1}$$

where a is a constant, and q is a C^2 function on M. The motivation to study equation (1.1) is to understand the Ricci flow introduced by Hamilton (cf. [1, 2] and references therein). It is pointed out by Ma that it would be interesting to consider the local gradient estimate for positive solutions to the nonlinear evolution equation (1.1). For this purpose, Ma proved the following result (cf. Theorem 1 in [1]):

Theorem 1.1 (Ma [1]) *Let (M, g) be a complete noncompact Riemannian manifold of dimension n with Ricci curvature bounded below by the constant $-K := -K(2R)$, where $R > 0$ and*

Received September 24, 2008, accepted October 27, 2009
Supported by the China Scholarship Council of China (Grant No. 2007U13020) during the 2007–2009 Academic Year

$K(2R) \geq 0$, in the geodesic ball $B_p(2R)$ for some fixed point $p \in M$. Let u be a positive smooth solution to the elliptic equation

$$\Delta u + au \log u = 0, \ \ in \ M,$$

where $a < 0$ is a real constant. Let $f = \log u$ and $(f, 2f)$ be the maximum among f and $2f$. Then there are two uniform positive constants c_1 and c_2 such that

$$|\nabla f|^2 + a(f, 2f) \leq \frac{n((n+2)c_1^2 + (n-1)c_1^2(1 + R\sqrt{K}) + c_2)}{R^2} + 2n(K - a)$$

on $B_p(2R)$.

More recently, Yang has obtained the following result:

Theorem 1.2 (Yang [2]) *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Suppose the Ricci curvature of M is bounded below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the geodesic ball $B_p(2R)$ with radius $2R$ around $p \in M$. If $u(x,t)$ is a positive smooth solution to the equation (1.1) with $q \equiv b$ on $M \times [0, \infty)$, where a, b are constants, letting $f(x,t) = \log u(x,t)$, then*

(i) *If $a < 0$, we have for any $\alpha > 1$ and $0 < \delta < 1$,*

$$\begin{aligned} &|\nabla f|^2(x,t) + a\alpha f(x,t) + \alpha b - \alpha f_t(x,t) \\ &\leq \frac{n\alpha^2}{2\delta t} + \frac{n\alpha^2}{2\delta}\left[\frac{2\varepsilon^2}{R^2} + \frac{\nu}{R^2} - \frac{a}{2} + \frac{\varepsilon^2}{R^2}(n-1)(1 + R\sqrt{K}) \right. \\ &\quad \left. + \frac{K}{\alpha - 1} + \frac{n\alpha^2\varepsilon^2}{8(1-\delta)(\alpha-1)R^2}\right] \end{aligned}$$

on $B_p(R) \times (0, \infty)$, where $\varepsilon > 0$ and $\nu > 0$ are constants;

(ii) *If $a > 0$, we have for any $\alpha > 1$ and $0 < \delta < 1$,*

$$\begin{aligned} &|\nabla f|^2(x,t) + \alpha a f(x,t) + \alpha b - \alpha f_t(x,t) \\ &\leq \frac{n\alpha^2}{2\delta t} + \frac{n\alpha^2}{2\delta}\left[\frac{2\varepsilon^2}{R^2} + \frac{\nu}{R^2} + a + \frac{\varepsilon^2}{R^2}(n-1)(1 + R\sqrt{K}) \right. \\ &\quad \left. + \frac{K}{\alpha - 1} + \frac{n\alpha^2\varepsilon^2}{8(1-\delta)(\alpha-1)R^2}\right] \end{aligned}$$

on $B_p(R) \times (0, \infty)$, where $\varepsilon > 0$ and $\nu > 0$ are constants.

In [3], Hamilton proved a new gradient estimate for the heat equation, see also [4] for the heat equation with potential, see also some further references [5, 6].

Theorem 1.3 (Hamilton [3]) *Let u be a positive solution of the heat equation $u_t = \Delta u$ on compact manifold M without boundary,* Ricci $\geq -K, K \geq 0$. *Then*

$$\mathrm{e}^{-2Kt}|\nabla \log u| - (\log u)_t - \mathrm{e}^{2Kt}\frac{n}{2t} \leq 0. \tag{1.2}$$

Stimulating by the Hamilton's idea, in [4], Ruan considered the parameter α as a function of the time t for proving the Hamilton type gradient estimate for Schrödinger equation, while in [7], Li and Yau introduced a constant parameter α in the proof of the well-known Li–Yau differential Harnack inequality. Note that in [8], Bakry and Ledoux initially proved a logarithmic Sobolev form of the Li–Yau parabolic inequality, in which hints the above idea, say "parameter function method", see also [9] for a bit further development of this method in the Hypoelliptic model.

Now it is natural to ask whether this idea is still valid for the nonlinear equation. Here we give an affirmative answer. The main result is

Theorem 1.4 *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Suppose that the Ricci curvature of M is bounded below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the geodesic ball $B_p(2R)$ for some fixed point $p \in M$. Also assume the potential function q satisfies, for all $x \in B_p(2R)$,*

$$|\nabla q|(x) \leq \gamma(2R), \quad \Delta q(x) \geq -\theta(2R). \tag{1.3}$$

If $u(x,t)$ is a positive smooth solution to the equation (1.1) on $M \times [0,\infty)$, let $f(x,t) = \log u(x,t)$. Then

(i) *If $a \leq 0$, we have for any $0 < \delta < 1$, $\lambda > 0$,*

$$\begin{aligned}&\sup_{x\in B_p(R)} \big(\alpha(t)|\nabla f|^2 - (f_t - af - q)\big)\\ &\leq \frac{n}{2\alpha(t)\delta}\Big[\frac{2C^2+nC}{R^2} + \frac{C\sqrt{(n-1)K}}{R} - \frac{a-1/\lambda}{4} + \frac{nC^2}{8\alpha(t)(1-\delta)R^2(1-\alpha)}\Big]\\ &\quad + \frac{n}{2\alpha(t)\delta t} + \Big(\frac{n}{2\alpha(t)\delta}\Big)^{\frac12}\big((1-\alpha)\lambda\gamma^2(2R)+\theta(2R)\big)^{\frac12},\end{aligned} \tag{1.4}$$

on $B_p(R) \times (0,\infty)$, where $\alpha = \mathrm{e}^{-2Kt}$ and C is a positive constant.

(ii) *If $a \geq 0$, we have for all $0 < \delta < 1$,*

$$\begin{aligned}&\sup_{x\in B_p(R)} \big(\alpha(t)|\nabla f|^2 - (f_t - af - q)\big)\\ &\leq \frac{n}{2\alpha\delta}\Big[\frac{2C^2+nC}{R^2} + \frac{C\sqrt{(n-1)K}}{R} + a + \frac{1}{4\lambda} + \frac{nC^2}{8\alpha(1-\delta)R^2(1-\alpha)}\Big]\\ &\quad + \frac{n}{2\alpha\delta t} + \Big(\frac{n}{2\alpha\delta}\Big)^{\frac12}\big((1-\alpha)\lambda\gamma^2(2R)+\theta(2R)\big)^{\frac12},\end{aligned}$$

on $B_p(R) \times (0,\infty)$, where $\alpha = \mathrm{e}^{-2Kt}$ and C is a positive constant.

Remark 1.5 If $a = 0$, we get the Hamilton type gradient estimates for Schrödinger equation $\partial_t u = \Delta u - qu$. If we assume further that $q = 0$, the result of Theorem 1.4 is exactly local version of (1.2). Note that a similar estimate has been obtained by Ruan in [4]. He proved this by using the Li–Yau gradient estimate method, while here we use a bit modified method introduced by Negrin (cf. [10]).

Now we give an interesting application of the above theorem, see Corollary 1.2 in [2].

Corollary 1.6 *Let M be a complete noncompact Riemannian manifold of dimension n without boundary. Suppose that the Ricci curvature of M is nonnegative. If $u(x)$ is a positive smooth solution to the equation*

$$\Delta u + au\log u = 0, \quad \text{on } M,$$

then if $a < 0$, we have $u(x) \geq \mathrm{e}^{-n/8}$; and if $a > 0$, we have $u(x) \leq \mathrm{e}^{n/2}$ for all $x \in M$.

Note that if $a < 0$, the above estimate of u in [2] is $u(x) \geq \mathrm{e}^{-n/4}$. In fact, it can be improved to $u(x) \geq \mathrm{e}^{-n/8}$.

2 Proofs

Proof of Theorem 1.4 Let $u(x,t)$ be a positive smooth solution of the heat equation (1.1) on $M\times[0,\infty)$. Define $f=\log u$ and

$$F=t\{\alpha(t)|\nabla f|^2-(f_t-af-q)\}$$

where $\alpha=\mathrm{e}^{-2Kt}$. Thus we have

$$\begin{aligned}\Delta f&=f_t-af-q-|\nabla f|^2\\&=\frac{(\alpha(t)-1)}{\alpha}(f_t-af-q)-\frac{F}{t\alpha}\\&=(\alpha-1)|\nabla f|^2-\frac{F}{t}.\end{aligned}\tag{2.1}$$

We will use the following notations (cf. [11]):

$$\Gamma(f,f)=\frac{1}{2}(\Delta f^2-2f\Delta f)=|\nabla f|^2,\quad \Gamma_2(f,f)=\frac{1}{2}(\Delta\Gamma(f,f)-2\Gamma(f,\Delta f)).$$

Under the assumption Ricci $\geq -K(2R)$, we have, by Bochner formula (cf. [11])

$$\Gamma_2(f,f)\geq -K(2R)|\nabla f|^2+\frac{1}{n}|\Delta f|^2,\tag{2.2}$$

on $B_p(2R)$. Through a direct computation,

$$\begin{aligned}\Delta F&=t\left(\alpha(t)\Delta\Gamma(f,f)-(\Delta f_t-a\Delta f-\Delta q)\right)\\&=t\left(2\alpha(t)\Gamma_2(f,f)+2\alpha(t)\Gamma(f,\Delta f)-(\Delta f_t-a\Delta f-\Delta q)\right)\\&=t\left(2\alpha(t)\Gamma_2(f,f)-2\frac{\nabla f\nabla F}{t}+2\alpha(t)\nabla f\nabla f_t-2a(\alpha(t)-1)|\nabla f|^2-2(\alpha(t)-1)\nabla f\nabla q\right)\\&\quad-\frac{a}{\alpha(t)}F+t\bigg(-\frac{a^2(\alpha(t)-1)}{\alpha(t)}f+\frac{a(2\alpha-1)}{\alpha(t)}f_t-f_{tt}-\frac{a(\alpha(t)-1)}{\alpha(t)}q+q_t+\Delta q\bigg),\\\partial_tF&=\frac{F}{t}+t(\alpha'(t)|\nabla f|^2+2\alpha(t)\nabla f\nabla f_t-f_{tt}+af_t+q_t),\end{aligned}$$

thus, combining the conditions (2.2), (1.3) and the equation (2.1),

$$\begin{aligned}(\Delta-\partial_t)F=&\,t\bigg(2\alpha(t)\Gamma_2(f,f)-2\frac{\nabla f\nabla F}{t}+(a(1-\alpha(t))-\alpha'(t))|\nabla f|^2\\&+2(1-\alpha(t))\nabla f\nabla q+\Delta q\bigg)-\frac{F}{t}-aF\\\geq&\,t\bigg[\frac{2\alpha(t)}{n}\left((1-\alpha(t))|\nabla f|^2+\frac{F}{t}\right)^2-2\frac{\nabla f\nabla F}{t}+a(1-\alpha(t))|\nabla f|^2\\&+2(1-\alpha(t))\gamma(2R)|\nabla f|-\theta(2R)\bigg]-\frac{F}{t}-aF.\end{aligned}\tag{2.3}$$

Here we use the fact $\alpha(t)=\mathrm{e}^{-2Kt}$. Let η be a C^2 function on $[0,\infty)$ satisfying (cf. [7])

$$\eta(t)=1,\ \text{for } 0\leq t\leq 1,\quad \eta(t)=0,\ \text{for } t\geq 2,$$

and

$$0\leq\eta(t)\leq 1,\quad -C\eta^{\frac{1}{2}}\leq\eta'(t)\leq 0,\quad \eta''(t)\geq -C,\ \text{for all}\ \ t\geq 0.$$

Let $d(p,x)$ be the geodesic distance between p and x, and put $\psi(x)=\eta(\frac{d(p,x)}{R})$, for $x\in B_p(R)$. Then we have

$$|\nabla\psi|^2\le\frac{C^2\psi}{R^2}, \tag{2.4}$$

and

$$\Delta\psi(x)=\frac{\eta''|\nabla d(p,x)|^2}{R^2}+\frac{\eta'\Delta d(p,x)}{R}\ge-\frac{nC}{R^2}-\frac{C\sqrt{(n-1)K}}{R},\quad x\notin\mathrm{cut}(p), \tag{2.5}$$

where we have used the classic Laplacian comparison $\Delta d\le\frac{n-1}{d}+\sqrt{(n-1)K}$. For fixed $T>0$, suppose that ψF attains its maximum at the point $(x_0,t_0)\in B_p(2R)\times[0,T]$. According to the well known argument of Calabi [12], we can assume that x_0 is not in the cut locus of p. Then at (x_0,t_0), by the maximum principle, we have

$$\nabla(\psi F)=0,\quad \Delta(\psi F)\le 0,\quad \partial_t(\psi F)\ge 0. \tag{2.6}$$

It follows that

$$F\Delta\psi+\psi(\Delta-\partial_t)F-\frac{2F|\nabla\psi|^2}{\psi}\le 0,$$

at (x_0,t_0). Combining the estimates (2.4) and (2.5), we have

$$\psi(\Delta-\partial_t)F\le\left(\frac{2C^2+nC}{R^2}+\frac{C\sqrt{(n-1)K}}{R}\right)F. \tag{2.7}$$

Multiplying both sides of (2.3) by ψ, we have by (2.6),

$$\begin{aligned}\psi(\Delta-\partial_t)F\ge t\psi\Bigg[&\frac{2\alpha(t)}{n}\left((1-\alpha(t))|\nabla f|^2+\frac{F}{t}\right)^2+a(1-\alpha(t))|\nabla f|^2\\&-2(1-\alpha(t))\gamma(2R)|\nabla f|-\theta(2R)\Bigg]-\frac{2C|\nabla f|F\sqrt{\psi}}{R}\\&-\frac{\psi F}{t}-a\psi F,\quad \text{at } (x_0,t_0).\end{aligned} \tag{2.8}$$

Following Negrin [10], we denote

$$\mu=|\nabla f|^2(x_0,t_0)/F(x_0,t_0)\quad\text{and}\quad B:=\left(\frac{2C^2+nC}{R^2}+\frac{C\sqrt{(n-1)K}}{R}\right).$$

Combining (2.7) and (2.8), we have at (x_0,t_0),

$$\begin{aligned}BF&+\frac{2CF^{\frac32}\mu^{\frac12}\psi^{\frac12}}{R}+\frac{\psi F}{t}+a\psi F\\&\ge t\psi\left\{\frac{2\alpha(t)((1-\alpha(t))\mu t+1)^2F^2}{nt^2}+a(1-\alpha(t))\mu F\right.\\&\quad\left.-2(1-\alpha(t))\gamma(2R)\mu^{\frac12}F^{\frac12}-\theta(2R)\right\}.\end{aligned} \tag{2.9}$$

In the following, let us divide into two cases.

Case (i) $a\le 0$. Multiplying both sides of the above inequality (2.9) by $s\psi$ and using the fact that $\psi \in [0,1]$ and $a \le 0$, we obtain at (x_0, t_0),

$$\begin{aligned} Bt\psi F + \frac{2C(F\psi)^{\frac{3}{2}}\mu^{\frac{1}{2}}t}{R} + \psi F \ge{}& \frac{2\alpha(t)((1-\alpha)\mu t+1)^2(\psi F)^2}{n} + a(1-\alpha)\mu(\psi F)t^2 \\ &- 2(1-\alpha)\gamma(2R)t^2\mu^{\frac{1}{2}}(F\psi)^{\frac{1}{2}} - t^2\theta(2R) \\ \ge{}& \frac{2\alpha(t)((1-\alpha)\mu t+1)^2(\psi F)^2}{n} + (a-1/\lambda)(1-\alpha)\mu(\psi F)t^2 \\ &- (1-\alpha)\lambda t^2\gamma^2(2R) - t^2\theta(2R), \end{aligned} \tag{2.10}$$

where λ is a positive constant. For any fixed $\delta \in (0,1)$, at the point (x_0, t_0), we have

$$\frac{2C(F\psi)^{\frac{3}{2}}\mu^{\frac{1}{2}}t}{R} \le (1-\delta)\frac{2\alpha(t)(\psi F)^2((1-\alpha)\mu t+1)^2}{n} + \frac{n\psi F\mu t^2C^2/R^2}{2\alpha(t)(1-\delta)((1-\alpha)\mu t+1)^2}.$$

Denoting $\omega = \psi(x_0)F(x_0,t_0)$, equation (2.10) becomes, at (x_0,t_0),

$$A_1\omega^2 - 2A_2\omega - A_3 \le 0,$$

where

$$\begin{aligned} A_1 &= \frac{2\alpha(t)\delta((1-\alpha)\mu t+1)^2}{n}, \\ A_2 &= \frac{1}{2}\left(Bt + 1 + \frac{n\mu t^2C^2}{2\alpha(t)(1-\delta)R^2((1-\alpha)\mu t+1)^2} - (a-1/\lambda)(1-\alpha)\mu t^2\right), \\ A_3 &= (1-\alpha)\lambda t^2\gamma^2(2R) + t^2\theta(2R). \end{aligned}$$

We deduce that, at (x_0,t_0),

$$\omega \le \frac{2A_2}{A_1} + \left(\frac{A_3}{A_1}\right)^{\frac{1}{2}}. \tag{2.11}$$

Note that

$$\frac{2A_2}{A_1} \le \frac{n}{2\alpha(t)\delta}\left(1 + Bt - \frac{a-1/\lambda}{4}t + \frac{nC^2t}{8\alpha(t)(1-\delta)R^2(1-\alpha)}\right),$$

and

$$\left(\frac{A_3}{A_1}\right)^{\frac{1}{2}} \le \left(\frac{n}{2\alpha(t)\delta}\right)^{\frac{1}{2}}\left((1-\alpha)\lambda\gamma^2(2R) + \theta(2R)\right)^{\frac{1}{2}}t.$$

Substituting the above two inequalities into (2.11), we have

$$\begin{aligned} \omega &= \psi(x_0)F(x_0,t_0) \\ &\le \frac{n}{2\alpha(t_0)\delta}\left(1 + Bt_0 - \frac{a-1/\lambda}{4}t_0 + \frac{nC^2t_0}{8\alpha(t_0)(1-\delta)R^2(1-\alpha(t_0))}\right) \\ &\quad + \left(\frac{n}{2\alpha(t_0)\delta}\right)^{\frac{1}{2}}\left((1-\alpha(t_0))\lambda\gamma^2(2R) + \theta(2R)\right)^{\frac{1}{2}}t_0 \\ &\le \frac{n}{2\alpha(T)\delta}\left(1 + BT - \frac{a-1/\lambda}{4}T + \frac{nC^2T}{8\alpha(T)(1-\delta)R^2(1-\alpha(T))}\right) \\ &\quad + \left(\frac{n}{2\alpha(T)\delta}\right)^{\frac{1}{2}}\left((1-\alpha(T))\lambda\gamma^2(2R) + \theta(2R)\right)^{\frac{1}{2}}T, \end{aligned}$$

where the last inequality follows from the fact that $\frac{t}{\alpha(t)(1-\alpha(t))}$ with $\alpha(t) = \mathrm{e}^{-2Kt}$ is increasing in $t \in [0,T]$.

Since ψF reaches the maximum at (x_0, t_0), T is arbitrary and moreover $\psi|_{B_p(R)} = 1$, we have for all $t > 0$ and $\lambda > 0$,

$$\sup_{x\in B_p(R)} \big(\alpha(t)|\nabla f|^2 - (f_t - af - q)\big) \le \frac{n}{2\alpha(t)\delta}\bigg[\frac{2C^2+nC}{R^2} + \frac{C\sqrt{(n-1)K}}{R} - \frac{a-1/\lambda}{4} + \frac{nC^2}{8\alpha(t)(1-\delta)R^2(1-\alpha)}\bigg] + \frac{n}{2\alpha(t)\delta t} + \left(\frac{n}{2\alpha(t)\delta}\right)^{\frac{1}{2}} \big((1-\alpha)\lambda\gamma^2(2R)+\theta(2R)\big)^{\frac{1}{2}}.$$

Case (ii) $a \ge 0$. Multiplying both sides of the inequality (2.9) by $t\psi$ and using the fact that $\psi \in [0, 1]$ and $a \ge 0$, we obtain at (x_0, t_0),

$$Bt\psi F + \frac{2C(F\psi)^{\frac{3}{2}}\mu^{\frac{1}{2}}t}{R} + \psi F + at(\psi F) \ge \frac{2\alpha((\alpha-1)\mu t+1)^2(\psi F)^2}{n} - 2(1-\alpha)\gamma(2R)t^2\mu^{\frac{1}{2}}(F\psi)^{\frac{1}{2}} - t^2\theta(2R).$$

Doing as in Case (i), we can obtain, for $\alpha = \mathrm{e}^{-2Kt}$, $\lambda > 0$,

$$\sup_{x\in B_p(R)} \big(\alpha(t)|\nabla f|^2 - (f_t - af - q)\big) \le \frac{n}{2\alpha\delta}\bigg[\frac{2C^2+nC}{R^2} + \frac{C\sqrt{(n-1)K}}{R} + a + \frac{1}{4\lambda} + \frac{nC^2}{8\alpha(1-\delta)R^2(1-\alpha(t)\alpha)}\bigg] + \frac{n}{2\alpha\delta t} + \left(\frac{n}{2\alpha\delta}\right)^{\frac{1}{2}} \big((1-\alpha)\lambda\gamma^2(2R)+\theta(2R)\big)^{\frac{1}{2}}. \quad \square$$

Proof of Corollary 1.6 The proof follows easily from Theorem 1.4. □

Acknowledgements The author would like to thank the referees for their helpful suggestions.

References

[1] Ma, L.: Gradient estimates for a simple elliptic equation on non-compact Riemannian manifolds. *J. Funct. Anal.*, **241**, 374–382 (2006)

[2] Yang, Y. Y.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. *Proc. Amer. Math. Soc.*, **136**, 4095–4102 (2008)

[3] Hamiltom, R. S.: A matrix harnack estimate for the heat equation. *Comm. Anal. Geom.*, **1**, 113–126 (1993)

[4] Ruan, Q. H.: Gradient estimate for Schrödinger operators on manifolds. *J. Geom. Phys.*, **58**, 962–966 (2008)

[5] Li, J.: Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds. *J. Funct. Anal.*, **100**, 233–256 (1991)

[6] Ma, L., Zhao, L., Song, X. F.: Gradient estimate for the degenerate parabolic equation $u_t = \Delta F(u) + H(u)$ on manifolds. *J. Diff. Equ.*, **224**, 1157–1177 (2008)

[7] Li, P., Yau, S. T.: On the parabolic kernel of the Schrödinger operator. *Acta Math.*, **156**, 153–201 (1986)

[8] Bakry, D., Ledoux, M.: A logarithmic Sobolev form of the Li–Yau parabolic inequality. *Revista Mat. Iberoamericana*, **22**, 683–702 (2006)

[9] Bakry, D., Baudoin, F., Bonnefont, M., et al.: Subelliptic Li–Yau estimates on three dimensional model spaces. In: Potential Theory and Stochastics in Albac: Aurel Cornea Memorial Volume, 2009

[10] Negrin, E.: Gradient estimates and a Liouville type theorem for the Schrödinger operator. *J. Funct. Anal.*, **127**, 198–203 (1995)

[11] Bakry, D.: On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. In: New trends in stochastic analysis (Charingworth, 1994), Taniguchi symposium, World Sci. Publ. River Edge, NJ, 1997, 43–75

[12] Calabi, E.: An extension of E. Hopf's maximum principle with application to Riemannian geometry. *Duke Math. J.*, **25**, 45–56 (1958)