Acta Mathematica Sinica, English Series
May, 2012, Vol. 28, No. 5, pp. 957–968
Published online: December 7, 2011
DOI: 10.1007/s10114-011-0106-8
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

$*$-Regular Leavitt Path Algebras of Arbitrary Graphs

Gonzalo ARANDA PINO
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, 29071 *Málaga, Spain*
E-mail: *g.aranda@uma.es*

Kulumani RANGASWAMY
Department of Mathematics, University of Colorado, Colorado Springs, CO 80933, *USA*
E-mail: *ranga@uccs.edu*

Lia VAŠ
Department of Mathematics, Physics and Statistics, University of the Sciences in Philadelphia, Philadelphia, PA 19104, *USA*
E-mail: *l.vas@usp.edu*

Abstract If K is a field with involution and E an arbitrary graph, the involution from K naturally induces an involution of the Leavitt path algebra $L_K(E)$. We show that the involution on $L_K(E)$ is proper if the involution on K is positive-definite, even in the case when the graph E is not necessarily finite or row-finite. It has been shown that the Leavitt path algebra $L_K(E)$ is regular if and only if E is acyclic. We give necessary and sufficient conditions for $L_K(E)$ to be $*$-regular (i.e., regular with proper involution). This characterization of $*$-regularity of a Leavitt path algebra is given in terms of an algebraic property of K, not just a graph-theoretic property of E. This differs from the known characterizations of various other algebraic properties of a Leavitt path algebra in terms of graph-theoretic properties of E alone. As a corollary, we show that Handelman's conjecture (stating that every $*$-regular ring is unit-regular) holds for Leavitt path algebras. Moreover, its generalized version for rings with local units also continues to hold for Leavitt path algebras over arbitrary graphs.

Keywords Leavitt path algebra, $*$-regular, involution, arbitrary graph

MR(2000) Subject Classification 16D70, 16W10, 16S99

1 Introduction

Leavitt path algebras can be regarded as the algebraic counterparts of the graph C^*-algebras, the descendants from the algebras investigated by Cuntz in [1]. Leavitt path algebras can also be viewed as a broad generalization of the algebras constructed by Leavitt in [2] to produce rings without the Invariant Basis Number property.

The Leavitt path algebra $L_K(E)$ was introduced in the papers [3] and [4]. The algebra $L_K(E)$ was first defined for a row-finite graph E (countable graph such that every vertex emits only a finite number of edges) and a field K. Although their history is very recent, a flurry of

Received February 5, 2010, accepted February 22, 2011
The first author is partially supported by the Spanish MEC and Fondos FEDER through project MTM2007-60333, and by the Junta de Andalucía and Fondos FEDER, jointly, through projects FQM-336 and FQM-2467

activity has followed the papers [3] and [4]. The main directions of research include: characterization of algebraic properties of a Leavitt path algebra $L_K(E)$ in terms of graph-theoretic properties of E; study of the modules over $L_K(E)$; computation of various substructures (such as the Jacobson radical, the center, the socle and the singular ideal); investigation of the relationship and connections with $C^*(E)$ and general C^*-algebras; classification programs; study of the K-theory; and generalization of the constructions and results first from row-finite to countable graphs and finally, from countable to completely arbitrary graphs. For examples of each of these directions see [3, 5–11].

The base field K is naturally endowed with an involution $^-$ (the identity involution can always be considered in the absence of other possibilities). A given involution on K naturally induces an involution of a Leavitt path algebra $L_K(E)$. The presence of an involution on a ring yields some favorable features: the ring is isomorphic to its opposite ring and a certain dose of symmetry is present. For example, a left Rickart $*$-ring is also a right Rickart $*$-ring while this is not the case for one-sided Rickart rings. Also, consideration of a complex Leavitt path algebra $L_{\mathbb{C}}(E)$ as an algebra with involution (induced from the complex-conjugate involution on $\mathbb{C}$), brings $L_{\mathbb{C}}(E)$ a step closer to its analytic counterpart $C^*(E)$. These facts justify our interest in the study of the involution on a Leavitt path algebra.

Raeburn [12] showed that the involution on $L_K(E)$ is proper if the involution on K is positive-definite and E is a row-finite countable graph without sinks. We extend this result to arbitrary graphs (Proposition 3.3). We also show that the converse holds: if the induced involution on $L_K(E)$ is positive-definite for every (equivalently some) graph E, then the involution on K is positive-definite (Proposition 3.4).

In [13], Abrams and Rangaswamy characterized the (von Neumann) regular Leavitt path algebras $L_K(E)$ as precisely those with acyclic underlying graphs E. In light of our consideration of the involution on $L_K(E)$, we wonder when is $L_K(E)$ $*$-regular (regular with proper involution). In Theorem 4.3, we characterize $*$-regular Leavitt path algebras as exactly those with E acyclic and K that is proper up to a certain extent (determined by the least upper bound of the numbers of all paths that end at any given vertex of E). Note that we do not impose any conditions on the cardinality of E: we work with completely arbitrary graphs.

Most existing characterization theorems for Leavitt path algebras have the following form:

$L_K(E)$ *has* (*ring-theoretic*) *property* (P) *if and only if* E *has* (*graph-theoretic*) *property* (P').

Such theorems have been formulated and proven for a good number of ring-theoretic properties. For example, simple, purely infinite simple, exchange, semisimple, regular and other Leavitt path algebras have been characterized. It is interesting that the underlying field K did not play any role in those characterization theorems. Theorem 4.3, however, has a different form:

$L_K(E)$ *has property* (P) *if and only if* E *has property* (P') and K *has property* (P'').

In other words, Theorem 4.3 is the first characterization theorem that involves a ring-theoretic property of the field K as well. Moreover, the characterization of $L_K(E)$ that is positive-definite (Proposition 3.4) also has the above form that features the field K as well.

The paper is organized as follows. In Section 2 we recall the basic definitions, examples and properties of Leavitt path algebras, whereas in Section 3 we focus on the involution of $L_K(E)$

and prove Propositions 3.3 and 3.4. We devote Section 4 to the proof of the characterization theorem for the $*$-regular Leavitt path algebras (Theorem 4.3). Finally, in Section 5 we consider [11, Problem 48, p. 380] that we shall refer to "Handelman's conjecture". This conjecture is stating that every $*$-regular ring is unit-regular. We prove that Handelman's conjecture holds for Leavitt path algebras. In fact, we formulate a generalized version of the conjecture for rings with local units and show that it holds for Leavitt path algebras over arbitrary graphs.

2 Definitions and Preliminaries

We recall some graph-theoretic concepts, the definition and standard examples of Leavitt path algebras.

A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 (with no restriction on their cardinals) together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the *source* and *range* of e. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite and E is row-finite, then E^1 must necessarily be finite as well; in this case we say simply that E is *finite*.

A vertex which emits (receives) no edges is called a *sink* (*source*). A vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set. A *path* μ in a graph E is a finite sequence of edges $\mu = e_1 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$ are the *source* and *range* of μ, respectively, and n is the *length* of μ. We view the elements of E^0 as paths of length 0.

If μ is a path in E, with $v = s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph which contains no cycles is called *acyclic*.

Let K denote an arbitrary base field and E an arbitrary graph. The *Leavitt path K-algebra* $L_K(E)$ is the K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* \mid e \in E^1\}$ with the following relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$.

(P1) $s(e)e = er(e) = e$ for all $e \in E^1$.

(P2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.

(CK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.

(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every $v \in E^0$ that is neither a sink nor an infinite emitter.

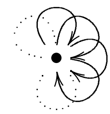
The first three relations are the path algebra relations. The last two are the so-called Cuntz–Krieger relations.

We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \cdots e_n$ is a path in E, we write μ^* for the element $e_n^* \cdots e_1^*$ of $L_K(E)$. With this notation, the Leavitt path algebra $L_K(E)$ can be viewed as a K-vector space span of $\{pq^* \mid p, q \,\text{are paths in}\, E\}$. (Recall that the elements of E^0 are viewed as paths of length 0, so that this set includes elements of the form v with $v \in E^0$.)

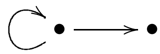
If E is a finite graph, then $L_K(E)$ is unital with $\sum_{v \in E^0} v = 1_{L_K(E)}$; otherwise, $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices of the graph.

Many well-known algebras can be realized as the Leavitt path algebra of a graph. The most basic graph configurations are shown below (the isomorphisms for the first three can be found in [3], the fourth in [14], and the last one in [15]).

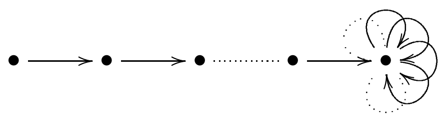
Example 2.1 The ring of Laurent polynomials $K[x, x^{-1}]$ is the Leavitt path algebra of the graph given by a single loop graph. Matrix algebras $\mathbb{M}_n(K)$ can be realized by the line graph with n vertices and $n-1$ edges. Classical Leavitt algebras $L(1, n)$ for $n \geq 2$ can be obtained by the n-rose — a graph with a single vertex and n loops. Namely, these three graphs are



The algebraic counterpart of the Toeplitz algebra T is the Leavitt path algebra of the graph having one loop and one exit:



Combinations of the previous examples are possible. For instance, the Leavitt path algebra of the graph



is $\mathbb{M}_n(L(1, m))$, where n denotes the number of vertices in the graph and m denotes the number of loops.

3 The Involution on a Leavitt Path Algebra

We recall some standard definitions first.

An *involution* $*$ on a ring R is an additive map $^* : R \to R$ that satisfies $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in R$. For any $a \in R$, the element a^* is called the *adjoint* of a. An element p in a ring with involution $*$ is called a *projection* if p is a self-adjoint ($p^* = p$) idempotent ($p^2 = p$).

If there is an involution defined on a ring R, then R is said to be a $*$-*ring*. If R is also an algebra over K with an involution $^-$, then R is a $*$-*algebra* if $(ax)^* = \overline{a} x^*$ for $a \in K$, and $x \in R$.

Let n be a positive integer. An involution $*$ on a ring R is said to be *n-proper* if

$$x_1^* x_1 + \cdots + x_n^* x_n = 0 \text{ implies } x_1 = \cdots = x_n = 0$$

for any n elements $x_1, \ldots, x_n$ in R. A ring with an n-proper involution will be referred to as an *n-proper* ring. This property is clearly left-right symmetric, since each element $x_i = a_i^*$ for some $a_i \in R$. A 1-proper involution is simply said to be *proper*.

The involution $*$ is said to be *positive-definite* if it is n-proper for every positive integer n. A $*$-ring with a positive-definite involution will be referred to as a *positive-definite* ring.

A field can have both an n-proper involution and an involution that is not n-proper. For example, consider the field $\mathbb{C}$: it is 2-proper (in fact positive-definite) for the conjugate involution ($a + \mathrm{i}b \mapsto a - \mathrm{i}b$) and not 2-proper for the identity involution. Also, the same involution can be n-proper and not $(n+1)$-proper (identity involution on $\mathbb{C}$ for $n = 1$).

Before we turn to Leavitt path algebras, let us recall one last fact about general $*$-rings. Recall that if R is a ring with involution $^-$, then the involution $^-$ induces the $*$-*transpose involution* on the ring $\mathbb{M}_n(R)$ of $n \times n$ matrices over R given by

$$A = (a_{ij}) \mapsto A^* = (\overline{a_{ji}}).$$

We believe that the following lemma is well known but we are not aware of a reference for it. For completeness, we provide a proof here.

Lemma 3.1 *Let n be a positive integer and let R be a ring with involution $^-$. Then the $*$-transpose involution on $\mathbb{M}_n(R)$ is proper if and only if the involution $^-$ in R is n-proper.*

Proof Assume that the involution $^-$ is n-proper in R. Suppose $A^*A = 0$ for some matrix $A = (a_{ij}) \in \mathbb{M}_n(R)$. Then the diagonal entries of the product A^*A are zero and so $\sum_{j=1}^{n} \overline{a_{ji}} a_{ji} = 0$ for every $i = 1, \dots, n$. Since $^-$ is n-proper, $a_{ij} = 0$ for all i, j. Hence $A = 0$.

Conversely, suppose $\sum_{i=1}^{n} \overline{a_i} a_i = 0$ for $a_i \in R$. Consider A to be the matrix of $\mathbb{M}_n(R)$ that has the elements $a_1, \dots, a_n$ in its first column and zeroes in the rest of its columns. Then $A^*A = 0$. Since the $*$-transpose involution is proper, $A = 0$. So $a_i = 0$ for every i. □

We turn now to Leavitt path algebras. Let K be a field with involution $^-$ and let E be an arbitrary graph. Recall that a typical element of the Leavitt path algebra $L_K(E)$ can be written as $\sum_{i=1}^{n} k_i p_i q_i^*$ where p_i and q_i are paths and $k_i \in K$. It is straightforward to see that the map $*$ given by

$$\left(\sum_{i=1}^{n} k_i p_i q_i^*\right)^* = \sum_{i=1}^{n} \overline{k_i} q_i p_i^*$$

defines the involution on $L_K(E)$ making it into a $*$-algebra.

If K is the field of complex numbers $\mathbb{C}$ and we consider the conjugate involution, the $*$-algebra structure of $L_{\mathbb{C}}(E)$ agrees with the *-algebra structure of $L_{\mathbb{C}}(E)$ (that is used for instance to see $L_{\mathbb{C}}(E)$ as a dense $*$-subalgebra of $C^*(E)$ as shown in [10, Theorem 7.3]).

As we will see in the next section, the $*$-regularity of a Leavitt path algebra $L_K(E)$ is closely related to the condition stating that the involution of $L_K(E)$ is n-proper or positive-definite. In this section, we characterize the positive-definiteness of $L_K(E)$.

The following proposition can be proved by easily adapting Raeburn's result [12, Lemma 1.3.1] to our notation and context.

Proposition 3.2 *Let E be a row-finite, countable graph without sinks. If the involution $^-$ on K is positive-definite, then the involution $*$ on $L_K(E)$ is proper.*

Proof The proof follows completely [12, Lemma 1.3.1]. One only needs to take into account that the axioms in [12] are given so that a path $e_1 e_2 \cdots e_n$ in [12] corresponds to the path $e_n e_{n-1} \cdots e_1$ here (i.e., the edges in [12] are multiplied so that the action of f precedes the action of e in the product ef, contrary to the action we consider here). Because of this difference, the assumptions of [12, Lemma 1.3.1] that $L_K(E)$ is column-finite with no sources, correspond exactly to our assumptions that $L_K(E)$ is row-finite with no sinks. □

It is noted in [12] that the condition that E does not have sinks (sources, in the terminology of that paper) can be avoided by using the so-called Yeend's trick. This assumption can also be avoided using a technique called the Desingularization Process. The added benefit of the desingularization is that it can help us also get rid of the row-finiteness assumption. The desingularization of a graph E is a new graph F obtained by adding a tail (more details can be found in [16] or [17]) to every sink or infinite emitter. The resulting graph F is a row-finite graph without sinks such that $L_K(E)$ embeds in $L_K(F)$ via an embedding that is a $*$-algebra homomorphism (see [17, Proposition 5.1] for more details).

Finally, the last remaining assumption (that the graph is countable) in Proposition 3.2 can be avoided by means of the Subalgebra Construction (see [13] for more details). We recall here the relevant concept E_F used in this construction. We shall use E_F in our main theorem too.

Let F be a finite set of edges in E. We define $s(F)$ (resp. $r(F)$) to be the sets of those vertices in E that appear as the source (resp. range) vertex of at least one element of F. The graph E_F is then defined as follows (see [13, Definition 2]):

$$E_F^0 = F \cup (r(F) \cap s(F) \cap s(E^1 \setminus F)) \cup (r(F) \setminus s(F)),$$
$$E_F^1 = \{(e,f) \in F \times E_F^0 \,|\, r(e) = s(f)\} \cup \{(e, r(e)) \,|\, e \in F \text{ with } r(e) \in (r(F) \setminus s(F))\},$$

where $s((x,y)) = x$ and $r((x,y)) = y$ for any $(x,y) \in E_F^1$.

The graph E_F is finite (see the comment after [13, Definition 2]). Also, by [13, Proposition 1], there is an algebra homomorphism $\theta : L_K(E_F) \to L_K(E)$. Furthermore, [13, Proposition 2] shows that for every finite set of elements S of $L_K(E)$, there is a subalgebra $B(S)$ of $L_K(E)$ containing S. The subalgebra $B(S)$ is of the form $L_K(E_F) \oplus (\bigoplus_{i=1}^m Kx_i)$ where F is a finite set of edges defined using S (see [13, p. 7]) and $x_i, i = 1, \dots, m$ is a finite set of vertices (defined in [13, p. 8]). By [13, Proposition 2], $L_K(E)$ is a directed union of subalgebras $B(S)$, where the S varies over all finite subsets of $L_K(E)$. Furthermore, and key to our current discussion, θ preserves the involution by the construction (as it can be seen from the proof of [13, Proposition 1]) so it is a $*$-algebra homomorphism.

Proposition 3.3 *Let E be an arbitrary graph. If the involution $^-$ on K is positive-definite, then the involution $*$ on $L_K(E)$ is proper.*

Proof We prove the claim first for the case when E is a countable. Let F be a desingularization of E. By the Desingularization Process, F is a row-finite graph without sinks and there is a $*$-algebra monomorphism $\phi : L_K(E) \to L_K(F)$. Now, suppose that $a^*a = 0$ in $L_K(E)$. Apply ϕ to get that $\phi(a)^*\phi(a) = 0$ in $L_K(F)$. Since F is row-finite and does not contain sinks, Proposition 3.2 can be applied and so $\phi(a) = 0$. Then $a = 0$ since ϕ is a monomorphism.

Now suppose that E is arbitrary and let $a \in L_K(E)$ be such that $a^*a = 0$. By the Subalgebra Construction, for the finite set $S = \{a, a^*\}$, there is a finite set of edges F and a finite number of vertices $x_i, i = 1, \dots, m$ such that the subalgebra $B(S)$ of $L_K(E)$ is of the form $L_K(E_F) \oplus (\bigoplus_{i=1}^m Kx_i)$ and $a, a^* \in B(S)$. Since $\bigoplus_{i=1}^m Kx_i$ is a direct summand in the previous equation for $B(S)$, we can actually attach a finite number of isolated vertices $v_1, \dots, v_m \notin E_F^0$ to the graph E_F so that we obtain a new finite graph G such that

$$L_K(G) \cong L_K(E_F) \oplus \left(\bigoplus_{i=1}^m Kv_i\right)$$

via a $*$-algebra isomorphism.

Since $B(S)$ is a subalgebra of $L_K(E)$, the equation $a^*a = 0$ holds in $B(S) \cong L_K(G)$. Apply the previous case to $L_K(G)$ in order to deal with possible sinks in G. Then we have $a = 0$. This finishes the proof. □

The last result of this section is a characterization of Leavitt path algebras which have positive-definite involutions in terms of the corresponding property in the field K.

Proposition 3.4 *Let K a field with involution. The following conditions are equivalent*:

(i) *The involution on K is positive-definite.*

(ii) *The involution on $L_K(E)$ is positive-definite for every graph E.*

(iii) *The involution on $L_K(E)$ is positive-definite for some graph E.*

Thus, if E is an arbitrary graph, $L_K(E)$ is positive-definite if and only if K is positive-definite.

Proof (i) $\Rightarrow$ (ii) Given E, let us consider the graph M_nE obtained from E by attaching a line of length $n-1$ to every vertex of E so that each line ends at the given vertex of the graph (see more details in [7]). The graph M_nE has the property that $\mathbb{M}_n(L_K(E))$ is isomorphic to $L_K(M_nE)$ as $*$-algebras by [7, Proposition 9.3].

By Proposition 3.3, we know that $L_K(M_nE)$ is $*$-proper. So, $\mathbb{M}_n(L_K(E))$ is $*$-proper. Now apply Lemma 3.1 to get that $L_K(E)$ is n-proper.

(ii) $\Rightarrow$ (iii) is a tautology.

(iii) $\Rightarrow$ (i) Suppose that $\sum_{i=1}^n \overline{k_i}k_i = 0$ for $k_i \in K$. Let E be a graph such that the involution on $L_K(E)$ is positive-definite. Let $v \in E^0$. Since v is a projection then $0 = (\sum_{i=1}^n \overline{k_i}k_i)v = \sum_{i=1}^n (k_iv)^*(k_iv)$ and therefore $k_iv = 0$ for all i by hypothesis. But E^0 is a set of linearly independent elements in $L_K(E)$ by [11, Lemma 1.5], so that $k_i = 0$ for all i, as needed. $\square$

4 $*$-Regular Leavitt Path Algebras

The (von Neumann) regular Leavitt path algebras $L_K(E)$ were characterized in [13] as those whose graphs E are acyclic. In light of the consideration of $L_K(E)$ as a ring with involution, we wonder which acyclic graphs have $L_K(E)$ that is $*$-regular. We provide an answer to this question in this section.

Recall that a ring R is (von Neumann) regular if for every $a \in R$ there exists $b \in R$ such that $aba = a$, or equivalently [18, Theorem 4.23], every right (resp. left) principal ideal is generated by an idempotent. This statement continues to hold if R is a ring with local units since $b \in bR$ (and $b \in Rb$) for all b in R so the principal right (and left) ideals of R have the same form as the principal right (left) ideals of a unital ring.

If R is a $*$-ring, the projections take over the role of idempotents. Thus, the concept of regularity for rings corresponds to $*$-regularity for $*$-rings: a $*$-ring R is said to be *$*$-regular* if every principal right ideal is generated by a projection. This definition naturally extends to rings with local units. Note that the condition of $*$-regularity is left-right symmetric since $aR = pR$ implies that $Ra^* = Rp$ for any $a \in R$ and a projection $p \in R$. So every principal left ideal of R is also generated by a projection in the case when every principal right ideal is.

A $*$-ring is $*$-regular if and only if it is regular and the involution $*$ is proper (see [19, Exercise 6A, §3]). In the next proposition we give a proof of this fact for rings with local units.

Proposition 4.1 *Let R be a ring with local units and with an involution $*$. Then R is $*$-regular if and only if R is regular and $*$ is proper.*

Proof If R is $*$-regular, then it is also regular because every projection is an idempotent. Now assume that $a^*a = 0$ for some $a \in R$. Then $aR = pR$ for some projection p so $a = pa$ ($a = px$ for some $x \in R$ implies that $pa = ppx = px = a$) and $p = ay$ for some $y \in R$. So, $a^* = a^*p = a^*ay = 0$. Hence $a = 0$. Thus $*$ is proper.

Conversely, suppose that R is regular and $*$ is proper. Since every principal right ideal is generated by an idempotent, it is enough to show that for an arbitrary idempotent x in R, $Rx^* = Rp$ for some projection $p \in R$. First observe that for any $x \in R$, $r_R(x) = r_R(x^*x)$ where $r_R(b)$ denotes the right annihilator of the element $b \in R$. This is because $x^*xy = 0$ implies that $(xy)^*xy = y^*(x^*xy) = 0$ so that $xy = 0$ for any $y \in R$. By the regularity of R, $Rx^*x = Rf$ for some idempotent $f \in R$. Thus $r_R(x) = r_R(x^*x) = r_R(f)$ and so the left annihilators $l_R(r_R(x))$ and $l_R(r_R(f))$ are also equal.

We claim that $Rx = l_R(r_R(x))$. To see this, first note that, since x is an idempotent, $r_R(x) = \{a - xa \mid a \in R\}$. So if $y \in l_R(r_R(x))$, then $y(a - xa) = 0$ for all $a \in R$, that is $ya = yxa$ for all $a \in R$. Since R is a ring with local units, there is an idempotent $u \in R$ such that $yu = y$ and $xu = x$. Hence $y = yu = yxu = yx \in Rx$. Thus $l_R(r_R(x)) \subseteq Rx$. Since the reverse inclusion is obvious, $Rx = l_R(r_R(x))$.

Similarly, $Rf = l_R(r_R(f))$. Thus $Rx = Rf = Rx^*x$. Hence $x = ax^*x$ for some $a \in R$. Let $p = ax^*$. We claim that p is a projection with $Rx^* = Rp$. To see this, note that $x = px$ and so $pp^* = pxa^* = xa^* = p^*$. Since $(pp^*)^* = pp^*$, we get $p = p^*$. From $pp^* = p^*$ and $p = p^*$ we have $p^2 = p$ and so p is a projection. From $p = ax^*$ and $x^* = x^*p$ we have that $Rx^* = Rp$. □

We turn to Leavitt path algebras now. For any vertex v in a graph E, let $\mu(v)$ denote the cardinality of the set of all the paths α in E with $r(\alpha) = v$ (including the trivial path v). With this notation, we recall the statement of [20, Lemma 3.4 and Proposition 3.5]. Let E be a finite acyclic graph and v a sink in E. The set

$$I_v = \left\{ \sum_i k_i \alpha_i \beta_i^* \,\middle|\, \alpha_i, \beta_i \text{ paths in } E \text{ with } r(\alpha_i) = r(\beta_i) = v, k_i \in K \right\}$$

is an ideal of $L_K(E)$ isomorphic to the matrix ring $\mathbb{M}_{\mu(v)}(K)$. If $\{v_1, \ldots, v_m\}$ is the set of all sinks in E, then $L_K(E) = \bigoplus_{i=1}^m I_{v_i} \cong \bigoplus_{i=1}^m \mathbb{M}_{\mu(v_i)}(K)$. Let us denote this isomorphism by ϕ and let us call it the *canonical isomorphism*.

From [20, Lemma 3.4 and Proposition 3.5] it can be seen that the restriction of ϕ on a direct summand I_v, for a vertex v, is given by $\phi(\sum_{i,j} k_{ij}\alpha_i\alpha_j^*) = (k_{ij}) \in \mathbb{M}_{\mu(v)}(K)$ where $i, j = 1, \ldots, \mu(v)$, α_i and α_j are paths ending at v and $k_{ij} \in K$.

Lemma 4.2 *Let E be a finite acyclic graph and let $\{v_1, \ldots, v_m\}$ be all the sinks in E. The canonical isomorphism $\phi : L_K(E) = \bigoplus_{i=1}^m I_{v_i} \to \bigoplus_{i=1}^m \mathbb{M}_{\mu(v_i)}(K)$ is a $*$-algebra isomorphism* (*with the standard involution on $L_K(E)$ and the $*$-transpose involution on the matrix algebras*).

Proof Since ϕ maps direct summands I_{v_i} on direct summands $\mathbb{M}_{\mu(v_i)}(K)$, it is enough if we prove the statement when E has only one sink v. If $\alpha_1, \ldots, \alpha_{\mu(v)}$ are all the different paths (including the trivial path) ending in v, then a typical element of $L_K(E) = I_v$ has the form $\sum_{i,j} k_{ij}\alpha_i\alpha_j^*$ for $i, j = 1, \ldots, \mu(v)$, α_i and α_j paths ending at v and $k_{ij} \in K$. Then we have

$$\phi\bigg(\bigg(\sum_{i,j} k_{ij}\alpha_i\alpha_j^*\bigg)^*\bigg) = \phi\bigg(\sum_{i,j} \overline{k_{ij}}\alpha_j\alpha_i^*\bigg) = \phi\bigg(\sum_{i,j} \overline{k_{ji}}\alpha_i\alpha_j^*\bigg) = (\overline{k_{ji}}) = (k_{ij})^*.$$

This proves the claim since $(k_{ij})^* = \big(\phi\big(\sum_{i,j} k_{ij}\alpha_i\alpha_j^*\big)\big)^*$. □

We finally have all the ingredients in hand to prove the main result of the paper.

Theorem 4.3 *Let E be an arbitrary graph, K be a field with involution $^-$ and let $\sigma = \sup\{\mu(v) : v \in E^0\}$ in case the supremum is finite or $\sigma = \omega$ otherwise. The following conditions are equivalent*:

(i) *$L_K(E)$ is $*$-regular.*

(ii) *$L_K(E)$ is regular and proper.*

(iii) *E is acyclic and K is n-proper for every finite $n \leq \sigma$.*

Proof (i) $\Leftrightarrow$ (ii) is Proposition 4.1.

(ii) $\Leftrightarrow$ (iii) By [13, Theorem 1], $L_K(E)$ is regular if and only if E is acyclic. So it is enough if we show, under the assumption that $L_K(E)$ is regular (equivalently, E is acyclic), that the involution $^-$ in K is n-proper for every finite $n \leq \sigma$ if and only if the involution $*$ in $L_K(E)$ is proper.

Now [13, Proposition 2 and Theorem 1] also state that, when E is acyclic, $L_K(E)$ is a directed union of subalgebras $B(S)$ where each $B(S) \overset{\theta}{\cong} L_K(E_F) \oplus (\bigoplus_{i=1}^m Kx_i)$ with E_F a finite acyclic graph constructed corresponding to various non-empty finite subsets F of edges in E. Moreover θ is a $*$-algebra isomorphism as we noted before. For a fixed F, E_F has a finite number of sinks. Let us denote them by $v_1, \ldots, v_k$. Then $L_K(E_F) \overset{\phi}{\cong} \bigoplus_{i=1}^k \mathbb{M}_{\mu_{E_F}(v_i)}(K)$ as $*$-algebras by Lemma 4.2. Thus, the involution $*$ in $L_K(E)$ is proper if and only if the $*$-transpose involution is proper in $\mathbb{M}_{\mu_{E_F}(v_i)}(K)$ with $v_i \in E_F$ for all the various graphs E_F corresponding to each $B(S)$ in the stated directed system of subalgebras of $L_K(E)$.

We distinguish two situations.

Case 1 Suppose that σ is infinite. Then either $\mu(v)$ is infinite for some vertex v or for every positive integer n, there is a vertex v_n with $\mu_E(v_n)$ an integer larger than n. In either case for each integer $n > 1$, we can choose a vertex v_n and a finite subset F_n of edges that appear in the $n-1$ distinct paths (other than the trivial path v_n) ending in v_n. The vertex v_n is a sink in E_{F_n} by the definition of the graph E_{F_n}.

Moreover, $e_1 \cdots e_k$ is a path of length k in E ending in v_n if and only if the path in E_{F_n} given by $(e_1, e_2)(e_2, e_3) \cdots (e_k, v_n)$ has length k and ends in v_n. Thus, v_n is a sink in E_{F_n} with $\mu_{E_{F_n}}(v_n) = n$. The graph E_{F_n} is finite acyclic (by [13, Lemma 1]). So, $L_K(E_{F_n})$ contains the ideal $I_{v_n} \cong \mathbb{M}_n(K)$.

Since this holds for every n, the involution $*$ is proper in each subalgebra $B(S)$ if and only if the $*$-transpose involution is proper in $\mathbb{M}_n(K)$ for each positive integer n. This is equivalent, by Lemma 3.1, to the statement that the involution $^-$ in K is n-proper for every n, that is, that K is positive-definite.

Case 2 Suppose that σ is finite, say $\sigma = n$ for some positive integer n. If $n = 1$, every vertex in E is isolated and $L_K(E)$ is isomorphic to $\bigoplus_{v \in E^0} Kv$ where $Kv \cong K$. Both of those algebras are proper, so we are done.

Suppose $n > 1$ and let v be a vertex for which $\mu(v) = n$. Let F_v be the non-empty finite set of edges in all the $\mu_E(v) - 1$ nontrivial paths ending in v. As noted in Case 1, v is a sink (and in this case, the only sink) in the finite acyclic graph E_{F_v} and, moreover, $\mu_{E_{F_v}}(v) = \mu_E(v)$.

So by Lemma 4.2, we have $L_K(E_{F_v}) \cong \mathbb{M}_{\mu_E(v)}(K)$ as $*$-algebras. Moreover, as n is the least upper bound of $\{\mu(v) : v \in E^0\}$, then all matrices $\mathbb{M}_{\mu_{E_{F_i}}(v_i)}(K)$ appearing in all other subalgebras $B(S)$ for various other finite subsets of edges F_i and sinks v_i will all have order that is less than or equal to n.

Therefore the involution $*$ is proper in each $B(S)$ if and only if the $*$-transpose involution is proper in $\mathbb{M}_{\mu_E(v)}(K)$. Since $\mu(v) = n$, the last statement holds exactly when the involution $^-$ in K is n-proper, again by Lemma 3.1. This finishes the proof. □

It is interesting to point out that the presence of involution gives a more prominent role to the field K than it had in the previous characterization theorems (e.g., simplicity [3], purely infinite simplicity [15], finite-dimensionality [20], just to cite a few). In particular, Theorem 4.3 also contrasts the characterization of regularity from [13] that was independent of the field K.

We further illustrate this behavior with an easy example. If E is the graph

$$\bullet \longrightarrow \bullet\ ,$$

then $L_K(E) \cong \mathbb{M}_2(K)$ as $*$-algebras for any field K. If $K = \mathbb{R}$ with the identity involution, $L_{\mathbb{R}}(E)$ is $*$-regular because $\mathbb{R}$ is positive-definite. However if $K = \mathbb{C}$ with the identity involution, then $L_{\mathbb{C}}(E) \cong \mathbb{M}_2(\mathbb{C})$ is regular but it is *not* $*$-regular (since the identity involution in $\mathbb{C}$ is not 2-proper). Furthermore, if $K = \mathbb{C}$ with the conjugate involution, then $L_{\mathbb{C}}(E)$ *is* $*$-regular, because the conjugation of complex numbers is positive-definite.

Also, since the identity involution on a field of characteristic $n > 0$ is not n-proper, the properness (thus also $*$-regularity) of a Leavitt path algebra over such field depends on the characteristic of the field. This fact also brings the field characteristic into spotlight.

Let us note the following corollary of Theorem 4.3.

Corollary 4.4 *Let K be a field with involution. The following conditions are equivalent*:

(i) *The involution on K is positive-definite.*

(ii) *$L_K(E)$ is $*$-regular for every acyclic graph E.*

Proof (i) ⇒ (ii) follows directly from Theorem 4.3.

(ii) ⇒ (i) Let us assume that $L_K(E)$ is $*$-regular for every acyclic E. Consider the line of length $n-1$ (see the second graph in Examples 2.1). The Leavitt path algebra of this graph is isomorphic to $\mathbb{M}_n(K)$. From the assumption that this algebra is $*$-regular, we obtain that K is n-proper by Lemma 3.1. Since this holds for every n, K is positive-definite. □

It is also interesting to note that the two equivalences of Corollary 4.4 parallel the first two equivalences of Proposition 3.4. The last equivalence of Proposition 3.4 in the $*$-setting would have the form: "$L_K(E)$ is $*$-regular for some acyclic graph E". However, this statement is weaker than the other two equivalences in Corollary 4.4 so we do not have complete analogy with Proposition 3.4. To see this, consider a graph consisting of a single vertex and the complex numbers with the identity involution. The Leavitt path algebra of this graph is $*$-regular but the field is not positive-definite.

5 Handelman's Conjecture for Leavitt Path Algebras

We close this paper by pointing out that Handelman's conjecture has a positive answer for the family of Leavitt path algebras of arbitrary graphs. The conjecture can be stated as follows.

Conjecture 5.1 (Handelman [21, Problem 48, p. 380]) *Every $*$-regular ring is unit-regular.*

This conjecture assumes that the ring is unital. First, we note that it is true for unital Leavitt path algebras. Let us assume that a unital $L_K(E)$ is $*$-regular. Then E is acyclic by Theorem 4.3. Then we have that $L_K(E)$ is unit-regular by [13, Theorem 2].

To prove that the conjecture remains true for Leavitt path algebras of arbitrary graphs, we adapt the notion of unit-regularity for rings with local units, as was done in [13] for instance.

Recall that a ring R with identity is said to be *unit-regular* if for each $a \in R$, there is a unit (an invertible element) u such that $aua = a$. If R is a ring with local units, then R is called *locally unit-regular* if for each $a \in R$ there is an idempotent (a local unit) v and local inverses u, u' such that $uu' = v = u'u$, $va = av = a$ and $aua = a$.

Clearly, a unit-regular (unital) ring is locally unit-regular (take the idempotent v from the definition of locally unit-regular to be the identity). Conversely, if a ring with identity is locally unit-regular, then it is unit-regular (see also [13, Lemma 3 (1)]). To see this, let $a \in R$. Then there is an idempotent v and local inverses u, u' in vRv such that $uu' = v = u'u$, $va = av = a$ and $aua = a$. Then $w = u + (1 - v)$ and $w' = u' + (1 - v)$ satisfy $ww' = 1 = w'w$ and $a = awa$. Hence R is unit-regular.

Corollary 5.2 *Let E be an arbitrary graph and let K be a field with involution. Suppose $L_K(E)$ is $*$-regular. Then*

(i) *$L_K(E)$ is locally unit-regular.*

(ii) *If $L_K(E)$ is a unital ring, then $L_K(E)$ is unit-regular.*

Proof (i) If $L_K(E)$ is $*$-regular, we have that E is acyclic by Theorem 4.3. Then $L_K(E)$ is locally unit-regular by [13, Theorem 2].

(ii) is a consequence of the fact that every unital locally unit-regular ring is unit-regular. □

Let us also note that the converse of Handelman's conjecture is not true. The examples of unit-regular and not $*$-regular rings can be found in the class of Leavitt path algebras as well. For instance, $\mathbb{M}_2(\mathbb{C})$ with the identity involution on $\mathbb{C}$ is such an example: we know that it is not $*$-regular but it is unit-regular (as a semisimple ring, see [21, p. 38]).

Acknowledgements The authors thank Gene Abrams for his valuable discussions during the preparation of this paper.

References

[1] Cuntz, J.: Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, **57**, 173–185 (1977)

[2] Leavitt, W. G.: Modules without invariant basis number. *Proc. Amer. Math. Soc.*, **8**, 322–328 (1957)

[3] Abrams, G., Aranda Pino, G.: The Leavitt path algebra of a graph. *J. Algebra*, **293**(2), 319–334 (2005)

[4] Ara, P., Moreno, M. A., Pardo, E.: Nonstable K-theory for graph algebras. *Algebra Represent. Theory*, **10**, 157–178 (2007)

[5] Aranda Pino, G., Pardo, E., Siles Molina, M.: Exchange Leavitt path algebras and stable rank. *J. Algebra*, **305**(2), 912–936 (2006)

[6] Aranda Pino, G., Martín Barquero, D., Martín González, C., et al.: The socle of a Leavitt path algebra. *J. Pure Appl. Algebra*, **212**(3), 500–509 (2008)

[7] Abrams, G., Tomforde, M.: Isomorphism and Morita equivalence of graph algebras. *Trans. Amer. Math. Soc.*, **363**(7), 3733–3767 (2011)

[8] Abrams, G., Ánh, P. N., Louly, A., et al.: The classification question for Leavitt algebras. *J. Algebra*, **320**, 1983–2026 (2008)

[9] Ara, P., Brustenga, M., Cortiñas, G.: K-theory for Leavitt path algebras. *Münster J. Math.*, **2**, 5–33 (2009)

[10] Tomforde, M.: Uniqueness theorems and ideal structure for Leavitt path algebras. *J. Algebra*, **318**(1), 270–299 (2007)

[11] Goodearl, K. R.: Leavitt path algebras and direct limits. *Contemp. Math.*, **480**, 165–187 (2009)

[12] Raeburn, I.: Chapter in "Graph Algebras: Bridging the Gap between Analysis and Algebra" (G. Aranda Pino, F. Perera, M. Siles Molina, eds.), ISBN: 978-84-9747-177-0, University of Málaga Press, Málaga, Spain, 2007

[13] Abrams, G., Rangaswamy, K. M.: Regularity conditions for arbitrary Leavitt path algebras. *Algebr. Represent. Theory*, **13**, 319–334 (2010)

[14] Siles Molina, M.: Algebras of quotients of Leavitt path algebras. *J. Algebra*, **319**(12), 5265–5278 (2008)

[15] Abrams, G., Aranda Pino, G.: Purely infinite simple Leavitt path algebras. *J. Pure Appl. Algebra*, **207**(3), 553–563 (2006)

[16] Drinen, D., Tomforde, M.: The C^*-algebras of arbitrary graphs. *Rocky Mountain J. Math.*, **35**(1), 105–135 (2005)

[17] Abrams, G., Aranda Pino, G.: The Leavitt path algebras of arbitrary graphs. *Houston J. Math.*, **34**(2), 423–442 (2008)

[18] Lam, T. Y.: A First Course on Noncommutative Rings, Springer-Verlag, New York, 1991

[19] Berberian, S. K.: Baer $*$-Rings, Die Grundlehren der mathematischen Wissenschaften 195, Springer-Verlag, Berlin-Heidelberg-New York, 1972

[20] Abrams, G., Aranda Pino, G., Siles Molina, M.: Finite-dimensional Leavitt path algebras. *J. Pure Appl. Algebra*, **209**(3), 753–762 (2007)

[21] Goodearl, K. R.: Von Neumann Regular Rings, Second ed., Krieger, Malabar, FL, 1991