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Weak Convergence Theorem for Lipschizian Pseudocontraction Semigroups in Banach Spaces

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Abstract The purpose of this paper is to study the weak convergence problems of the implicity iteration process for Lipschitzian pseudocontraction semigroups in general Banach spaces. The results presented in this paper extend and improve the corresponding results of Zhou [*Nonlinear Anal.*, **68**, 2977–2983 (2008)], Chen, et al. [*J. Math. Anal. Appl.*, **314**, 701–709 (2006)], Xu and Ori [*Numer. Funct. Anal. Optim*, **22**, 767–773 (2001)] and Osilike [*J. Math. Anal. Appl.*, **294**, 73–81 (2004)].

Keywords Lipschitzian pseudocontraction semigroup, demi-closed principle, common fixed point, Opial condition, strongly pseudocontractive mapping, implicit iteration process

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1 Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, $\Re^+$ is the set of nonnegative real numbers and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x,\ f\rangle = \|x\| \cdot \|f\|, \|x\| = \|f\|\},\ \ x \in E.$$

Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T. We also use "$\to$" to stand for strong convergence and "$\rightharpoonup$" for weak convergence. For a given sequence $\{x_n\} \subset C$, let $W_\omega(x_n)$ denote the weak ω-limit set, i.e.,

$$W_\omega(x_n) = \{z \in C : \text{there exists a subsequence } \{x_{n_i}\} \subset \{x_n\} \text{ such that } x_{n_i} \rightharpoonup z\}.$$

Definition 1.1 (1) *One-parameter family* $\boldsymbol{T} := \{T(t) : t \geq 0\}$ *of mappings from* C *into itself is said to be a pseudo-contraction semigroup on* C, *if the following conditions are satisfied*:

(i) $T(0)x = x$ *for each* $x \in C$;

(ii) $T(t + s)x = T(t)T(s)x$ *for any* $t,\ s \in \Re^+$ *and* $x \in C$;

(iii) *For each* $x \in C$, *the mapping* $t \mapsto T(t)x$ *is continuous*;

(iv) *For any* $x, y \in C$, *there exists* $j(x - y) \in J(x - y)$ *such that*

$$\langle T(t)x - T(t)y,\ j(x - y)\rangle \leq \|x - y\|^2,\ \text{for each } t > 0. \tag{1.1}$$

It is well known that [1] *the condition* (iv) *is equivalent to the following*:

$$\|x - y\| \leq \|x - y + s[(I - T(t))x - (I - T(t))y)]\|, \tag{1.2}$$

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for all $s > 0$ *and all* $x, y \in C$.

(2) *A pseudocontraction semigroup* $\boldsymbol{T} := \{T(t) : t \geq 0\} : C \to C$ *is said to be a strongly pseudocontractive semigroup, if the conditions* (i)–(iii) *and the following condition* (iv′) *are satisfied*:

(iv′) *There exists a bounded function* $k : [0, \infty) \to (0,\ 1)$ *with* $\sup_{t\geq 0} k(t) < 1$ *such that, for any given* $x, y \in C$, *there exists* $j(x-y) \in J(x-y)$ *such that*

$$\langle T(t)x - T(t)y,\ j(x-y)\rangle \leq k(t)\|x-y\|^2, \text{ for each } t > 0. \tag{1.3}$$

(3) *A pseudocontraction semigroup* $\boldsymbol{T} := \{T(t) : t \geq 0\} : C \to C$ *is said to be a strictly pseudocontractive semigroup, if the conditions* (i)–(iii) *and the following condition* (iv″) *are satisfied*:

(iv″) *There exists a bounded function* $\lambda : [0, \infty) \to (0,\ \frac{1}{2})$ *such that, for any given* $x, y \in C$, *there exists* $j(x-y) \in J(x-y)$ *such that*

$$\langle T(t)x - T(t)y,\ j(x-y)\rangle \leq \|x-y\|^2 - \lambda(t)\|(I-T(t))x - (I-T(t))y\|^2 \tag{1.4}$$

for each $t > 0$.

(4) *A pseudocontraction semigroup* $\boldsymbol{T} := \{T(t) : t \geq 0\} : C \to C$ *is said to be Lipschitzian* [6, 9], *if the conditions* (i)–(iv) *and the following condition* (v) *are satisfied*:

(v) *There exists a bounded measurable function* $L : (0,\ \infty) \to [0, \infty)$ *such that, for any* $x, y \in C$,

$$\|T(t)x - T(t)y\| \leq L(t)\|x-y\|, \text{ for each } t > 0.$$

In the sequel, we denote

$$M = \sup_{t\geq 0} L(t) < \infty. \tag{1.5}$$

Definition 1.2 (1) *A mapping* $T : C \to C$ *is said to be a pseudo-contraction* [1], *if for any* $x, y \in C$, *there exists* $j(x-y) \in J(x-y)$ *such that*

$$\langle Tx - Ty,\ j(x-y)\rangle \leq \|x-y\|^2.$$

(2) $T : C \to C$ *is said to be strongly pseudocontractive, if there exists* $k \in (0, 1)$ *such that*

$$\langle Tx - Ty,\ j(x-y)\rangle \leq k\|x-y\|^2,$$

for each $x, y \in C$ *and for some* $j(x-y) \in J(x-y)$.

(3) $T : C \to C$ *is said to be strictly pseudocontractive in the terminology of Browder and Petryshyn* [4, 10], *if there exists* $\lambda > 0$ *such that*

$$\langle Tx - Ty,\ j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2,$$

for every $x, y \in C$ *and for some* $j(x-y) \in J(x-y)$.

Lemma 1.3 (See [5] or [3]) *Let* E *be a real Banach space,* C *be a nonempty closed convex subset of* E *and* $T : C \to C$ *be a continuous strongly pseudocontractive mapping. Then* T *has a unique fixed point in* C.

Let E be a real Banach space, C be a nonempty closed convex subset of E and $\mathbf{T} := \{T(t) : T \geq 0\} : C \to C$ be a Lipschitzian pseudocontraction semigroup. For every $u \in C,\ t \in (0, \infty)$ and $s \in (0,\ 1)$, we define a mapping $U_s : C \to C$ by

$$U_s x = su + (1-s)T(t)x, \ \ x \in C.$$

It is easy to see that U_s is a continuous strongly pseudocontractive mapping. By using Lemma 1.3, there exists a unique fixed point $x_s \in C$ of U_s such that

$$x_s = su + (1-s)T(t)x_s. \tag{$**$}$$

Let $\mathbf{T} := \{T(t) : T \geq 0\} : C \to C$ be a Lipschitzian pseudocontraction semigroup, let $\{\alpha_n\}$ be a real sequence in $(0,1)$, and $\{t_n\}$ be a real sequence in $(0,\infty)$. By virtue of $(**)$, we can define an implicit iterative sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n)T(t_n)x_n, \ \ n \geq 1. \end{cases} \tag{1.6}$$

It should be pointed out that the following implicit iteration process:

$$\begin{cases} x_0 \in K, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \ n \geq 1 \end{cases} \tag{1.7}$$

was firstly introduced by Xu and Ori [8] for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ in a Hilbert space framework, where $T_n = T_{n(\text{mod } N)}$. In 2004, Osilike [7] extended the above sequence (1.7) from the class of nonexpansive mappings to more general class of strictly pseudocontractive mappings. In 2006, Chen, et al. [4] extended the results of Osilike [7] to more general Banach spaces.

Recently, Zhou [10] further extended the results of Chen et al. [4] from strictly pseudocontractive mapping to Lipschitzian pseudocontractions, and from q-uniformly smooth Banach space to uniformly convex Banach spaces with a Fréchet differentiable norm. Namely, he proved the following result

Theorem (Zhou [10]) *Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. Let K be a closed convex subset of E and $\{T_i\}_{i=1}^N$ be a finite family of Lipschitzian pseudocontractive self-mappings of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (1.7)*. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in (0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^N$.*

The purpose of this paper is to study the weak convergence problems of implicity iteration process (1.6) for Lipschitzian pseudocontraction semigroups and strictly pseudocontractive semigroups in general Banach spaces. The results presented in this paper extend and improve the corresponding results of Zhou [10], Chen, et al. [4], Osilike [7], Xu and Ori [8].

For this purpose, we firstly recall some concepts and conclusions.

A Banach space E is said to be uniformly convex, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in E$ with $\|x\|,\ \|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\|,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\right\},\ \ \forall \varepsilon \in [0,2].$$

Bruck [2] proved the following result:

Lemma 1.4 (Bruck [2]) *Let E be a uniformly convex Banach space with a modulus of convexity δ_E. Then $\delta_E : [0,2] \to [0,1]$ is continuous, increasing, $\delta_E(0) = 0$, $\delta_E(t) > 0$ for $t \in (0,2]$*

and

$$\|cu+(1-c)v\| \leq 1-2\min\{c,\ 1-c\}\delta_E(\|u-v\|),$$

$\forall\, c \in [0,1],$ *and* $u,\ v \in E$ *with* $\|u\|, \|v\| \leq 1.$

A Banach space E is said to satisfy the *Opial condition*, if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then the following inequality holds:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for any $y \in E$ with $y \neq x$.

It is well known that each Hilbert space and $l^p,\ p > 1$ satisfy the Opial condition, while L^p does not, unless $p = 2$.

Lemma 1.5 (Zhou [10]) *Let E be a real reflexive Banach space with Opial condition. Let C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup y$ and $\|(I-T)x_n\| \to 0$, then $(I-T)y = 0$.*

2 Main Results

Theorem 2.1 *Let E be a uniformly convex Banach space satisfying the Opial condition. Let C be a nonempty closed convex subset of E and $\boldsymbol{T} := \{T(t) : t \geq 0\} : C \to C$ be a Lipschitzian pseudocontractive semigroup such that $\boldsymbol{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0,\infty)$ satisfying the following conditions:*

(i) $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\sup_{x\in D} \|T(s+t_n)x - T(t_n)x\| \to 0,\ \ \forall\, s \in \Re^+$,

where $D = \{x \in E : \|x\| \leq \gamma\}$ and $\gamma = \sup_{n\geq 1} \|x_n\|$. Then the sequence $\{x_n\}$ defined by (1.6) converges weakly to a common fixed point of semigroup $\boldsymbol{T} := \{T(t) : t \geq 0\}$.

Proof The proof is divided into four steps.

(I) For each $p \in \mathbf{F}$, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

In fact,

$$\begin{aligned}\|x_n - p\|^2 &= \langle x_n - p,\ j(x_n - p)\rangle \\ &= \alpha_n\langle x_{n-1} - p,\ j(x_n - p)\rangle + (1-\alpha_n)\langle T(t_n)x_n - p,\ j(x_n - p)\rangle \\ &\leq \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n)\|x_n - p\|^2,\ \ \forall\, n \geq 1.\end{aligned}$$

Simplifying it, we have

$$\|x_n - p\| \leq \|x_{n-1} - p\|,\ \ \forall\, n \geq 1. \tag{2.1}$$

Consequently, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists, and so the sequence $\{x_n\}$ is bounded.

(II) Now we prove that

$$\|T(t_n)x_n - x_n\| \to 0,\ \ \text{as } n \to \infty. \tag{2.2}$$

In fact, by virtue of (1.2) and (1.6), we have

$$\begin{aligned}
\|x_n - p\| &\le \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n}(x_n - T(t_n)x_n) \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2}(x_{n-1} - T(t_n)x_n) \right\| \\
&= \left\| \alpha_n x_{n-1} + (1-\alpha_n)T(t_n)x_n - p + \frac{1-\alpha_n}{2}(x_{n-1} - T(t_n)x_n) \right\| \\
&= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\
&= \|x_{n-1} - p\| \cdot \left\| \frac{x_{n-1} - p}{2\|x_{n-1} - p\|} + \frac{x_n - p}{2\|x_{n-1} - p\|} \right\|.
\end{aligned} \tag{2.3}$$

Letting $u = \frac{x_{n-1}-p}{\|x_{n-1}-p\|}$ and $v = \frac{x_n - p}{\|x_{n-1}-p\|}$, from (2.1) we know that $\|u\| = 1,\ \ \|v\| \le 1$. It follows from (2.3) and Lemma 1.4 that

$$\|x_n - p\| \le \|x_{n-1} - p\| \left\{ 1 - \delta_E\left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \right\}.$$

Simplifying it we have

$$\|x_{n-1} - p\| \delta_E\left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \le \|x_{n-1} - p\| - \|x_n - p\|.$$

This implies that

$$\sum_{n=1}^{\infty} \|x_{n-1} - p\| \delta_E\left(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|} \right) \le \|x_0 - p\|.$$

Letting $\lim_{n\to\infty} \|x_n - p\| = r$, if $r = 0$, the conclusion of Theorem 2.1 is proved. If $r > 0$, it follows from the property of modulus of convexity δ_E that $\|x_{n-1} - x_n\| \to 0\,(n \to \infty)$. Therefore, from (1.6) and condition (i) we have that

$$\|x_{n-1} - T(t_n)x_n\| = \frac{1}{1-\alpha_n}\|x_n - x_{n-1}\| \to 0 \ \text{ as } n \to \infty. \tag{2.4}$$

Hence by virtue of (1.6) and (2.4), we have

$$\|x_n - T(t_n)x_n\| = \alpha_n \|x_{n-1} - T(t_n)x_n\| \to 0 \ \text{ as } n \to \infty.$$

The conclusion (2.2) is proved.

(III) Now we prove that for each $t > 0$

$$\lim_{n\to\infty} \|T(t)x_n - x_n\| = 0 \quad \text{as } n \to \infty. \tag{2.5}$$

In fact, for each $t > 0$, we have

$$\begin{aligned}
\|T(t)x_n - x_n\| &\le \|T(t)x_n - T(t)T(t_n)x_n\| + \|T(t)T(t_n)x_n - T(t_n)x_n\| \\
&\quad + \|T(t_n)x_n - x_n\| \\
&\le (1 + L(t))\|T(t_n)x_n - x_n\| + \|T(t)T(t_n)x_n - T(t_n)x_n\| \\
&\le (1 + M)\|T(t_n)x_n - x_n\| + \sup_{x\in D} \|T(t+t_n)x - T(t_n)x\|,
\end{aligned}$$

where $M = \sup_{t\ge 0} L(t) < \infty$ (by (1.5)). By condition (ii) and (2.2),

$$\lim_{n\to\infty} \|T(t)x_n - x_n\| = 0, \ \ \forall\, t \ge 0.$$

The conclusion (2.5) is proved.

(IV) Finally, we prove that $\{x_n\}$ converges weakly to a common fixed point of semigroup $\mathbf{T} := \{T(t) : t \geq 0\}$.

Since E is uniformly convex, so it is reflexive. Again since $\{x_n\} \subset C$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup u \in W_\omega(x_n)$. Hence from (2.5), for any $t > 0$, we have

$$\|T(t)x_{n_i} - x_{n_i}\| \to 0 \ \text{ as } n_i \to \infty.$$

By virtue of Lemma 1.5, $u \in F(T(t))$, $\forall\, t \geq 0$. This implies that

$$u \in \bigcap_{t \geq 0} F(T(t)) \cap W_\omega(x_n).$$

Next we prove that $W_\omega(x_n)$ is a singleton. Suppose to the contrary that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup q \in W_\omega(x_n)$ and $q \neq u$. By the same method as given above we can also prove that $q \in \bigcap_{t\geq 0} F(T(t)) \cap W_\omega(x_n)$. Taking $p = u$ and $p = q$ in (2.1), we know that the following limits

$$\lim_{n\to\infty} \|x_n - u\|, \quad \lim_{n\to\infty} \|x_n - q\|$$

exist. Since E satisfies the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \limsup_{n\to\infty} \|x_{n_i} - u\| < \limsup_{n_i\to\infty} \|x_{n_i} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| = \limsup_{n_j\to\infty} \|x_{n_j} - q\| \\ &< \limsup_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction, which shows that $q = u$. Hence

$$W_\omega(x_n) = \{u\} \subset \mathbf{F} := \bigcap_{t\geq 0} F(T(t)).$$

This implies that $x_n \rightharpoonup u$. The conclusion of Theorem 2.1 is proved.

Next we establish a weak convergence theorem for strictly pseudocontractive semigroups.

Theorem 2.2 *Let E be a reflexive Banach space satisfying the Opial condition. Let C be a nonempty closed convex subset of E and $\boldsymbol{T} := \{T(t) : t \geq 0\} : C \to C$ be a strictly pseudocontractive semigroup with a strictly pseudocontractive function $\lambda(t) : [0, \infty) \to (0, \frac{1}{2})$ such that $\boldsymbol{F} := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$, and $\{t_n\}$ be a sequence in $(0, \infty)$ satisfying the following conditions:*

(i) $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\sup_{x\in D} \|T(s + t_n)x - T(t_n)x\| \to 0, \quad \forall\, s \in \Re^+$, *where* $D = \{x \in E : \|x\| \leq \gamma\}$ *and* $\gamma = \sup_{n\geq 1} \|x_n\|$;

(iii) $\lim_{n\to\infty} \frac{\lambda(t_n)}{\alpha_n} = K$, *where K is a positive constant.*

Then the sequence $\{x_n\}$ defined by (1.6) converges weakly to a common fixed point of strictly pseudocontractive semigroup $\boldsymbol{T} := \{T(t) : t \geq 0\}$.

Proof It follows from (1.4) and (1.6) that for any given $p \in \mathbf{F} := \bigcap_{t \geq 0} F(T(t))$,

$$\begin{aligned}\|x_n - p\|^2 &= \langle x_n - p,\ j(x_n - p)\rangle \\ &= \alpha_n \langle x_{n-1} - p,\ j(x_n - p)\rangle + (1 - \alpha_n)\langle T(t_n)x_n - p,\ j(x_n - p)\rangle \\ &\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n)\|x_n - p\|^2 \\ &\quad - \lambda(t_n)(1 - \alpha_n)\|x_n - T(t_n)x_n\|^2, \end{aligned} \tag{2.6}$$

which implies that

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\| \|x_n - p\| - \frac{\lambda(t_n)}{\alpha_n}(1 - \alpha_n)\|x_n - T(t_n)x_n\|^2. \tag{2.7}$$

This shows that

$$\|x_n - p\| \leq \|x_{n-1} - p\|, \quad \forall\, n \geq 1.$$

Therefore the limit $\lim_{n\to\infty} \|x_n - p\|$ exists and so $\{\|x_n\|\}$ is bounded. Denote $\beta = \sup_{n \geq 0} \|x_n - p\|$.

From (2.7) we have

$$\frac{\lambda(t_n)}{\alpha_n}(1 - \alpha_n)\|x_n - T(t_n)x_n\|^2 \leq \beta\{\|x_{n-1} - p\| - \|x_n - p\|\}. \tag{2.8}$$

Letting $n \to \infty$ and taking the limit on both sides of (2.8) and by using condition (i) and condition (iii) we have

$$\lim_{n\to\infty} \|x_n - T(t_n)x_n\| = 0. \tag{2.9}$$

By the same method as given in the proof of Theorem 2.1, from (2.9) and condition (ii) we can prove that $\{x_n\}$ converges weakly to some common fixed point of strictly pseudocontractive semigroup $\mathbf{T} := \{T(t) : t \geq 0\}$.

The proof of Theorem 2.2 is completed.

Remark Theorems 2.1 and 2.2 extend and improve the corresponding results of Zhou [10], Chen, et al. [4], Osilike [7], Xu and Ori [8].

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