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Viscosity Approximation Methods for Nonexpansive Multimaps in Banach Spaces

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Abstract We prove strong convergence of the viscosity approximation process for nonexpansive nonself multimaps. Furthermore, an explicit iteration process which converges strongly to a fixed point of multimap T is constructed. It is worth mentioning that, unlike other authors, we do not impose the condition "$Tz = \{z\}$" on the map T.

Keywords nonexpansive retract, Banach spaces, fixed point, inwardness, nonexpansive multimap

MR(2000) Subject Classification 47H10, 47H09

1 Introduction

Let D be a nonempty closed convex subset of a Banach space $E := (E, \|\cdot\|)$ and let $K(D)$ and $KC(D)$ denote the family of nonempty compact subsets and nonempty compact convex subsets of D, respectively. Let $T : D \to K(E)$. Then T is said to be a *contraction* if there exists $0 \le k < 1$ such that $H(Tx, Ty) \le k\|x - y\|$ for $x, y \in D$, where H is the Hausdorff metric induced by $\|\cdot\|$. If $k = 1$, then T is called *nonexpansive*. A point x^* is a *fixed point of* T if $x^* \in Tx^*$. The set of fixed points of T is denoted by $F(T)$.

Let $T : D \to K(D)$ be nonexpansive. Given a $u \in D$ and a $t \in (0, 1)$, let $G_t : D \to K(D)$ be defined by

$$G_t x := tTx + (1 - t)u, \quad x \in D.$$

Then G_t is a contraction and, by the Nadler contraction principle [1], has a fixed point $x_t \in D$, that is,

$$x_t \in tTx_t + (1 - t)u. \tag{1.1}$$

Let

$$P_T(x) = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\},$$

where $d(x, A) := \inf\{\|x - a\| : a \in A \subset E\}$. Then $P_T : D \to K(E)$ is a multimap having nonempty compact values.

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Instead of

$$G_t x := tTx + (1-t)u, \quad u \in D, \tag{1.2}$$

we consider, for $t \in (0,1)$,

$$S_t x := tP_T(x) + (1-t)u, \quad u \in D. \tag{1.3}$$

It is clear that $S_t x \subseteq G_t x$ for all $x \in D$, and if P_T is nonexpansive and T is weakly inward, then S_t is a weakly inward contraction. Now Theorem 1 of Lim [2] guarantees that S_t has a fixed point, say x_t, that is,

$$x_t \in tP_T(x_t) + (1-t)u \subseteq tTx_t + (1-t)u. \tag{1.4}$$

If T is single-valued, then (1.1) and hence (1.4) reduces to

$$x_t = tTx_t + (1-t)u. \tag{1.5}$$

The strong convergence of the net $\{x_t\}$ for a self or non-self nonexpansive single-valued map T has been studied by a number of authors, see, for instance, the work of Browder [3], Halpern [4], Jung and Kim [5], Kim and Takahashi [6], Reich [7], Singh and Watson [8], Takahashi and Kim [9], Xu [10], Xu and Yin [11] etc. Details on various iterative methods can be found in [12].

In 1967, Browder [3] proved the following strong convergence result.

Theorem B [3] *Let D be a closed bounded convex subset of a Hilbert space E, and T a nonexpansive self-mapping of D. Let $\{t_n\}$ be a sequence in $(0,1)$ converging to 1. Fix $u \in D$ and define a sequence $\{x_n\}$ in D by*

$$x_n = t_n T x_n + (1-t_n)u, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u.

Reich [7] extended Browder's result to uniformly smooth Banach spaces.

Pietramala [13] gave an example of multivalued selfmap defined on a closed convex bounded subset of a finite-dimensional Hilbert space, which illustrates that Browder's theorem cannot be extended to genuine multivalued case without assuming extra conditions. López and Xu [14] established the strong convergence of $\{x_t\}$ in a Hilbert space for nonexpansive multimaps T satisfying $Tz = \{z\}$. Later on, Kim and Jung [15] extended López and Xu's result to a Banach space having a sequentially continuous duality map. Sahu [16] studied this problem in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. Recently, Jung [17] noted that the condition $Tz = \{z\}$ should be added in the main results of Sahu [16] and he further established the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u,\ u \in D$ for a nonexpansive nonself multimap T with $Tz = \{z\}$ in a uniformly convex and reflexive Banach space having a uniformly Gâteaux differentiable norm. More recently, Shahzad and Zegeye [18] extended the results of Jung [17] to a class of multimaps under mild conditions. More precisely, they obtained the following extension of Browder's theorem.

Theorem SZ [18, Theorem 1] *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to K(E)$ be such that P_T is nonexpansive. Suppose that D is a nonexpansive retract of E and that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T(x) + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1^-$ and in this case, $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

Let $T : D \to D$ be a nonexpansive selfmap on *closed convex* subset K of a Banach space E. For a given contraction $f : K \to K$ with a suitable contraction constant and for a given $t \in (0, 1)$, define a contraction $T_t : D \to D$ by

$$T_t x = tTx + (1 - t)fx, \quad x \in D.$$

By the Banach contraction principle it yields a fixed point $z_t \in D$ of T_t, that is, z_t is the unique solution of the equation

$$z_t = tTz_t + (1 - t)fz_t. \tag{1.6}$$

If $f = u$, for $u \in D$, then (1.6) reduces to (1.5)

Concerning convergence $\{z_t\}$ of (1.6), in 2000, Moudafi [19] introduced as viscosity approximation method and proved that if E is a real Hilbert space, the sequence $\{z_t\}$ converges strongly to a fixed point of T in K. It should be pointed out that Moudafi's result generalizes Browder's and Halpern's theorems in the direction of viscosity approximations. Viscosity approximations are very important becuase they are applied to convex optimization, linear programming, monotone inclusions and elliptic differential equations.

In 2004, Xu [20] studied further the viscosity approximation method for a nonexpansive map and proved the following result:

Theorem 1.1 [20, Theorem 4.1] *Let D be a nonempty closed convex and bounded subset of a real uniformly smooth Banach space E. Let $T : D \to D$ be a nonexpansive map with $F(T) \neq \emptyset$ and $f : D \to D$ be a contraction. Then for $t \in (0, 1)$, the viscosity approximation process $\{y_t\}$ defined by* (1.6) *converges strongly to a fixed point of T.*

The above theorem of Xu [20] improves Theorem 2.1 of Moudafi [19]. Moreover, Xu [20] has proved the convergence of the viscosity iterative process

$$x_{n+1} = \lambda_{n+1} fx_n + (1 - \lambda_{n+1})Tx_n, \quad n \geq 1, \tag{1.7}$$

where $\lambda_n \in (0, 1)$ satisfies certain conditions, to the fixed point of T and to a solution of a certain variational inequality. This result of Xu [20] extends Theorem 2.2 of Moudafi [19] to a Banach space setting.

It is our purpose in this paper to prove the convergence of the viscosity approximation process for nonexpansive nonself multimaps. Furthermore, an explicit iteration process which converges strongly to a fixed point of multimap T is constructed. It is worth mentioning that, unlike other authors, we do not impose the condition "$Tz = \{z\}$" on the map T. Our result

(Theorem 3.1) extends Theorem 1 and Corollary 1 of Jung and Kim [5], Corollary 2 of Jung and Kim [5] and Theorem 1 and Corollary 2 of Xu and Yu [11] to multimaps. Theorem SZ, Theorem 1 of Jung [17], Theorem 4.1 of Kim and Jung [15] and Theorem 1 of Sahu [16] are special cases of Theorem 3.1 under which $f = u$ or T is selfmap.

2 Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued normalized duality map by j.

The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$ the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly for $x \in S_1(0)$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see, e.g., [22]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality map is norm-to-w^* uniformly continuous on bounded subsets of E.

The inward set of D at x is defined by

$$I_D(x) = \{z \in E : z = x + \lambda(y - x) : y \in D, \lambda \geq 0\}.$$

It is known that if D is convex, then the closure of $I_D(x)$, is $\overline{I_D(x)} = x + T_D(x)$ for any $x \in D$, where

$$T_D(x) = \left\{ y \in E : \lim \inf_{\lambda \to 0^+} \frac{d(x + \lambda y, D)}{\lambda} = 0 \right\}.$$

A multimap $T : D \to CB(E)$ is said to satisfy (i) *weak inwardness condition* if $Tx \subset \overline{I_D(x)}$ for all $x \in D$; (ii) *inwardness condition* if $Tx \subset I_D(x)$ for all $x \in D$.

Let $K \subseteq E$ be closed convex and Q a map of E onto K. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, *the range of* Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

We shall let LIM be a Banach limit. Recall that LIM $\in (\ell^\infty)^*$ such that $\|\text{LIM}\| = 1$, $\liminf_{n\to\infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n\to\infty} a_n$, and $\text{LIM}_n a_n = \text{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in \ell^\infty$.

In what follows, we shall make use of the following lemmas.

Lemma 2.1 (See, e.g., [23]) *Let E be a Banach space having a uniformly Gâteaux differentiable norm and D a nonempty closed convex subset of E. Let $\{x_n\}$ be a bounded sequence in E,* LIM *a Banach limit, and $u \in D$. Then the following are equivalent:*

(i) $\mathrm{LIM}_n\|x_n - u\|^2 = \min_{y\in D} \mathrm{LIM}_n\|x_n - y\|^2$;

(ii) $\mathrm{LIM}_n\langle x - u, J(x_n - u)\rangle \le 0$ *for all* $x \in D$.

Lemma 2.2 (See, e.g., [24]) *Let E be a uniformly convex Banach space, D a nonempty closed convex subset of E, and $\{x_n\}$ a bounded sequence in E. Then the set*

$$C = \left\{u \in D : \mathrm{LIM}_n\|x_n - u\|^2 = \min_{y\in D} \mathrm{LIM}_n\|x_n - y\|^2\right\}$$

consists of one point.

Lemma 2.3 *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\gamma_n)a_n + \sigma_n, \quad n \ge 0,$$

where (i) $0 < \gamma_n < 1$; (ii) $\sum_{n=1}^{\infty}\gamma_n = \infty$.

Suppose, either

(a) $\sigma_n = o(\gamma_n)$, *or* (b) $\limsup_n \sigma_n \le 0$.

Then $a_n \to 0$ as $n \to \infty$.

The proof of the above lemma is standard and is therefore omitted.

3 Main Results

Theorem 3.1 *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to K(E)$ be such that P_T is nonexpansive. Suppose that D is a nonexpansive retract of E and that for each $t \in (0,1)$, the contraction S_t defined by $S_tx = tP_Tx + (1-t)f(x)$ has a fixed point $x_t \in D$, where $f : D \to D$ is a contraction with constant β. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1^-$; in this case, $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

Proof Given any $x_t \in D$, we can find some $y_t \in P_T(x_t)$ such that

$$x_t = ty_t + (1-t)fx_t. \tag{3.1}$$

Let $p \in F(T) \neq \emptyset$. Then $p \in P_T(p)$ and then we obtain for all $t \in (0,1)$ that

$$\|y_t - p\| = d(y_t, P_T(p)) \le H(P_T(x_t), P_T(p)) \le \|x_t - p\|. \tag{3.2}$$

Thus,

$$\begin{aligned}\|x_t - p\| &= \|ty_t + (1-t)fx_t - p\| \\ &\le t\|y_t - p\| + (1-t)\|fx_t - fp\| + (1-t)\|fp - p\| \\ &\le t\|x_t - p\| + (1-t)\beta\|x_t - p\| + (1-t)\|fp - p\|,\end{aligned}$$

and so

$$\|x_t - p\| \le \frac{1}{1-\beta}\|fp - p\|$$

for $t \in (0,1)$. Hence $\{x_t\}$ is uniformly bounded.

Suppose that $\{x_t\}$ remains bounded as $t \to 1^-$. Now we show that $F(T) \neq \emptyset$ and x_t converges to a fixed point of T as $t \to 1^-$. Let $t_n \to 1$ and set $x_n := x_{t_n}$. Define the mapping $\phi : E \to \mathbb{R}$ by

$$\phi(x) := \text{LIM}_n \|x_n - x\|^2, \quad \forall\, x \in E.$$

Since E is reflexive, $\phi(x) \to \infty$ as $\|x\| \to \infty$, ϕ is continuous and convex, it follows that ϕ attains its infimum over D at z (say) (see [25, p. 79]) and so the set $C := \{z \in D : \phi(z) = \inf_{x \in D} \phi(x)\} \neq \emptyset$ is also a closed, bounded and convex subset of D. Let Q be a nonexpansive retraction of E onto D. Then

$$\begin{aligned} \phi(y) &= \text{LIM}_n \|x_n - y\|^2 \\ &\geq \text{LIM}_n \|Qx_n - Qy\|^2 \\ &= \text{LIM}_n \|x_n - Qy\|^2 \\ &\geq \text{LIM}_n \|x_n - z\|^2 = \phi(z) \end{aligned}$$

for any $y \in E$. This implies that z is the global minimum point over all of E. Furthermore, z is unique by Lemma 2.2. Notice that $x_n = t_n y_n + (1 - t_n) f x_n$ for some $y_n \in P_T(x_n)$ gives that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Since T and hence P_T is compact-valued we have for each $n \geq 1$, some $w_n \in P_T(z)$ such that

$$\|y_n - w_n\| = d(y_n, P_T(z)) \leq H(P_T(x_n), P_T(z)) \leq \|x_n - z\|.$$

Without loss of generality, let $w_n \to w \in P_T(z)$. Then

$$\text{LIM}_n \|x_n - w\|^2 \leq \text{LIM}_n \|y_n - w_n\|^2 \leq \text{LIM}_n \|x_n - z\|^2.$$

But z is the unique global minimum. Therefore $z = w \in P_T(z) \subset Tz$. Furthermore, for any $p \in F(T)$ we have from (3.1) and (3.2) that

$$\begin{aligned} \langle x_n - y_n, j(x_n - p)\rangle &= \langle x_n - p + p - y_n, j(x_n - p)\rangle \\ &\geq \|x_n - p\|^2 - \|y_n - p\| \|x_n - p\| \geq 0, \end{aligned}$$

and so

$$0 \leq \langle x_n - y_n, j(x_n - p)\rangle = (1 - t_n)\langle f(x_n) - y_n, j(x_n - p)\rangle.$$

This together with $\|x_n - y_n\| \to 0$ implies that

$$\text{LIM}_n \langle x_n - f x_n, j(x_n - p)\rangle \leq 0. \tag{3.3}$$

In particular,

$$\text{LIM}_n \langle x_n - f x_n, j(x_n - z)\rangle \leq 0. \tag{3.4}$$

Also, by Lemma 2.1, we have

$$\text{LIM}_n \langle x - z, j(x_n - z)\rangle \leq 0$$

for all $x \in D$. In particular,

$$\mathrm{LIM}_n \langle fz - z, j(x_n - z)\rangle \leq 0. \tag{3.5}$$

Thus using (3.4) and (3.5) we find $\mathrm{LIM}_n\|x_n - z\| = 0$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z$ as $k \to \infty$. Assume that there is another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \to q$ as $l \to \infty$. Since

$$d(x_{n_l}, P_T(x_{n_l})) \leq \|x_{n_l} - y_{n_l}\| \leq (1 - t_{n_l})\|fx_{n_l} - y_{n_l}\| \to 0 \quad \text{as } l \to \infty,$$

it follows that $d(q, P_T(q)) = 0$, and so $q \in P_T(q) \subseteq Tq$. Moreover, $x_{n_k} \to z$, implies from (3.3) that

$$\langle z - fz, j(z - q)\rangle \leq 0. \tag{3.6}$$

Also, with $x_{n_l} \to q$, we have that

$$\langle q - fq, j(q - z)\rangle \leq 0. \tag{3.7}$$

Inequalities (3.6) and (3.7) yield that

$$\|z - q\|^2 \leq \beta\|z - q\|^2,$$

which implies that $z = q$. Thus $x_n \to z$ as $n \to \infty$. This completes the proof. □

Under the hypothesis of Theorem 3.1 if T is weakly inward then by Theorem 1 of Lim [2], the contraction S_t defined by $S_t x = tP_T(x) + (1 - t)fx$ has a fixed point $x_t \in D$. Thus, we have the following corollary.

Corollary 3.2 *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to K(E)$ a weakly inward multimap such that P_T is nonexpansive. Suppose that D is nonexpansive retract of E. Then T has a fixed point if and only if the fixed point $\{x_t\}$ of $\{S_t\}$ remains bounded as $t \to 1^-$; in this case, $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

If, in Corollary 3.2, T is a self map, then the method of proving Theorem 3.1 gives the required result without the requirement that D is a nonexpansive retraction. Thus, we have the following corollary.

Corollary 3.3 *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to K(D)$ be such that P_T is nonexpansive. Then T has a fixed point if and only if the fixed point $\{x_t\}$ of $\{S_t\}$ remains bounded as $t \to 1^-$; in this case, $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

It is known that a closed convex subset of a Hilbert space is a nonexpansive retraction with the proximity map as a nonexpansive retraction. Thus, the requirement that D is a nonexpansive retraction of E is not needed. In fact, we have the following corollary.

Corollary 3.4 *Let E be a Hilbert space, D a nonempty closed convex subset of E, and $T : D \to K(E)$ be such that P_T is nonexpansive. Suppose that for each $t \in (0, 1)$, the contraction S_t*

defined by $S_tx = tP_T(x)+(1-t)fx$ has a fixed point $x_t \in D$, where $f : K \to K$ is a contraction. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1^-$ and in this case $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.

We know that if D is a closed bounded convex subset of a uniformly convex Banach space E and $T : D \to KC(E)$ is an nonexpansive mapping satisfying the weak inwardness condition, then T has a fixed point. So we get the following corollaries.

Corollary 3.5 *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed bounded convex subset of E, and $T : D \to KC(E)$ a multimap satisfying the weak inwardness condition such that P_T is nonexpansive. Suppose that D is a nonexpansive retraction of E. Then, for $t \in (0,1)$, the fixed point $\{x_t\}$ of $\{S_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

Corollary 3.6 *Let E be a Hilbert space, D a nonempty closed bounded convex subset of E, and $T : D \to KC(E)$ a multimap satisfying the weak inwardness condition such that P_T is nonexpansive. Then, for $t \in (0,1)$, the fixed point $\{x_t\}$ of $\{S_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

Essentially the same arguments as above and as in Theorem 2 of Jung [17] yield the following result.

Theorem 3.7 *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to KC(E)$ a multimap satisfying the inwardness condition such that P_T is nonexpansive. Suppose that every closed bounded convex subset of D is compact and T has a fixed point. Then, for $t \in (0,1)$, the fixed point $\{x_t\}$ of $\{S_t\}$ converges strongly as $t \to 1^-$ to a fixed point of T.*

Remark 3.8 (1) In the above theorem and corollaries, we observe that, if we assume that T is $*$-nonexpansive (for the definition, see [26]) then P_T is nonexpansive and hence the results are valid. We now give an example of a multimap T which is not nonexpansive but P_T is nonexpansive. Let $D = [0,\infty)$ and T be defined by $Tx = [x, 3x]$ for $x \in D$. Then $P_T(x) = \{x\}$ for $x \in D$. On the other hand one easily checks that T is *not nonexpansive.* Note that T is $*$-nonexpansive (see [27]).

(2) Theorem 3.1 extends Theorem SZ and hence Theorem 1 and Corollary 1 of Jung and Kim [21], Corollary 2 of Jung and Kim [5] and Theorem 1 and Corollary 2 of Xu and Yu [11] either to viscosity approximation or to multimaps. Theorem 1 of Jung [17], Theorem 4.1 of Kim and Jung [15] and Theorem 1 of Sahu [16] are special cases of Theorem 3.1 under which $f = u$ or T is selfmap.

(3) Our results apply to L^p and l^p spaces for $1 < p < \infty$.

4 Applications

Theorem 4.1 *Let E be a uniformly convex Banach space having a uniformly Gâteaux differ-*

entiable norm, D a nonempty closed convex subset of E, and $T : D \to K(D)$ a multimap such that P_T is nonexpansive. For given $x_0 \in D, y_0 \in P_T(x_0)$, let $\{x_n\}$ be generated by the algorithm (*see, e.g.,* [28])

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \\ y_n \in P_T(x_n) \text{ such that } \|y_{n-1} - y_n\| = d(y_{n-1}, P_T(x_n)),\ n \geq 1, \end{cases} \tag{$*$}$$

where $f : D \to D$ is a contraction with constant β and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum \alpha_n = \infty$ *and*

(iii) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since T is compact-valued, for any $y_{n-1} \in P_T(x_{n-1})$, we can find some $y_n \in P_T(x_n)$ such that $\|y_{n-1} - y_n\| = d(y_{n-1}, P_T(x_n)) = \inf\{\|y_{n-1} - z\| : z \in P_T(x_n)\}$, and hence scheme $(*)$ is well defined. Let $p \in F(T)$ and $y_n \in P_T(x_n)$. Then we have that

$$\|y_n - p\| = d(y_n, P_T(p)) \leq H(P_T(x_n), P_T(p)) \leq \|x_n - p\|. \tag{4.1}$$

Thus, for $y_n \in P_T(x_n)$ and $y_{n-1} \in P_T(x_{n-1})$ satisfying $(*)$ we get that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(fx_n - p) + (1-\alpha_n)(y_n - p)\| \\ &\leq \alpha_n(\|fx_n - fp\| + \|fp - p\|) + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n(\beta\|x_n - p\| + \|fp - p\|) + (1-\alpha_n)\|x_n - p\| \\ &\leq (1-(1-\beta)\alpha_n)\|x_n - p\| + \alpha_n\|fp - p\| \\ &\leq \max\left\{\|x_n - p\|, \frac{1}{1-\beta}\|fp - p\|\right\}, \end{aligned}$$

and hence $\|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-\beta}\|fp - p\|\}$ which gives that $\{x_n\}$, $\{fx_n\}$ and $\{y_n\}$ are bounded. But this implies

$$\|x_{n+1} - y_n\| = \alpha_n\|fx_n - y_n\| \to 0 \quad \text{as } n \to \infty.$$

Moreover, from $(*)$ we get

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})(fx_{n-1} - y_{n-1}) + (1-\alpha_n)(y_n - y_{n-1}) \\ &\quad +\alpha_n(fx_n - fx_{n-1})\| \\ &\leq (1-\alpha_n)d(y_{n-1}, P_T(x_n)) + |\alpha_n - \alpha_{n-1}|M \\ &\quad +\beta\alpha_n\|x_n - x_{n-1}\| \\ &\leq (1-\alpha_n)H(P_T(x_n), P_T(x_{n-1})) + |\alpha_n - \alpha_{n-1}|M \\ &\quad +\beta\alpha_n\|x_n - x_{n-1}\| \\ &\leq (1-(1-\beta)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M, \end{aligned}$$

where $M := \sup \|fx_{n-1} - y_{n-1}\| < \infty$ as $n \to \infty$. Therefore, by assumption and Lemma 2.3 we conclude that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and hence

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty.$$

Let $g(x) := LIM_n\|x_n - x\|, \forall x \in D$. Then g is continuous and convex on D. Define a set $D_0 = \{x \in D : g(x) = \inf_{y\in D} g(y)\}$. Then using the fact that $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$, $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and the method of proof of Theorem 3.1 there exists $z \in D_0$ such that $z \in P_T z \subset F(T)$. Now, using Lemma 2.1 and the definition of D_0, we get

$$LIM_n\langle f(z) - z, j(x_n - z)\rangle \leq 0.$$

On the other hand, $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ together with the norm to ω^* uniform continuity of j imply

$$\lim_{n\to\infty} (\langle f(z) - z, j(x_{n+1} - z)\rangle - \langle f(z) - z, j(x_n - z)\rangle) = 0.$$

Hence, by Proposition 2 of [29] we obtain

$$\limsup_{n\to\infty}\langle f(z) - z, j(x_{n+1} - z)\rangle \leq 0.$$

Finally we show that $x_n \to z$. Now for $y_n \in P_T(x_n)$ from $(*)$ and (4.1) we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \alpha_n\langle fx_n - z, j(x_{n+1} - z)\rangle + (1 - \alpha_n)\langle y_n - z, j(x_{n+1} - z)\rangle \\
&\leq \alpha_n\langle fz - z, j(x_{n+1} - z)\rangle + \alpha_n\langle fx_n - fz, j(x_{n+1} - z)\rangle \\
&\quad +(1 - \alpha_n)\|y_n - z\| \times \|j(x_{n+1} - z)\| \\
&\leq \alpha_n\langle fz - z, j(x_{n+1} - z)\rangle + \alpha_n\|fx_n - fz\| \times \|j(x_{n+1} - z)\| \\
&\quad +(1 - \alpha_n)\|x_n - z\| \times \|j(x_{n+1} - z)\| \\
&\leq \alpha_n\langle fz - z, j(x_{n+1} - z)\rangle + \alpha_n\frac{\beta^2\|x_n - z\|^2 + \|x_{n+1} - z\|^2}{2} \\
&\quad +(1 - \alpha_n)\frac{\|x_n - z\|^2 + \|x_{n+1} - z\|^2}{2}.
\end{aligned} \tag{4.2}$$

Thus we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n(1 - \beta^2))\|x_n - z\|^2 + 2\alpha_n\langle fz - z, j(x_{n+1} - z)\rangle \\
&= (1 - \gamma_n)\|x_n - z\|^2 + \sigma_n,
\end{aligned}$$

where $\gamma_n := \alpha_n(1 - \beta^2)$ and $\limsup_n \sigma_n \leq 0$, for $\sigma_n := 2\alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle$. Thus, by Lemma 2.3, $\{x_n\}$ converges strongly to a fixed point z of T. $\square$

The following corollary follows from Theorem 4.1 with the use of Corollaries 3.3 and 3.4.

Corollary 4.2 *Let E be a Hilbert space, D a nonempty closed convex subset of E, and $T : D \to K(D)$ a multimap such that P_T is nonexpansive. Then if $F(T) \neq \emptyset$ then $\{x_n\}$ defined by $(*)$ converges strongly to a fixed point of T.*

If E is Chebyshev and T maps from D to $KC(D)$, then P_T is a singleton and hence the second condition of equation $(*)$ is not needed. In fact, we have the following corollary.

Corollary 4.3 *Let E be uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E, and $T : D \to KC(D)$ a multimap such that P_T is nonexpansive. For given $x_0 \in D, y_0 \in T(x_0)$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_T(x_n) \ n \geq 1, \tag{$**$}$$

where f and $\{\alpha_n\}$ are as in Theorem 4.1. Then $\{x_n\}$ converges strongly to a fixed point of T provided that $F(T) \neq \emptyset$.

Remark 4.4 If in Theorem 4.1 we have that T is single-valued, then the iteration scheme $(*)$ reduces to the scheme studied by Halpern [4], Moudafi [19], Xu [20] and the references therein.

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