Acta Mathematica Sinica, English Series
Apr., 2009, Vol. 25, No. 4, pp. 553–564
Published online: March 25, 2009
DOI: 10.1007/s10114-009-7362-x
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Viscosity Approximations by Generalized Contractions for Resolvents of Accretive Operators in Banach Spaces

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Abstract In this paper, we prove a strong convergence theorem for resolvents of accretive operators in a Banach space by the viscosity approximation method with a generalized contraction mapping. The proximal point algorithm in a Banach space is also considered. The results extend some very recent theorems of W. Takahashi.

Keywords viscosity approximation method, accretive operator, generalized contraction, resolvent, proximal point algorithm

MR(2000) Subject Classification 47H06, 47H10, 47J25

1 Introduction

Let E be a real Banach space and let $A \subset E \times E$ be a maximal monotone operator. Then the problem of finding a solution $v \in E$ with $0 \in Av$ has been investigated by many researchers.

The following method for solving the equation

$$0 \in Av \text{ in } E$$

is called the proximal point algorithm:

$$x_0 = x \in E, \quad x_{n+1} = J_{\lambda_n} x_n, \quad n \in \{0, 1, 2, \dots, \}, \tag{1}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n} = (I + \lambda_n A)^{-1}$.

Rockafellar [1] proved that if E is a Hilbert space, $\liminf_{n\to\infty} \lambda_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1) converges weakly to an element of $A^{-1}0$.

Other results (weak and strong convergence) on this topic were given by Brézis and Lions [2], Güler [3], Reich [4–5], Pazy [6], Nevanlinna and Reich [7], Jung and Takahashi [8], Kamimura and Takahashi [9–10], Bruck [11], Bruck and Reich [12], Reich [13–16], Passty [17], Bruck and Passty [18], etc. See also the references mentioned therein.

Received July 17, 2007, Accepted July 12, 2008
This research is partially supported by a grant from the National Science Council
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On the other hand, if E is a reflexive Banach space with a uniformly Gateaux differentiable norm and C is a closed convex subset of E which has normal structure, then, very recently, Takahashi [19] introduced the following viscosity approximation scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{\lambda_n} x_n, \quad n \in \{0, 1, 2, \ldots\}, \tag{2}$$

where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $]0, 1[$ and $\{\lambda_n\}$ is a sequence in $(0, \infty)$.

Then, under additional conditions, he proved that the sequence $\{x_n\}$ generated by (2) converges strongly to some $u \in A^{-1}0$, where $u = Pf(u)$ and P is a sunny nonexpansive retraction of C onto $A^{-1}0$. See also [20] for a recent result.

In this paper, motivated by Takahashi's results, we will consider the viscosity approximation scheme (2) with a generalized contraction mapping f. Strong convergence theorems will be proved. Our theorems extend to the case of a generalized contraction mapping the results in Takahashi [19].

2 Preliminaries

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$ and denote $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Let E be a real Banach space with norm $\|\cdot\|$, I be the identity operator on E and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. If $\{x_n\}$ is a sequence in E, then we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. We also know that if C is a closed convex subset of a uniformly convex Banach space E, then for each $x \in E$, there exists a unique element $u = Px \in C$ with $\|x - u\| = \inf\{\|x - y\| : y \in C\}$. Such a P is called the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, \quad x \in E.$$

Let $S(E) = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists. In this case, E is said to be smooth. If the above limit is attained uniformly for $x \in S(E)$, then the norm of E is called uniformly Gâteaux differentiable. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single-valued and uniformly norm to weak* continuous on each bounded subset of E. If E is uniformly smooth, then the duality mapping J is uniformly norm to norm continuous on each bounded subset of E.

Let C be a closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A self nonexpansive on a nonempty bounded closed convex subset C of E which has normal structure has a fixed point in C, see Kirk [21].

Let D be a subset of C. A mapping P of C into D is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \ge 0$. A mapping P of C into itself is called a retraction if $P^2 = P$.

An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. If A is accretive, then we have $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $x_1, x_2 \in D(A)$, $y_1 \in Ax_1, y_2 \in Ax_2$ and $r > 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I+rA)$. We also know that for an m-accretive operator A, we have $A^{-1}0 = F(J_r)$ for all $r > 0$. An operator $A \subset E \times E^*$ is called monotone if for any $(x_1, y_1), (x_2, y_2) \in A, \langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. A monotone operator $A \subset E \times E^*$ is called maximal if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

3 Main Results

In this section, we prove a strong convergence theorem by the viscosity approximation method with generalized contractions for resolvents of accretive operators.

We need first some auxiliary results.

A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an L-function if $\psi(0) = 0$, $\psi(t) > 0$, for each $t > 0$ and for every $s > 0$ there exists $u > s$ such that $\psi(t) \leq s$, for all $t \in [s, u]$. In consequence, every L-function ψ satisfies $\psi(t) < t$, for each $t > 0$.

Note that Rus in [22] (see also [23]) uses, in order to establish a fixed point result, the concept of comparison function. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. In consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0. Also note that in their fixed point theorem, Boyd and Wong (see [24]) work with a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions $\phi(0) = 0, \phi(t) < t$ for all $t > 0$ and ϕ is right upper semicontinuous. It is easy to see that the function φ, as well as the function ϕ, is an L-function.

Definition 3.1 *Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be*

(i) *a (ψ, L)-contraction if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$, for all $x, y \in X$, with $x \neq y$;*

(ii) *an Meir–Keeler type mapping if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \delta$ we have $d(f(x), f(y)) < \epsilon$.*

Remark 3.1 It is known that the condition (ii) from the above definition is equivalent to the following one:

(ii)$'$ for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in X$ with $d(x, y) < \epsilon + \delta$ we have $d(f(x), f(y)) < \epsilon$.

Remark 3.2 If, in Definition 3.2 we consider $\psi(t) = at$, for each $t \in \mathbb{R}_+$ (where $a \in [0, 1[$), then we get the usual contraction mapping with coefficient a. Other examples of L-functions are $\psi(t) = \frac{t}{1+t}$ and $\psi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$. However under the usual contraction condition, as well as under the Matkowski–Rus and Boyd–Wong conditions below (see Theorem 2.1 and

Theorem 2.2) the inequality is not strict. To get a strict one as under the (ψ, L)-contraction condition just replace the function φ or ϕ by $\frac{t+\varphi(t)}{2}$ or respectively by $\frac{t+\phi(t)}{2}$.

The following fixed point results are known.

Theorem 3.1 (Matkowski, see e.g. [25], Rus [22]) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$. Then $F(f) = \{x^*\}$.*

Theorem 3.2 (Boyd–Wong [24]) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions $\phi(0) = 0, \phi(t) < t$ for all $t > 0$ and ϕ is right upper semicontinuous, such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$. Then $F(f) = \{x^*\}$.*

Theorem 3.3 (Reich [26]) *Let (X, d) be a complete metric space and $f : X \to X$ be a (ψ, L)-contraction. Then $F(f) = \{x^*\}$.*

Theorem 3.4 (Meir–Keeler [27]) *Let (X, d) be a complete metric space and $f : X \to X$ be a Meir–Keeler type mapping. Then $F(f) = \{x^*\}$.*

Remark 3.3 The fixed point theorems presented above are both significant extensions of the celebrated Banach–Caccioppoli contraction principle, see Kirk, Sims [28].

A very useful characterization of Meir–Keeler mappings in terms of (ψ, L)-functions was given by Lim.

Theorem 3.5 (Lim [29]) *Let (X, d) be a metric space and $f : X \to X$ be a mapping. The following assertions are equivalent*:

(i) *f is a Meir–Keeler type mapping*;

(ii) *There exists an L-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that f is a (ψ, L)-contraction.*

The following results are important for the main theorems of this paper.

Proposition 3.1 (Suzuki [30]) *Let X be a Banach space and C a convex subset of it. Let $f : C \to C$ be a Meir–Keeler type mapping. Then for each $\epsilon > 0$ there exists $k \in]0, 1[$ such that*

$$\text{for each } x, y \in C \text{ with } \|x - y\| \geq \epsilon \text{ we have } \|f(x) - f(y)\| \leq k\|x - y\|.$$

Proposition 3.2 (See [30]) *Let X be a Banach space and C a convex subset of X. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ be a Meir–Keeler type mapping. Then the following assertions hold*:

(i) *$T \circ f$ is a Meir–Keeler type mapping on C*;

(ii) *For each $\alpha \in]0, 1[$ the mapping $x \mapsto \alpha f(x) + (1 - \alpha)T(x)$ is a Meir–Keeler type mapping on C.*

Proposition 3.3 (See [31]) *Let X be a Banach space and C a convex subset of it. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ be a (ψ, L)-contraction. Then the following assertions hold*:

(i) *$T \circ f$ is a (ψ, L)-contraction on C and has a unique fixed point in C*;

(ii) *For each $\alpha \in]0, 1[$ the mapping $x \mapsto \alpha f(x) + (1 - \alpha)T(x)$ is of Meir–Keeler type and it has a unique fixed point in C.*

From now on by a generalized contraction mapping we mean a Meir–Keeler type mapping or a (ψ, L)-contraction. In the rest of the paper we suppose that the L-function from the characterization theorem (see Theorem 3.5), as well as, the function ψ from the definition of the (ψ, L)-contraction is strictly increasing and the function $\eta(t) := t - \psi(t)$, $t \in \mathbb{R}_+$ is strictly increasing and onto. In consequence, we have that η is a bijection on $\mathbb{R}_+$. It's worth noting that all the functions ψ given in Remark 3.2 satisfy the above additional assumptions.

Our first main result is the following theorem.

Theorem 3.6 *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm and C is a closed convex subset of E which has normal structure. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and satisfying the condition*

$$\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA).$$

Let J_r be the resolvent of A for all $r > 0$ and $f : C \to C$ be a generalized contraction.

Then the following assertions hold:

(i) *For each $r > 0$ the operator $J_r \circ f$ has a unique fixed point $u_r \in C$;*

(ii) *The net $\{u_r\} \to u \in A^{-1}0$ as $r \to +\infty$, where $u = P_{A^{-1}0} f(u)$, with $P_{A^{-1}0}$ the unique sunny nonexpansive retraction of C onto $A^{-1}0$.*

Proof (i) follows from Proposition 3.2 (i) and Proposition 3.3 (i).

(ii) Consider $z \in A^{-1}0$. then we have

$$\begin{aligned} \|u_r - z\| &= \|J_r f(u_r) - J_r z\| \leq \|f(u_r) - z\| \\ &\leq \|f(u_r) - f(z)\| + \|f(z) - z\| \\ &\leq \psi(\|u_r - z\|) + \|f(z) - z\|. \end{aligned}$$

Hence

$$\|u_r - z\| \leq \eta^{-1}(\|f(z) - z\|)$$

and thus the sequence $\{u_r\}$ is bounded. Also, we have

$$\begin{aligned} \|f(u_r) - z\| &\leq \|f(u_r) - f(z)\| + \|f(z) - z\| \\ &\leq \psi(\|u_r - z\|) + \|f(z) - z\| \\ &\leq \psi(\eta^{-1}(\|f(z) - z\|)) + \|f(z) - z\|. \end{aligned}$$

Hence the sequence $\{f(u_r)\}$ is bounded too.

Next, as in the proof of Theorem 3.1 in [19], we consider for any sequence $\{r_i\}$ of positive real numbers with $r_i \to +\infty$, the functional $g : C \to \mathbb{R}$ defined by

$$g(x) = \mu_i \|u_{r_i} - x\|^2,$$

where μ_i is a Banach limit. Put $u_i := u_{r_i}$. The functional g is continuous and convex and has the properties:

a) $\{g(x_n)\} \to +\infty$ as $\|x_n\| \to +\infty$;

b) g has a minimizer in C.

Define $K := \{v \in C : g(v) := \min\{g(x) : x \in C\}\}$. Then, from [32], the set K is nonempty bounded closed and convex. Let us show now that K is invariant under J_r, for any $r > 0$. We have

$$\begin{aligned}\|J_r u_i - u_i\| &= r\|A_r u_i\| \le r|Au_i| \\ &= r|AJ_{r_i} f(u_i)| \le r\|A_{r_i} f(u_i)\| \\ &= \frac{r}{r_i}\|f(u_i) - J_{r_i} f(u_i)\| = \frac{r}{r_i}\|f(u_i) - u_i\| \to 0, \text{ as } i \to \infty.\end{aligned}$$

Then, for any $v \in K$,

$$\begin{aligned}\mu_i\|u_{r_i} - J_r v\|^2 &\le \mu_i(\|u_{r_i} - J_r u_i\| + \|J_r u_i - J_r v\|)^2 \\ &= \mu_i\|J_r u_i - J_r v\|^2 \le \mu_i\|u_i - v\|^2.\end{aligned}$$

Thus $J_r v \in K$. By Kirk's theorem [21], we get that J_r has a fixed point $v_0 \in K$. Since v_0 is a minimizer for g on C, by a theorem of Takahashi and Ueda in [33], we have that

$$\mu_i\langle x - v_0, J(u_i - v_0)\rangle \le 0, \text{ for all } x \in C.$$

Putting $x := f(v_0)$, we obtain

$$\mu_i\langle f(v_0) - v_0, J(u_i - v_0)\rangle \le 0.$$

Since A is accretive we can write

$$\begin{aligned}\|u_i - v_0\|^2 &= \langle u_i - v_0, J(u_i - v_0)\rangle \\ &= \langle J_{r_i} f(u_i) - f(u_i) + f(u_i) - v_0, J(u_i - v_0)\rangle \\ &= -r_i\langle A_{r_i} f(u_i), J(u_i - v_0)\rangle + \langle f(u_i) - v_0, J(u_i - v_0)\rangle \\ &\le \langle f(u_i) - v_0, J(u_i - v_0)\rangle.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\mu_i\|u_i - v_0\|^2 &\le \mu_i\langle f(u_i) - v_0, J(u_i - v_0)\rangle \\ &= \mu_i\langle f(u_i) - f(v_0) + f(v_0) - v_0, J(u_i - v_0)\rangle \\ &\le \mu_i\langle f(u_i) - f(v_0), J(u_i - v_0)\rangle \\ &\le \mu_i(\|f(u_i) - f(v_0)\|\|u_i - v_0\|).\end{aligned}$$

We will prove now that there exists a subsequence $\{u_{i_j}\}$ of $\{u_i\}$ such that $\{u_{i_j}\} \to v_0$ as $j \to +\infty$. Suppose by contradiction that each subsequence of $\{u_i\}$ does not converge to v_0. Then there exists $\varepsilon > 0$ such that $\|u_{i_j} - v_0\| \ge \varepsilon$ for all $j \in \mathbb{N}$. Then, using Proposition 3.1, we have

$$\begin{aligned}\mu_{i_j}\|u_{i_j} - v_0\|^2 &\le \mu_{i_j}(\|f(u_{i_j}) - f(v_0)\|\|u_{i_j} - v_0\|) \\ &\le \mu_{i_j}(k\|u_{i_j} - v_0\|^2) = k\mu_{i_j}\|u_{i_j} - v_0\|^2,\end{aligned}$$

so we obtain $(1-k)\mu_{i_j}\|u_{i_j} - v_0\|^2 \le 0$ and hence $\mu_{i_j}\|u_{i_j} - v_0\|^2 = 0$. Next, let us observe that

$$0 \le \liminf_{j\to\infty} \mu_{i_j}\|u_{i_j} - v_0\|^2 \le \mu_{i_j}\|u_{i_j} - v_0\|^2 = 0.$$

Thus, there exists a subsequence of $\{u_{i_j}\}$ that converges strongly to v_0, which is a contradiction to our hypothesis. Hence, we have proved that there exists a subsequence $\{u_{i_j}\}$ of $\{u_i\}$ such that $\{u_{i_j}\} \to v_0$ as $j \to +\infty$.

Since

$$A_r f(u_r) = \tfrac{1}{r}(f(u_r) - J_r f(u_r)) \in AJ_r f(u_r) = Au_r,$$

taking into account that A is accretive, we get that

$$0 \leq \langle A_r f(u_r), J(u_r - v)\rangle = \frac{1}{r}\langle f(u_r) - u_r, J(u_r - v)\rangle \text{ for any } v \in A^{-1}0.$$

Hence we obtain

$$\langle f(v_0) - v_0, J(v_0 - v)\rangle \geq 0, \text{ for all } v \in A^{-1}0.$$

In order to prove that $\{u_r\}$ converges strongly to an element $u \in A^{-1}0$, we consider a subsequence $\{u_{r_j}\}$ of $\{u_r\}$ such that $\{u_{r_j}\} \to z$ as $r_j \to +\infty$. Then, for any $r > 0$ we have

$$\begin{aligned} \|z - J_r z\| &\leq \|z - u_{r_j}\| + \|u_{r_j} - J_r u_{r_j}\| + \|J_r u_{r_j} - J_r z\| \\ &\leq 2\|z - u_{r_j}\| + \|u_{r_j} - J_r u_{r_j}\|. \end{aligned}$$

Since $\|u_{r_j} - J_r u_{r_j}\| \to 0$ as $j \to +\infty$ we get that $z = J_r z$. From $z \in A^{-1}0$ and $\langle f(v_0) - v_0, J(v_0 - v)\rangle \geq 0$, for all $v \in A^{-1}0$ we have

$$\langle f(v_0) - v_0, J(v_0 - z)\rangle \geq 0.$$

From $v_0 \in A^{-1}0$ and $\langle f(u_r) - u_r, J(u_r - v_0)\rangle \geq 0$ we also have that

$$\langle f(z) - z, J(z - v_0)\rangle \geq 0.$$

Hence from the above two relations we obtain

$$\langle f(v_0) - v_0 - f(z), J(v_0 - z)\rangle \geq 0,$$

and hence

$$\|v_0 - z\|^2 \leq \langle f(v_0) - f(z), J(v_0 - z)\rangle \leq \|f(v_0) - f(z)\|\|v_0 - z\| \leq \psi(\|v_0 - z\|)\|v_0 - z\|.$$

This implies, using the properties of ψ, that $\|v_0 - z\| = 0$ and hence $v_0 = z$. Therefore, $\{u_r\}$ converges strongly to an element $u \in A^{-1}0$.

Since $\langle f(u) - u, J(u - z)\rangle \geq 0$, for all $z \in A^{-1}0$ and taking into account that P is a sunny nonexpansive retraction of C onto $A^{-1}0$ we get $Pf(u) = u$. The proof is complete. □

Using Theorem 3.6 we obtain the following strong convergence theorem for the proximal point algorithm by the viscosity approximation method for generalized contraction for resolvents of accretive operators in Banach spaces. This is our second main result. The proof follows from the approach given in Theorem 4.1 in [19]. See also [16] and [34].

Theorem 3.7 *Let E be a reflexive Banach space with a uniformly Gateaux differentiable norm, and C be a closed convex subset of E which has normal structure. Let $A \subset E \times E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and satisfies the condition*

$$\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA).$$

Let J_r be the resolvent of A for all $r > 0$ and $f : C \to C$ be a generalized contraction.

Consider the sequence $\{x_n\}$ in C defined by

$$x_1 := x \in C \text{ and } x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n, \text{ for all } n \in \mathbb{N}^*,$$

where $\{\alpha_n\} \subset]0,1[$ and $\{r_n\} \subset]0,+\infty[$ satisfy the conditions:

a) $\lim_{n\to+\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

b) $\lim_{n\to+\infty} r_n = +\infty$.

Then the sequence $\{x_n\} \to u \in A^{-1}0$ as $n \to +\infty$, where $u = P_{A^{-1}0} f(u)$, with $P_{A^{-1}0}$ the unique sunny nonexpansive retraction of C onto $A^{-1}0$.

Proof Let $z \in A^{-1}0$ and define $M := \max\{\|x_1 - z\|, \eta^{-1}(\|f(z) - z\|)\}$. We will prove by induction that

$$\|x_n - z\| \le M, \text{ for all } n \in \mathbb{N}^*.$$

The case $n = 1$ is obvious. Suppose that $\|x_n - z\| \le M$. We have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n - z\| \\ &\le \alpha_n \|f(x_n) - z\| + (1-\alpha_n)\|J_{r_n} x_n - z\| \\ &\le \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\ &\le \alpha_n \psi(M) + \alpha_n \eta(\eta^{-1}(\|f(z) - z\|)) + (1-\alpha_n) M \\ &\le \alpha_n \psi(M) + \alpha_n \eta(M) + (1-\alpha_n) M = M. \end{aligned}$$

The induction is complete and hence the sequence $\{x_n\}$ is bounded. In consequence, the sequences $\{f(x_n)\}$ and $\{J_{r_n} x_n\}$ are bounded too.

For any $r > 0$ we know from Theorem 2.6 that there exists a unique $u_r \in C$ such that $u_r = J_r f u_r$ and $u_r \to u$ as $r \to +\infty$, where $u = P_{A^{-1}0} f(u)$, with $P_{A^{-1}0}$ the unique sunny nonexpansive retraction of C onto $A^{-1}0$.

In a similar way to the proof of Theorem 4.1 in [19] we obtain that

$$\limsup_{n\to+\infty} \langle f(u) - u, J(x_n - u)\rangle \le 0.$$

Indeed, let us show first that the above inequality is equivalent to the following one:

$$\limsup_{n\to+\infty} \langle f(u) - u, J(J_{r_n} x_n - u)\rangle \le 0. \qquad (*)$$

Since

$$x_{n+1} - J_{r_n} x_n = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n - J_{r_n} x_n = \alpha_n (f(x_n) - J_{r_n} x_n),$$

we obtain, by letting $n \to +\infty$, that

$$\lim_{n\to+\infty} (x_{n+1} - J_{r_n} x_n) = 0.$$

Since the map J is uniformly continuous on any bounded subset of the space E, which has a uniformly Gateaux differentiable norm, we get that

$$\lim_{n\to+\infty} |\langle f(u) - u, J(x_{n+1} - u)\rangle - \langle f(u) - u, J(J_{r_n} x_n - u)\rangle| = 0,$$

and thus the equivalence is proved.

Thus, it is sufficient to prove that

$$\limsup_{n\to\infty}\langle f(u)-u, J(J_{r_n}x_n-u)\rangle \le 0.$$

In this respect, let $\epsilon > 0$. Taking into account that $u_r \to u$ as $r \to +\infty$ and the fact that J is uniformly continuous on any bounded subset of the space E, there exists $r_0 > 0$ such that

$$|\langle f(u)-u, J(J_{r_n}x_n-u)\rangle - \langle f(u)-u, J(J_{r_n}x_n-u_r)\rangle| < \frac{\epsilon}{3},$$

for all $r \ge r_0$ and all $n \in \mathbb{N}$.

Also, since the sequences $\{u_r\}$ and $\{J_{r_n}x_n\}$ are bounded and f is continuous, there exists $s_0 > 0$ such that

$$|\langle f(u)-u, J(J_{r_n}x_n-u_r)\rangle - \langle f(u_r)-u_r, J(J_{r_n}x_n-u_r)\rangle| < \frac{\epsilon}{3},$$

for all $r \ge s_0$ and all $n \in \mathbb{N}$.

On the other hand, since $A_r f(u_r) \in AJ_r f(u_r) = Au_r$ and $A_{r_n}x_n \in AJ_{r_n}x_n$, we have

$$0 \le \langle A_{r_n}x_n - A_r f(u_r), J(J_{r_n}x_n-u_r)\rangle = \left\langle A_{r_n}x_n - \frac{1}{r}(f(u_r)-u_r), J(J_{r_n}x_n-u_r)\right\rangle.$$

Hence we get

$$\langle f(u_r)-u_r, J(J_{r_n}x_n-u_r)\rangle \le r\langle A_{r_n}x_n, J(J_{r_n}x_n-u_r)\rangle.$$

We also know that

$$\|A_{r_n}x_n\| = \frac{1}{r_n}\|x_n - J_{r_n}x_n\| \to 0 \text{ as } n \to +\infty.$$

Hence, for a fixed $r \ge \max\{r_0, s_0\}$, there exists $n_0 \in \mathbb{N}$ such that

$$\langle f(u_r)-u_r, J(J_{r_n}x_n-u_r)\rangle < \frac{\epsilon}{3}, \text{ for all } n \ge n_0.$$

Thus, for all $n \ge n_0$ we successively have

$$\begin{aligned}\langle f(u)-u, J(J_{r_n}x_n-u)\rangle &= \langle f(u)-u, J(J_{r_n}x_n-u)\rangle - \langle f(u)-u, J(J_{r_n}x_n-u_r)\rangle \\ &\quad + \langle f(u)-u, J(J_{r_n}x_n-u_r)\rangle - \langle f(u_r)-u_r, J(J_{r_n}x_n-u_r)\rangle \\ &\quad + \langle f(u_r)-u_r, J(J_{r_n}x_n-u_r)\rangle < \epsilon.\end{aligned}$$

Hence, by passing to the $\limsup_{n\to+\infty}$ and using the fact that ϵ was arbitrarily chosen, the inequality $(*)$ is proved.

Next, from

$$x_{n+1}-u = \alpha_n(f(x_n)-u) + (1-\alpha_n)(J_{r_n}x_n-u),$$

we have

$$(1-\alpha_n)^2\|J_{r_n}x_n-u\|^2 \ge \|x_{n+1}-u\|^2 - 2\alpha_n\langle f(x_n)-u, J(x_{n+1}-u)\rangle.$$

Hence

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le (1-\alpha_n)^2\|J_{r_n}x_n-u\|^2 + 2\alpha_n\langle f(x_n)-u, J(x_{n+1}-u)\rangle \\ &\le (1-\alpha_n)^2\|x_n-u\|^2 + 2\alpha_n\langle f(x_n)-f(u), J(x_{n+1}-u)\rangle \\ &\quad + 2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle\end{aligned}$$

$$\leq (1-\alpha_n)^2\|x_n-u\|^2 + 2\alpha_n\psi(\|x_n-u\|)\|x_{n+1}-u\| \\ + 2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle.$$

We show now that the sequence $\{x_n\}$ converges strongly to u. Assume by contradiction that it does't converge. Then there exist $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}-u\| \geq \varepsilon$, for each $i \in \mathbb{N}$. Then there exists $k \in]0,1[$ such that

$$\|f(x_{n_i}) - f(u)\| \leq k\|x_{n_i}-u\|, \text{ for all } i \in \mathbb{N}.$$

For the sake of simplicity we also denote this subsequence by $\{x_n\}$. Then

$$\begin{aligned}\|x_{n+1}-u\|^2 &\leq (1-\alpha_n)^2\|x_n-u\|^2 + 2\alpha_n k\|x_n-u\|\|x_{n+1}-u\| \\ &\quad + 2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle \\ &\leq (1-\alpha_n)^2\|x_n-u\|^2 + \alpha_n k[\|x_n-u\|^2 + \|x_{n+1}-u\|^2] \\ &\quad + 2\alpha_n\langle f(u)-u, J(x_{n+1}-u)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\|x_{n+1}-u\|^2 \\ &\leq \frac{(1-\alpha_n)^2+\alpha_n k}{1-\alpha_n k}\|x_n-u\|^2 + \frac{2\alpha_n}{1-\alpha_n k}\langle f(u)-u, J(x_{n+1}-u)\rangle \\ &= \frac{(1-2\alpha_n+\alpha_n k}{1-\alpha_n k}\|x_n-u\|^2 + \frac{\alpha_n^2}{1-\alpha_n k}\|x_n-u\|^2 + \frac{2\alpha_n}{1-\alpha_n k}\langle f(u)-u, J(x_{n+1}-u)\rangle \\ &\leq \left(1-\frac{2(1-k)\alpha_n}{1-\alpha_n k}\right)\|x_n-u\|^2 + \frac{2(1-k)\alpha_n}{1-\alpha_n k}\left[\frac{\alpha_n S}{2(1-k)} + \frac{1}{1-k}\langle f(u)-u, J(x_{n+1}-u)\rangle\right],\end{aligned}$$

where $S := \sup\|x_n-u\|^2$.

Denote $\beta_n := \frac{2(1-k)\alpha_n}{1-k\alpha_n}$. Then $\lim_{n\to+\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$.

Let $\delta > 0$. From the above relation and from

$$\limsup_{n\to\infty}\langle f(u)-u, J(x_n-u)\rangle \leq 0,$$

we have that there exists $m \in \mathbb{N}^*$ such that for each $n \geq m$ we have

$$\frac{\alpha_n S}{2(1-k)} \leq \frac{\delta}{2} \text{ and } \frac{1}{1-k}\langle f(u)-u, J(x_{n+1}-u)\rangle \leq \frac{\delta}{2}.$$

Therefore we have

$$\|x_{n+1}-u\|^2 \quad \leq (1-\beta_n)\|x_n-u\|^2 + [1-(1-\beta_n)]\delta.$$

Then we obtain

$$\|x_{n+m}-u\|^2 \leq \prod_{k=m}^{m+n-1}(1-\beta_k)\|x_m-u\|^2 + \left[1-\prod_{k=m}^{m+n-1}(1-\beta_k)\right]\delta.$$

Hence

$$\limsup_{n\to\infty}\|x_n-u\|^2 = \limsup_{n\to\infty}\|x_{n+m}-u\|^2 \leq \delta,$$

and thus, since δ is arbitrary,

$$\limsup_{n\to\infty}\|x_n-u\|^2 \leq 0.$$

This shows that the subsequence $\{x_{n_i}\}$ converges strongly to u. The contradiction permits us to conclude that $\{x_n\}$ converges strongly to u. □

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