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Strong Commutativity-preserving Generalized Derivations on Semiprime Rings

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Abstract Let R be a ring with a subset S. A mapping of R into itself is called strong commutativity-preserving (scp) on S, if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. The main purpose of this paper is to describe the structure of the generalized derivations which are scp on some ideals and right ideals of a prime ring, respectively. The semiprime case is also considered.

Keywords (semi-)prime ring, generalized derivation, GPI, strong commutativity-preserving

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Mappings which preserve certain algebraic properties of elements in a ring are widely studied in functional analysis and linear algebra. A map f of a ring R is called commutativity-preserving, if $[f(x), f(y)] = 0$ whenever x and y are commuting elements of R. For example, all Lie endomorphisms are commutativity-preserving. In 1994, Bell and Mason generalized the definition [1]. If R is a ring and $S \subseteq R$, a mapping of R into itself is called strong commutativity-preserving (scp) on S, if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [2], Bell and Daif proved that the nonzero right ideal ρ of a semiprime ring R is a central ideal if there exists a derivation d which is scp on ρ. Further results about scp mappings can be found in [3–4].

As derivations are widely studied, in recent years, many authors began to study the properties of generalized derivations, and some of their results generalize the former on derivations, see [5–12]. In this paper, which can be thought of as belonging to both of these categories, we study generalized derivations which are scp on some subsets of prime or semiprime rings, and one of our results generalizes the theorem of [2, Theorem 1].

Throughout this paper let R denote a semiprime ring with center $Z(R) = Z$, ρ a nonzero right ideal, I a nonzero ideal, C its extended centroid, U its Utumi quotient ring, Q_s its symmetric quotient ring. D always denotes a generalized derivation on R. Since every generalized

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derivation of R can be uniquely extended to a generalized derivation of U [6], we may assume $D(x) = xa + \delta(x)$, where $a \in U$, δ is a derivation of R, and a, δ are uniquely determined by D.

Theorem 1 *Let R be a semiprime ring with a right ideal ρ and a generalized derivation $D(x) = xa + \delta(x)$, where $a \in U$ and δ is a derivation of R. If D is scp on ρ, then $\delta(\rho)\rho \subseteq Z$.*

Proof Let $d(x) = [x, a] + \delta(x)$. Then d is a derivation of R and $D(x) = ax + d(x)$. Since D is scp on ρ, for all $x, y \in \rho$, we have

$$[x, xy] = [D(x), D(xy)] = D(x)[D(x), y] + [D(x), x]d(y) + x[D(x), d(y)].$$

On the other hand,

$$[x, xy] = x[x, y] = x[D(x), D(y)].$$

Making use of the last two equations and $D(x) = ax + d(x)$, we get

$$x[D(x), ay] = D(x)[D(x), y] + [D(x), x]d(y).$$

Replace y by yr for $r \in R$ to obtain

$$xay[D(x), r] = D(x)y[D(x), r] + [D(x), x]yd(r). \tag{1}$$

Replacing r by $D(x)$, $x \in \rho$, gives

$$[D(x), x]ydD(x) = 0, \qquad x, y \in \rho. \tag{2}$$

We claim that $[D(x), x]\rho = 0$ for all $x \in \rho$. In fact, let P be an arbitrary prime ideal of R. By (2), for all $x \in \rho$, we have

$$[D(x), x]\rho \subseteq P \qquad \text{or} \qquad dD(x) \in P. \tag{3}$$

Assume that there exists $x_0 \in \rho$ such that $[D(x_0), x_0]\rho \nsubseteq P$. Then $dD(x_0) \in P$ and for all $y \in \rho$,

$$\begin{aligned}[x_0, yD(x_0)] &= [D(x_0), D(yD(x_0))] = [D(x_0), D(y)D(x_0) + ydD(x_0)] \\ &\equiv [D(x_0), D(y)]D(x_0) \mod P \\ &\equiv [x_0, y]D(x_0) \mod P.\end{aligned}$$

On the other hand,

$$[x_0, yD(x_0)] = [x_0, y]D(x_0) + y[x_0, D(x_0)], \qquad y \in \rho,$$

so we have $\rho[x_0, D(x_0)] \subseteq P$. Then $\rho \subseteq P$ or $[x_0, D(x_0)] \in P$. Either of these conditions implies $[x_0, D(x_0)]\rho \subseteq P$, a contradiction, so $[D(x), x]\rho \subseteq P$ for all $x \in \rho$. Since R is a semiprime ring, the intersection of the prime ideals of R is zero. Thus

$$[D(x), x]\rho = 0. \tag{4}$$

Recalling (1) and $D(x) - xa = \delta(x)$ gives

$$\delta(x)y[D(x), r] = 0, \qquad x, y \in \rho, \ r \in R. \tag{5}$$

Replace r by $D(z)$ for $z \in \rho$ to get $\delta(x)y[x, z] = 0$. Replacing z by zr, $r \in R$, gives

$$\delta(x)yxzr - \delta(x)yzrx = 0, \qquad x, y, z \in \rho, \ r \in R.$$

Let P be a prime ideal of R. Use Martindale's theorem [13, Theorem 2] in R/P to get that $\delta(x)\rho^2 \subseteq P$ or $[x, R] \subseteq P$ for all $x \in \rho$. If $\delta(x)\rho^2 \subseteq P$, then $\delta(x)\rho \subseteq P$. Hence $\delta(x)\rho \subseteq P$ or $[x, R] \subseteq P$ for all $x \in \rho$. Then ρ is the union of its two subgroups A and B, where $A = \{x \in \rho \mid \delta(x)\rho \subseteq P\}$, $B = \{x \in \rho \mid [x, R] \subseteq P\}$, so $\delta(\rho)\rho \subseteq P$ or $[\rho, R] \subseteq P$. Either of these cases yields $[\delta(x)y, z] \in P$, $x, y, z \in \rho$. Since P is an arbitrary prime ideal of R, $[\delta(x)y, z] = 0$. Replace z by $z = zr$ for $r \in R$ to see that $z[\delta(x)y, r] = 0$. Then $\rho \subseteq P$ or $[\delta(\rho)\rho,\ R] \subseteq P$ for an arbitrary prime ideal P of R. Either case leads to $[\delta(\rho)\rho, R] = 0$, i.e., $\delta(\rho)\rho \subseteq Z$.

As a corollary of Theorem 1 we get [2, Theorem 1].

Corollary 2 *Let R be a semiprime ring, ρ a nonzero right ideal of R and δ a derivation of R. If δ is scp on ρ, then $\rho \subseteq Z$.*

Proof By Theorem 1, $\delta(\rho)\rho \subseteq Z$. Replacing x, y by xs, yt with $s, t \in \rho$ in $[\delta(x), \delta(y)] = [x, y]$, respectively, we get

$$\begin{aligned}[xs, yt] &= [\delta(xs), \delta(yt)] = [x\delta(s), y\delta(t)] = x\delta(s)y\delta(t) - y\delta(t)x\delta(s) \\ &= \delta(s)y\delta(t)x - \delta(t)x\delta(s)y = \delta(t)x\delta(s)y - \delta(t)x\delta(s)y = 0.\end{aligned}$$

Then $[\rho^2, \rho^2] = 0$. By the corollary of Herstein's [14, Lemma 1.1.5], the commutative one-sided ideals of a semiprime ring are central. Then $\rho^2 \subseteq Z$. Hence $0 = [xy, r] = [x, r]y + x[y, r]$ for all $x, y \in \rho$, $r \in R$. Replacing y by yz for $z \in \rho$ yields $xy[z, r] = 0$, i.e., $xRy[z, r] = 0$. Taking $x = y[z, r]$ leads to $y[z, r] = 0$. Then replace r by rr_1 for $r_1 \in R$ to see that $[x, R]Ry = 0$. Thus $y \in P$ or $[x, R] \subseteq P$, $x, y, z \in \rho$, where P is an arbitrary prime ideal of R. Either case leads to $[\rho, R] \subseteq P$, and hence $\rho \subseteq Z$.

In Theorem 1, if R is a prime ring, the form of the scp generalized derivations could be described much clearer. The following lemma shows that there is only a special kind of generalized inner derivations may be scp on a nonzero right ideal of noncommutative prime rings.

Lemma 3 *Let R be a noncommutative prime ring with a nonzero right ideal ρ and a generalized derivation D. If D is scp on ρ, then there exist some $a \in Q_s$ and $b \in U$ such that $D(x) = ax + xb$, $x \in R$, and $a\rho = 0$.*

Proof Let $D(x) = xm + \delta(x)$, where δ is a derivation of R and $m \in U$. By Theorem 1, if $\delta(\rho)\rho \neq 0$, R contains a nonzero central ideal, and therefore R is commutative. Otherwise $\delta(\rho)\rho = 0$. If $\delta = 0$, we are done. If $\delta \neq 0$, $\delta(x) = [q, x]$ for some $q \in Q_s$ satisfying $q\rho = 0$, which is based on [15, Theorem 4], so

$$D(x) = xm + [q, x] = qx + x(m - q), \qquad x \in R.$$

Take $a = q$, $b = m - q$ to complete the proof.

In Lemma 3, if ρ is a nonzero ideal of R, we can describe the scp generalized derivations much more precisely.

Theorem 4 *Let R be a noncommutative prime ring with a nonzero ideal I and a nonzero generalized derivation D. If D is scp on I, then $D(x) = x$ or $D(x) = -x$.*

Proof By Lemma 3, there exist some $a \in Q_s$, $b \in U$ such that $D(x) = ax + xb$, $x \in R$, and $aI = 0$. Then $a = 0$ and $D(x) = xb$. By [16, Theorem 2], I and U satisfy the same differential identities.

Since D is scp on I, we may assume

$$[x, y] = [xb, yb], \qquad x, y \in U. \tag{6}$$

In particular, taking $y = 1$ yields $[xb, b] = 0$ for all $x \in U$. Replace x by x_1x_2 for x_1, $x_2 \in U$ to get $[U, b]Ub = 0$. Then $b = 0$ or $b \in C$. But $D \neq 0$, so $b \in C$. It follows from (6) that $b^2 = 1$. Since R is a prime ring, its extended centroid C is a field. Thus $b = 1$ or $b = -1$. Consequently, $D(x) = x$, $x \in R$, or $D(x) = -x$, $x \in R$.

In Theorem 8 we will give all of the possibilities for a generalized derivation to be scp on a nonzero right ideal of a prime ring. To prove Theorem 8, the following lemmas are needed. Lemma 5 is based on [16–17] and [18, Theorem 6.1.6 and Theorem 4.3.7]. It has been discussed in many papers, for example [19, p. 341].

Lemma 5 *Let R be a prime ring with a dense right ideal ρ, $f(x_1, \ldots, x_m)$ be a nontrivial generalized polynomial with coefficients in U. If $f(x_1, \ldots, x_m)$ is a GPI for ρ, then there exists a vector space V_F over a field F so that $R \subseteq U \subseteq \mathrm{End}(V_F)$ and $\mathrm{End}(V_F)$ satisfies $f(x_1, \ldots, x_m)$. Moreover, the C-independent elements in U are also F-independent.*

Lemma 6 *Let V be a vector space over a field F with $\dim_F V \geq 3$, P a nontrivial right ideal of $\mathrm{End}(V_F)$ and $b \in \mathrm{End}(V_F)$. If $[X, Y] = [Xb, Yb]$ is a GPI for P, then $\{v, w, bw\}$ is F-dependent for all $w \in V$. Moreover, there are $e \in P$, $v \in V$ such that $ev = v$ and e is a minimal idempotent element in $\mathrm{End}(V_F)$.*

Proof Since P is nontrivial, there are $e \in P$, $v \in V$ such that $ev = v$ with e minimal and idempotent. Assume that there is a $w \in V$ such that $\{v, w, bw\}$ is F-independent. Choose $x_0, y_0 \in \mathrm{End}(V_F)$ so that $x_0bw = 0$, $x_0w = 0$, $x_0v = v$, $y_0bw = 0$, $y_0w = v$. Then $[ex_0b, ey_0b]w = 0$, $[ex_0, ey_0]w = v \neq 0$, a contradiction. Therefore $\{v, w, bw\}$ is F-dependent for all $w \in V$.

Lemma 7 *Let V be a vector space over a field F with $\dim_F V \geq 2$, P be a nontrivial right ideal of $\mathrm{End}(V_F)$ and $b \in \mathrm{End}(V_F)$. Then $[X, Y] = [Xb, Yb]$ is a GPI for P if and only if b is invertible, P is a minimal right ideal of $\mathrm{End}(V_F)$ and one of the following situations happens:*

(i) $\dim_F V = 2$ *and* $\det(b) = 1$.

(ii) $\dim_F V \geq 3$, $P = e\mathrm{End}(V_F)$, $ev = v$, $ew_\alpha = 0$, $bw_\alpha = c_\alpha v + dw_\alpha$ *and* $bv = d^{-1}v$. *Here e is a minimal idempotent element in $\mathrm{End}(V_F)$, $\{v\} \cup \{w_\alpha\}_{\alpha \in A}$ is a basis of V_F, $d, c_\alpha \in F$ and $d \neq 0$.*

Proof First suppose P satisfies $[X, Y] = [Xb, Yb]$. If $\dim_F(V) = 2$, any nontrivial right ideal of $M_2(F)$ is minimal. Assume $P = \{e_{11}x_1 + e_{12}x_2 \mid x_1, x_2 \in F\}$. Let $b = e_{11}b_1 + e_{12}b_2 + e_{21}b_3 + e_{22}b_4$, $b_i \in F$, $x, y \in P$ with $x = e_{11}x_1 + e_{12}x_2$ and $y = e_{11}y_1 + e_{12}y_2$. Then $[x, y] = e_{12}(x_1y_2 - x_2y_1)$ and

$$[xb, yb] = e_{12}(x_1y_2 - x_2y_1)(b_1b_4 - b_2b_3).$$

Since P satisfies $[X, Y] = [Xb, Yb]$, $\det(b) = 1$. For any nontrivial right ideal P' of $M_2(F)$, there is an invertible element $q \in M_2(F)$ such that $P' = qPq^{-1}$. From the discussion above we have $\det(q^{-1}bq) = 1$. Thus $\det(b) = 1$ and b is invertible.

Suppose $\dim_F(V) \geq 3$. By Lemma 6, there are $e \in P$, $v \in V$ such that e is a minimal idempotent element in $\mathrm{End}(V_F)$, $ev = v$ and $\{v, w, bw\}$ is F-dependent for all $w \in V$.

We claim that $bV = V$. Assume $r(b) = 1$, where $r(b)$ denotes the rank of b. Let $bV = Fu$. Then u and v are F-independent. Otherwise $bV = Fv$. Take $v_1 \in V$ such that v_1 is F-independent of v. Let $bv_1 = c_1 v$. Choose $x_0, y_0 \in \mathrm{End}(V_F)$ such that $x_0 v = v, y_0 v = v, y_0 v_1 = v, x_0 v_1 = 0$. Then

$$[ex_0 b, ey_0 b]v_1 = 0, \qquad [ex_0, ey_0]v_1 = v \neq 0,$$

a contradiction. Thus u and v are F-independent. Then $V = Fv \oplus Fu \oplus V_0$ and $\dim(V_0) \geqslant 1$. Lemma 6 shows that $bV_0 = 0$. Since $r(b) \neq 0$, there are $c_1, c_2 \in F$ and $c_1 c_2 \neq 0$ such that $b(c_1 v + c_2 u) \neq 0$. Then $\{v, w = c_1 v + c_2 u + v_1, bw = b(c_1 v + c_2 u)\}$ is F-independent for any $0 \neq v_1 \in V_0$, contradicting Lemma 6. Thus $r(b) \geqslant 2$. Set $bV = V_1$. Then $v \in V_1$. Otherwise, set $V = Fv \oplus V_1 \oplus V_2$. Then $bV_1 \subseteq V_1$. By Lemma 6, there exists $c \in F$ such that $bu = cu$ for each $u \in V_1$. If $bv \neq 0$, we may choose $u \in V_1$ so that bv and u are F-independent, since $r(b) \geqslant 2$. Then $\{v, w = u + v, bw = cu + bv\}$ is F-independent, contradicting Lemma 6. Thus $bv = 0$. For each $0 \neq w \in V_1$, choose $x_1, y_1 \in \mathrm{End}(V_F)$ such that $x_1 w = 0,\ y_1 w = v$ and $x_1 v = v$, we get

$$[ex_1 b, ey_1 b]w = 0, \qquad [ex_1, ey_1]w = v \neq 0,$$

a contradiction, so $v \in V_1$ and $V = V_1 \oplus V_2$. Assume $V_1 \subsetneq V$. Then $bV_2 \subseteq Fv$, by Lemma 6. Since $r(b) \geqslant 2$, there exists $u \in V_1$ such that $bu \notin Fv$. Take $w = u + v_2$ and let $bv_2 = cv$ with $0 \neq v_2 \in V_2$ and $c \in F$. Then $\{v, w, bw = bu + cv\}$ is F-independent, contradicting Lemma 6. Thus $bV = V_1 = V$.

Next, we will show that b is invertible. Let $V = Fv \oplus V_0$. Then $\dim_F(V_0) \geqslant 2$. We may assume without loss of generality that $eV_0 = 0$. Let $\{w_\alpha\}_{\alpha \in A}$ be a basis of V_0. By Lemma 6, for each $w \in V_0$, there exist $c_w, d_w \in F$ such that

$$bw = c_w v + d_w w.$$

Choose $u \in V_0$ such that u and w are F-independent. There are c_u, d_u, c_{w+u}, $d_{w+u} \in F$ such that $bu = c_u v + d_u u$ and $b(w+u) = c_{w+u} v + d_{w+u}(w+u)$. Then

$$c_{w+u} v + d_{w+u}(w+u) = b(w+u) = bw + bu = (c_w + c_u)v + d_w w + d_u u.$$

Since $\{v, w, u\}$ is F-independent, $c_{w+u} = c_w + c_u$ and $d_{w+u} = d_w = d_u$. Take $d = d_u$. Then for each $w \in V_0$ there exists $c_w \in F$ such that

$$bw = c_w v + dw,$$

where d is independent of w. It follows from $bV = V$ that $d \neq 0$. Let $bv = c_0 v + u_1$ with $u_1 \in V_0$. If $u_1 \neq 0$, take $v_1 \in V_0$ so that u_1 and v_1 are F-independent. Then $\{v, w = v + v_1, bw = (c_0 + c_{v_1})v + u_1 + dv_1\}$ is F-independent, contradicting Lemma 6. Thus $u_1 = 0$, $bv = c_0 v$, so

$$\begin{cases} bv = c_0 v, \\ bw_\alpha = c_\alpha v + dw_\alpha, \quad \alpha \in A, \end{cases}$$

where $c_0, d, c_\alpha \in F$. Since $bV = V$, $c_0, d \in F$ are each nonzero. For each $\alpha \in A$, choose $x_\alpha, y_\alpha \in \mathrm{End}(V_F)$ such that $x_\alpha v = v$, $y_\alpha v = v$, $x_\alpha w_\alpha = 0$, $y_\alpha w_\alpha = v$. Then $[x_\alpha, y_\alpha]w_\alpha = v$ and $[x_\alpha b, y_\alpha b]w_\alpha = dc_0 v$, so $c_0 = d^{-1}$. Now it is easy to verify that b is invertible.

Finally, we will prove that P is a minimal right ideal of $\mathrm{End}(V_F)$. Assert $P = e\mathrm{End}(V_F)$. Otherwise, assume that $P \nsubseteq e\mathrm{End}(V_F)$. Then there are $\bar{x} \in P \setminus e\mathrm{End}(V_F)$ and $v' \in V$ such that $\bar{x}v' \notin Fv$ and $(\bar{x}-e\bar{x})v'$ vanish in the direction of v while represented by the basis $\{v\} \cup \{w_\alpha\}_{\alpha\in A}$. Let

$$(\bar{x} - e\bar{x})v' = \sum_{i=1}^{k} d_i w_{\alpha_i} = w_0$$

be the representation of $(\bar{x} - e\bar{x})v'$ with respect to the basis $\{v\} \cup \{w_\alpha\}_{\alpha\in A}$, where $d_i \in F$, $1 \le i \le k$, are all nonzero. Replace w_{α_1} by w_0 to get that $\{v\} \cup \{w_0\} \cup \{w_\alpha\}_{\alpha\in A\setminus\{\alpha_1\}}$ is also a basis of V. Choose $r \in \mathrm{End}(V_F)$ such that $rw_0 = v'$ and $ru' = 0$ for all $u' \in \{v\} \cup \{w_\alpha\}_{\alpha\in A\setminus\{\alpha_1\}}$. Then $(\bar{x} - e\bar{x})rw_0 = w_0$, and $(\bar{x} - e\bar{x})r \in P$ is a minimal idempotent element in $\mathrm{End}(V_F)$. From the discussion above we conclude that there exist $\bar{d}$, $\bar{c}_v$, $\bar{c}_\alpha \in F$ with $\bar{d} \neq 0$ such that

$$\begin{cases} bu = \bar{d}^{-1}u, \\ bv = \bar{c}_v u + \bar{d}v, \\ bw_\alpha = \bar{c}_\alpha u + \bar{d}w_\alpha, \quad \alpha \in A \setminus \{\alpha_1\}. \end{cases}$$

Then

$$\bar{c}_v u + \bar{d}v = bv = d^{-1}v, \qquad \bar{d}^{-1}u = bu = c_u v + du,$$

so we have $d^{-1} = \bar{d}$, $\bar{d}^{-1} = d$ and $\bar{c}_v = c_u = 0$. For any $\beta \in A \setminus \{\alpha_1\}$,

$$\bar{c}_\beta u + \bar{d}w_\beta = bw_\beta = c_\beta v + dw_\beta.$$

Then $d = \bar{d} = d^{-1}$ and $c_\beta = \bar{c}_\beta = 0$. Since $b \notin C$, there exists $\alpha \in A \setminus \{\alpha_1\}$ so that $c_\alpha \neq 0$, a contradiction. Thus $P = e\mathrm{End}(V_F)$ is a minimal right ideal of $\mathrm{End}(V_F)$.

Conversely, it is easy to verify that P satisfies the GPI $[X, Y] = [Xb, Yb]$ when either of the two situations happens.

Theorem 8 *Let R be a noncommutative prime ring, ρ a nonzero right ideal of R, and D a generalized derivation of R. If D is scp on ρ, then there exists some $b \in U$ which is invertible and $a \in Q_s$ such that $D(x) = ax + xb$ and $a\rho = 0$. Moreover, $b = \pm 1$, or else $b \notin C$, ρC is a minimal right ideal of RC, $RC \subseteq U \subseteq \mathrm{End}(V_F)$, where V_F is a vector space over a field F, and one of the following situations holds*:

(i) $\dim_F V = 2$ *and* $\det(b) = 1$.

(ii) $\dim_F V \geq 3$, $\rho C \subseteq e\mathrm{End}(V_F)$, $ev = v$, $ew_\alpha = 0$, $\alpha \in A$, $bv = d^{-1}v$, *and* $bw_\alpha = c_\alpha v + dw_\alpha$. *Here e is a minimal idempotent element in* $\mathrm{End}(V_F)$, $\{v\} \cup \{w_\alpha\}_{\alpha\in A}$ *is a basis of* V_F, $d, c_\alpha \in F$ *and* $d \neq 0$.

Conversely, D is scp on ρC when either of the situations happens.

Proof By Lemma 3, there are some $a \in Q_s, b \in U$ such that $D(x) = ax + xb$, $x \in R$, and $a\rho = 0$, so $[x, y] = [xb, yb]$ for all $x, y \in \rho$. If $b \in C$, $[x, y](b^2 - 1) = 0$. Since R is noncommutative and C is a field, $b = 1$ or $b = -1$. Now we may assume $b \notin C$. Then for any nonzero $x_0, y_0 \in \rho$,

$$[x_0 X, y_0 Y] = [x_0 X b, y_0 Y b] \tag{7}$$

is a nontrivial GPI for R. By Lemma 5, there exists a vector space V_F over a field F such that $RC \subseteq U \subseteq \mathrm{End}(V_F)$ and (7) is also a GPI for $\mathrm{End}(V_F)$. Set $P = \rho\mathrm{End}(V_F)$. Then P satisfies $[X, Y] = [Xb, Yb]$ and $\rho C \subseteq P$. Since R is noncommutative, $\dim_F(V) \geqslant 2$. By Lemma 7, there only remains to show that ρC is a minimal right ideal of RC. By [18, Theorem 4.3.7], RC is a primitive ring with nonzero socle and there is a minimal idempotent element f of RC which lies in ρC. Since P is a minimal right ideal of $\mathrm{End}(V_F)$ and $f \neq 0$, we have $f\mathrm{End}(V_F) = P$. Since $\rho C \subseteq P = f\mathrm{End}(V_F)$, $\rho C \subseteq fRC$. Since fRC is a minimal right ideal of RC, $\rho C = fRC$ is a minimal right ideal of RC.

Theorem 4 gives all of the possible forms of the generalized derivations which are scp on a nonzero ideal of a prime ring. And Theorem 8 shows that the forms of the generalized derivations which are scp on a nonzero right ideal of a prime ring are not so simple. The following example shows that the last case in Theorem 8 really happens.

Example Let R be a matrix ring over a field F, say, $R = M_n(F)$, $n \geq 2$. Then R is a prime ring. Set $e = e_{11}$, $b = \sum_{i=1}^{n} e_{ii} + \sum_{i=2}^{n} e_{1i}$ and $\rho = eR$. Then ρ is a nonzero minimal right ideal of R and $[x, y] = [xb, yb]$ for all $x, y \in \rho$. But $b \notin C$.

References

[1] Bell, H. E., Mason, G.: On derivations in near rings and rings. *Math. J. Okayama Univ.*, **34**, 135–144 (1992)

[2] Bell, H. E., Daif, M. N.: On commutative and strong commutativity-preserving maps. *Canad. Math. Bull.*, **37**(4), 443–447 (1994)

[3] Brešar, M., Miers, C. R.: Strong commutativity preserving maps of semiprime rings. *Canad. Math. Bull.*, **37**(4), 457–460 (1994)

[4] Deng, Q.: On strong commutativity preserving mappings. *Results Math.,* **30**(3–4), 259–263 (1996)

[5] Hvala, B.: Generalized derivations in rings. *Comm. Algebra,* **26**(4), 1147–1166 (1998)

[6] Lee, T. K.: Generalized derivations of left faithful rings. *Comm. Algebra,* **27**(8), 4057–4073 (1999)

[7] Lee, T. K., Shiue, W. K.: Identities with generalized derivations. *Comm. Algebra,* **29**(10), 4437–4450 (2001)

[8] Argac, N., Nakajima, A., Albas, E.: On orthogonal generalized derivations of semiprime rings. *Turk. J. Math.,* **28**, 185–194 (2004)

[9] Wei, F.: Generalized derivations with nilpotent values on semiprime rings. *Acta Mathematica Sinica, English Series,* **20**(3), 453–462 (2004)

[10] Ma, J. and Xu, X. W.: Cocentralizing generalized derivations in prime rings. *Northeast. Math. J.,* **22**(1), 105–113 (2006)

[11] Xu, X. W., Ma, J., Niu, F. W.: Generalized derivations with skew nilpotent values on Lie ideals. *Northeast. Math. J.* **22**(2), 241–252 (2006)

[12] Xu, X. W., Ma, J., Niu, F. W.: Annihilators of power central values of generalized derivations. *Chinese Ann. Math.,* **28**(A), 131–140 (2007)

[13] Martindale III, W. S.: Prime rings satisfying a generalized polynomial identity. *J. Algebra,* **12**, 576–584 (1969)

[14] Herstein, I. N.: *Rings with involution.* Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, Ill.-London, 1976

[15] Lanski, C.: Left deals and derivations in semiprime rings. *J. Algebra,* **277**, 658–667 (2004)

[16] Chuang, C. L.: GPIs having coefficients in Utumi quotient rings. *Proc. Amer. Math. Soc.,* **103**(3), 723–728 (1988)

[17] Erickson, T. S., Martindale 3rd, W. S. and Osborn, J. M.: Prime nonassociative algebras. *Pacific J. Math.,* **60**, 49–63 (1975)

[18] Beidar, K. I., Martindale III, W. S., Mikhalev, A. V.: Rings with generalized identities, New York, Marcel Dekker, 1996

[19] Lanski, C.: An Engel condition with derivation for left ideal. *Proc. Amer. Math. Soc.*, **125**, 339–345 (1997)