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Symmetric Lévy Type Operator

Jian WANG
School of Mathematical and Computer Science, Fujian Normal University, Fuzhou 350007, *P. R. China*
E-mail: jianwang@fjnu.edu.cn

Abstract The study of symmetric property in the $\mathbb{L}^2$-sense for the non-positive definite operator is motivated by the theory of probability and analysis. This paper presents some sufficient conditions for the existence of symmetric measure for Lévy type operator. Some new examples are illustrated. The present study is an important step for considering various ergodic properties and functional inequalities of Lévy type operator.

Keywords Lévy type generator, symmetric measure

MR(2000) Subject Classification 60J35, 60J75

1 Introduction and Main Results

In this paper, we consider a symmetric measure of Lévy type operator, which is given by

$$\begin{aligned} Lf(x) = & \sum_{i,j=1}^d a_{ij}(x)\partial_{ij}f(x) + \sum_{i=1}^d b_i(x)\partial_i f(x) \\ & + \int_{\mathbb{R}^d\setminus\{0\}} \Big(f(x+z) - f(x) - \mathbf{1}_{\{|z|\le 1\}} \sum_{i=1}^d z_i\partial_i f(x)\Big)\nu(x,dz), \end{aligned} \tag{1.1}$$

where for each $x \in \mathbb{R}^d$, $a(x) := (a_{ij}(x))$ is non-negative definite, $a_{ij}(x) \in C^2(\mathbb{R}^d)$ for $1 \le i, j \le d$, $b(x) := (b_i(x))$ is a vector of measurable functions and $\nu(x, dz)$ is a Lévy-jump measure kernel, i.e. $\nu(x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d \setminus \{0\}$ and satisfies $\int(1 \wedge |z|^2)\nu(x, dz) < +\infty$ for every $x \in \mathbb{R}^d$. So far as we know, there is a very limited literature about symmetric measure for Lévy type operator. One can see [1] for recent study on this topic. On the other hand, we know that the ordinary Lévy process, whose infinitesimal generator is just (1.1) with constant coefficients (that is, $a(x)$, $b(x)$ and $\nu(x, dz)$ are independent of x), is symmetric with respect to the Lebesgue measure dx if and only if the drift term in the characteristic exponent vanishes and the Lévy measure is symmetric, see [2] for details. We are interested in those Lévy type operators, which possess the symmetric probability measure. Then, we can consider various ergodic properties and functional inequalities of such operators. This is actually a starting point of this paper.

Denote $\mathscr{D}(L)$ as the domain of operator L and $C_0^\infty(\mathbb{R}^d)$ as the infinite times differentiable functions on $\mathbb{R}^d$ with compact support. It is clear that $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(L)$. Recall that the

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operator $(L, C_0^\infty(\mathbb{R}^d))$ is symmetric with respect to a measure μ if

$$(f, Lg) = (Lf, g) \quad \text{for every } f, g \in C_0^\infty(\mathbb{R}^d) \cap \mathbb{L}^2(\mu), \tag{1.2}$$

where (,) indicates the scalar product in $\mathbb{L}^2(\mu)$.

Firstly, we need some assumptions on the symmetric measure μ and the Lévy measure $\nu(x, dz)$.

Assumptions (H) (1) *The functions* a_{ij}, b_i *and* $\int(1 \wedge |z|^2)\nu(x, dz)$ *all belong to* $\mathbb{L}^2_{\text{loc}}(\mu)$. *The function* $f_r(x) := \mathbf{1}_{B_{2r}(0)}(x) \int_{\{|z|>r\}} \nu(x, dz)$ *belongs to* $\mathbb{L}^2(\mu)$ *for large enough* r.

(2) $\mu(dx)$ *is absolutely continuous with respect to the Lebesgue measure* dx, *i.e. there exists a nonnegative function* $\rho(x)$ *such that* $\mu(dx) = \rho(x)dx$.

(3) *For each* $x \in \mathbb{R}^d$, $\nu(x, dz)$ *satisfies the following condition*:

$$\int_{\{|z|\le 1\}} |z||\nu(x, dz) - \nu(x, d(-z))| < +\infty, \tag{1.3}$$

where $|m(dz)|$ *denotes the total variational measure of a signed measure* $m(dz)$.

Now, it is time to present our main results in this paper.

Theorem 1.1 *If Assumptions* (H) *hold and the density function* $\rho \in C^1(\mathbb{R}^d)$ *is positive everywhere, furthermore,* $\nu(x, dz)$ *and* $b_i(x) (i = 1, 2, \dots, d)$ *have the following form*

$$\begin{aligned} &\nu(x, dz) = j(x, x + z)\rho(x + z)dz, \\ &b_i(x) = \sum_{j=1}^d (a_{ij}(x)\rho^{-1}(x)\partial_j\rho(x) + \partial_j a_{ij}(x)) + \frac{1}{2}\int_{\{|z|\le 1\}} z_i(\nu(x, dz) - \nu(x, -dz)), \end{aligned} \tag{1.4}$$

where $j(x, y)$ *is a non-negative measurable function in* $\mathbb{R}^{2d}$ *such that* $j(x, x) = 0$ *and* $j(x, y) = j(y, x)$ *for every* $x, y \in \mathbb{R}^d$, *then the Lévy type operator* L *defined by* (1.1) *is symmetric with respect to the measure* μ.

Remark 1.1 First, condition (1) in Assumptions (H) assures that L maps $C_0^\infty(\mathbb{R}^d)$ into $\mathbb{L}^2(\mu)$, which will be proved in Lemma 2.1 in Section 2. Second, condition (1.3) in Assumptions (H) ensures that the integral term of $b_i(x)$ in Theorem 1.1 is well defined. On the other hand, by Theorem 1.1, (1.3) is equivalent to

$$\int_{\{|z|\le 1\}} |z||j(x, x + z)\rho(x + z) - j(x, x - z)\rho(x - z)|dz < +\infty. \tag{1.5}$$

Now, if $\nu(x, dz) = 0$ for each $x \in \mathbb{R}^d$, the operator (1.1) becomes the diffusion operator

$$L_0 := \sum_{i,j=1}^d a_{ij}\partial_{ij} + \sum_{i=1}^d b_i\partial_i, \tag{1.6}$$

and the condition (1.4) in Theorem 1.1 becomes

$$b_i(x) = \sum_{j=1}^d (a_{ij}(x)\rho^{-1}(x)\partial_j\rho(x) + \partial_j a_{ij}(x)), \qquad 1 \le i \le d. \tag{1.7}$$

This is a well-known fact for elliptic operators, see [3] for more examples. Therefore, what we need to do is to consider a symmetric measure of the perturbation of drift type for pure Lévy type jump operator:

$$L_1 f(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x + z) - f(x) - \mathbf{1}_{\{|z|\le 1\}}\nabla f(x) \cdot z)\nu(x, dz) + c(x) \cdot \nabla f(x), \tag{1.8}$$

where $c(x) = (c_1(x), c_2(x), \dots, c_n(x))$ is a vector of measurable function. For this, we have the following result.

Theorem 1.2 *If Assumptions* (H) *hold and* $\nu(x,dz)$ *and* $c_i(x)(i=1,2,\dots,d)$ *have the following form*:

$$\begin{aligned}\nu(x,dz) &= j(x,x+z)\rho(x+z)dz,\\ c_i(x) &= \frac{1}{2}\int_{\{|z|\le 1\}} z_i(\nu(x,dz)-\nu(x,-dz)),\end{aligned}\tag{1.9}$$

where $j(x,y)$ *is defined as in Theorem* 1.1, *then the pure Lévy type operator* L_1 *given by* (1.8) *is symmetric with respect to the measure* μ.

The paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 are presented in Section 2. In Section 3, we study the regularity of the Dirichlet forms corresponding to the pure Lévy jump operator L_1 with the symmetric measure μ. To conclude this section, some examples are presented to illustrate the power of Theorem 1.1 and Theorem 1.2. The first two examples are used for Theorem 1.2.

Example 1.1 If the symmetric measure $\mu(dx)=dx$, then $\rho=1$. The condition (1.3) turns to

$$\int_{\{|z|\le 1\}} |z||j(x,x+z)-j(x,x-z)|dz<+\infty,\quad x\in\mathbb{R}^d.\tag{1.10}$$

There are many nonnegative symmetric functions $j(x,y)$ satisfying (1.10). For example, it suffices that for every $x\in\mathbb{R}^d$, there exist $c(x)\in(0,+\infty)$ and $\alpha(x)>-d-1$ such that

$$|j(x,x+z)-j(x,x-z)|\le c(x)|z|^{\alpha(x)}.$$

Example 1.2 This example is a pure Lévy type operator possessing a symmetric probability measure. Set

$$\begin{aligned}L_0f(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^d\setminus\{0\}}(f(x+z)-f(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla f(x)\cdot z)\frac{c(x,x+z)}{|z|^{n+\alpha}}\mathrm{e}^{-\frac{|x+z|^2}{2}}dz,\\ b(x) &= \frac{1}{2(2\pi)^{n/2}}\int_{\{|z|\le 1\}} z\left(\frac{c(x,x+z)}{|z|^{n+\alpha}}\mathrm{e}^{-\frac{|x+z|^2}{2}}-\frac{c(x,x-z)}{|z|^{n+\alpha}}\mathrm{e}^{-\frac{|x-z|^2}{2}}\right)dz,\end{aligned}$$

and

$$Lf(x)=L_0f(x)+b(x)\cdot\nabla f(x),\tag{1.11}$$

where $\alpha\in(0,2)$ and $c(x,y)$ is a nonnegative symmetric bounded function. Then the operator defined by (1.11) is symmetric with respect to the standard Gaussian measure $\mu(dx)=(2\pi)^{-n/2}\mathrm{e}^{-\frac{|x|^2}{2}}dx$. Furthermore, by Proposition 3.1 below, the operator L corresponds to a conservative Dirichlet form D, which has a simple form as follows:

$$D(f)=\frac{1}{(2\pi)^n}\int\int\frac{c(x,y)}{|x-y|^{n+\alpha}}(f(x)-f(y))^2\mathrm{e}^{-\frac{|x|^2+|y|^2}{2}}dxdy.\tag{1.12}$$

The Dirichlet form (1.12) is compared with the classical Dirichlet form

$$D_1(f)=\frac{\alpha 2^{\alpha-1}\Gamma((n+\alpha)/2)}{\pi^{n/2}\Gamma(1-\alpha/2)}\int\int\frac{1}{|x-y|^{n+\alpha}}(f(x)-f(y))^2dxdy,$$

which is closely associated with symmetric α-stable process, see [4] for details. Note that the symmetric measure for $D_1(f)$ is Lebesgue measure dx, which is not finite.

Lastly, we present an example for Theorem 1.1.

Example 1.3 We begin with the classical Ornstein–Uhlenbeck operator on $\mathbb{R}$ with $L_0:=\frac{d^2}{dx^2}-x\frac{d}{dx}$, We know that L_0 has a unique symmetric probability measure $\mu(dx):=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}dx$. Now, following Theorem 1.1 or Example 1.2 above, we can construct the general Ornstein–Uhlenbeck type operator associated with Lévy type operator. Set $\nu(x,dz)=\frac{1}{\sqrt{2\pi}}j(|z|)\mathrm{e}^{-\frac{(x+z)^2}{2}}dz$, and

$$\begin{aligned}L_1f(x)=&\frac{1}{\sqrt{2\pi}}\int(f(x+z)-f(x)-\mathbf{1}_{\{|z|\le1\}}f'(x)\cdot z)j(|z|)\mathrm{e}^{-\frac{(x+z)^2}{2}}dz\\&+\frac{1}{2\sqrt{2\pi}}\int_{\{|z|\le1\}}zj(|z|)(\mathrm{e}^{-\frac{(x+z)^2}{2}}-\mathrm{e}^{-\frac{(x-z)^2}{2}})dz\cdot f'(x),\end{aligned}$$

where $j(x)$ is a nonnegative Borel measurable function on $\mathbb{R}$, such that

$$\int j(|z|)(1\wedge|z|^2)\mathrm{e}^{-\frac{(x+z)^2}{2}}dz<\infty\qquad\text{for } x\in\mathbb{R}.$$

Then the operator $L:=L_0+L_1$ is symmetric with respect with to μ.

2 Proofs of Theorems 1.1 and 1.2

Firstly, we prove the following lemma.

Lemma 2.1 *Under the condition* (1) *in Assumptions* (H), *the operator L given in* (1.1) *maps $C_0^\infty(\mathbb{R}^d)$ into $\mathbb{L}^2(\mu)$.*

Proof For any $f\in C_0^\infty(\mathbb{R}^d)$, we choose $r>0$ so that $\mathrm{supp}(f)\subset B_r(0)$. Then

$$||Lf||_{\mathbb{L}^2(\mu)}\le||\mathbf{1}_{B_{2r}(0)}Lf||_{\mathbb{L}^2(\mu)}+||\mathbf{1}_{B_{2r}^c(0)}Lf||_{\mathbb{L}^2(\mu)}.\tag{2.1}$$

First, we have

$$\begin{aligned}\mathbf{1}_{B_{2r}(0)}(x)Lf(x)=&\mathbf{1}_{B_{2r}(0)}(x)\sum_{i,j=1}^d a_{ij}(x)\partial_{ij}f(x)+\mathbf{1}_{B_{2r}(0)}(x)\sum_{i=1}^d b_i(x)\partial_i f(x)\\&+\mathbf{1}_{B_{2r}(0)}(x)\int_{\{|z|>1\}}(f(x+z)-f(x))\nu(x,dz)\\&+\mathbf{1}_{B_{2r}(0)}(x)\int_{\{|z|\le1\}}\Big(f(x+z)-f(x)-\sum_{i=1}^d z_i\partial_i f(x)\Big)\nu(x,dz)\\\le&\sum_{|\alpha|\le2}||\partial^\alpha f||_\infty\mathbf{1}_{B_{2r}(0)}(x)\Big(\sum_{i,j=1}^d|a_{ij}|(x)+\sum_{i=1}^d|b_i|(x)\\&+2\int_{\{|z|>1\}}\nu(x,dz)+\frac12\int_{\{|z|\le1\}}z^2\nu(x,dz)\Big).\end{aligned}\tag{2.2}$$

On the other hand,

$$\begin{aligned}\mathbf{1}_{B_{2r}^c(0)}(x)Lf(x)&=\mathbf{1}_{B_{2r}^c(0)}(x)\int_{\{|z|>1\}}f(x+z)\nu(x,dz)\\&\le||f||_\infty\mathbf{1}_{B_{2r}^c(0)}(x)\int_{\{|z|>1\}}\mathbf{1}_{B_r(0)}(x+z)\nu(x,dz).\end{aligned}$$

Note that if $|x|>2r$ and $|x+z|\le r$, then $|z|\ge|x|-|x+z|>2r-r=r$. Therefore,

$$\mathbf{1}_{B_{2r}^c(0)}(x)Lf(x)\le||f||_\infty\mathbf{1}_{B_{2r}^c(0)}(x)\int_{\{|z|>r\}}\nu(x,dz).\tag{2.3}$$

Thus, combining (2.1), (2.2), (2.3) with the condition (1) in Assumptions (H), we have

$$\begin{aligned}||Lf||_{\mathbb{L}^2(\mu)}\le&2\sum_{|\alpha|\le2}||\partial^\alpha f||_\infty\Big[\int_{B_{2r}(0)}\Big(\sum_{i,j=1}^d|a_{ij}|(x)+\sum_{i=1}^d|b_i|(x)+\int1\wedge|z|^2\nu(x,dz)\Big)^2\mu(dx)\Big]^{1/2}\\&+||f||_\infty\Big[\int_{B_{2r}^c(0)}\Big(\int_{\{|z|>r\}}\nu(x,dz)\Big)^2\mu(dx)\Big]^{1/2}\\<&\infty,\end{aligned}$$

which means that L maps $C_0^\infty(\mathbb{R}^d)$ into $\mathbb{L}^2(\mu)$.

Now, we present the proof of Theorem 1.2.

Proof of Theorem 1.2 Let $f, g \in C_0^\infty(\mathbb{R}^d)$. We need to prove the following

$$(f, L_1 g) = -\frac{1}{2}\int\int (f(x)-f(y))(g(x)-g(y))j(x,y)\mu(dx)\mu(dy) =: -D(f,g). \tag{2.1}$$

We begin the proof from the right hand. First, set $J_0(dz,dx) = j(x,x+z)\rho(x+z)\rho(x)dzdx$, then, replacing y by $x+z$, we have

$$\begin{aligned}
D(f,g) &= \frac{1}{2}\int\int (f(y)-f(x))(g(y)-g(x))j(x,y)\mu(dx)\mu(dy)\\
&= \frac{1}{2}\int\int (f(x+z)-f(x))(g(x+z)-g(x))j(x,x+z)\rho(x)\rho(x+z)dxdz\\
&= \frac{1}{2}\int\int (f(x+z)-f(x))(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x+z)\cdot z)J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int (f(x+z)-f(x))(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int_{\{|z|\le 1\}} (f(x+z)-f(x))[(\nabla g(x+z)+\nabla g(x))\cdot z]J_0(dz,dx)\\
&= -\int\int f(x)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int f(x+z)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x+z)\cdot z)J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int f(x+z)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int_{\{|z|\le 1\}} f(x)[(\nabla g(x+z)-\nabla g(x))\cdot z]J_0(dz,dx)\\
&\quad + \frac{1}{2}\int\int_{\{|z|\le 1\}} (f(x+z)-f(x))[(\nabla g(x+z)+\nabla g(x))\cdot z]J_0(dz,dx)\\
&=: I_1+I_2+I_3+I_4+I_5,
\end{aligned} \tag{2.2}$$

where the third equality is obtained by inserting $\mathbf{1}_{\{|z|\le 1\}}\nabla g(x+z)\cdot z$ and $\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z$ in second term in second equality. Note that the expressions above are well defined, because of

$$\int (1\wedge |z|^2)j(x,x+z)\rho(x+z)dz < +\infty \qquad \text{for each } x\in\mathbb{R}^d. \tag{2.3}$$

For I_2, by changing variables and using the symmetry of $j(x,y)$, we have

$$\begin{aligned}
I_2 &= \frac{1}{2}\int\int f(x)(g(x)-g(x-z)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)j(x-z,x)\rho(x-z)\rho(x)dzdx\\
&= \frac{1}{2}\int\int f(x)(g(x)-g(x+z)+\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)j(x+z,x)\rho(x+z)\rho(x)dzdx\\
&= -\frac{1}{2}\int\int f(x)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx).
\end{aligned} \tag{2.4}$$

And using the same method, we also get

$$\begin{aligned}
I_3 &= \frac{1}{2}\int\int f(x)(g(x)-g(x-z)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x-z)\cdot z)j(x-z,x)\rho(x-z)\rho(x)dzdx\\
&= \frac{1}{2}\int\int f(x)(g(x)-g(x+z)+\mathbf{1}_{\{|z|\le 1\}}\nabla g(x+z)\cdot z)j(x+z,x)\rho(x+z)\rho(x)dzdx\\
&= -\frac{1}{2}\int\int f(x)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx)
\end{aligned}$$

$$+\frac{1}{2}\int\int_{\{|z|\le 1\}} f(x)[(\nabla g(x+z)-\nabla g(x))\cdot z]J_0(dz,dx). \tag{2.5}$$

Therefore, from (2.2), (2.4) and (2.5), we have

$$\begin{aligned}D(f,g) &= -2\int\int f(x)(g(x+z)-g(x)-\mathbf{1}_{\{|z|\le 1\}}\nabla g(x)\cdot z)J_0(dz,dx)\\ &\quad+\int\int_{\{|z|\le 1\}} f(x)[(\nabla g(x+z)-\nabla g(x))\cdot z]J_0(dz,dx)\\ &\quad+\frac{1}{2}\int\int_{\{|z|\le 1\}}(f(x+z)-f(x))[(\nabla g(x+z)+\nabla g(x))\cdot z]J_0(dz,dx)\\ &= 2I_1+2I_4+I_5.\end{aligned} \tag{2.6}$$

Now, we turn to considering I_4. Again, changing variables and using the symmetry of $j(x,y)$, we obtain

$$\begin{aligned}I_4 &= \frac{1}{2}\int\int_{\{|z|\le 1\}} f(x-z)[(\nabla g(x)-\nabla g(x-z))\cdot z]j(x-z,x)\rho(x-z)\rho(x)dzdx\\ &= \frac{1}{2}\int\int_{\{|z|\le 1\}} f(x+z)[(\nabla g(x+z)-\nabla g(x))\cdot z]j(x+z,x)\rho(x+z)\rho(x)dzdx\\ &= \frac{1}{2}\int\int_{\{|z|\le 1\}} f(x+z)[(\nabla g(x+z)-\nabla g(x))\cdot z]J_0(dz,dx),\end{aligned} \tag{2.7}$$

so

$$2I_4 = \frac{1}{2}\int\int_{\{|z|\le 1\}}(f(x+z)+f(x))[(\nabla g(x+z)-\nabla g(x))\cdot z]J_0(dz,dx). \tag{2.8}$$

Finally, we consider the last two terms in (2.6):

$$\begin{aligned}2I_4+I_5 &= \int\int_{\{|z|\le 1\}}[(f(x+z)\nabla g(x+z)-f(x)\nabla g(x))\cdot z]J_0(dz,dx)\\ &= \lim_{\epsilon\to 0}\int\int_{\{\epsilon<|z|\le 1\}}[(f(x+z)\nabla g(x+z)-f(x)\nabla g(x))\cdot z]J_0(dz,dx).\end{aligned}$$

Since for $\epsilon>0$,

$$\begin{aligned}&\int\int_{\{\epsilon<|z|\le 1\}}(f(x+z)\nabla g(x+z)\cdot z)J_0(dz,dx)\\ &\quad= \int\int_{\{\epsilon<|z|\le 1\}}(f(x)\nabla g(x)\cdot z)j(x-z,x)\rho(x-z)\rho(x)dzdx\\ &\quad= -\int\int_{\{\epsilon<|z|\le 1\}}(f(x)\nabla g(x)\cdot z)J(dx,d(-z)),\end{aligned}$$

we have

$$2I_4+I_5 = -\lim_{\epsilon\to 0}\int\int_{\{\epsilon<|z|\le 1\}}(f(x)\nabla g(x)\cdot z)(J_0(dx,dz)+J_0(dx,d(-z))).$$

Now according to (1.3), we can change the order of limit and integrals and obtain

$$\begin{aligned}2I_4+I_5 &= -\int\int_{\{|z|\le 1\}}(f(x)\nabla g(x)\cdot z)(J_0(dx,dz)+J_0(dx,d(-z)))\\ &= -(f,\nabla g\cdot c),\end{aligned} \tag{2.9}$$

where $c(x)$ is defined by (1.9).

Combining (2.6) and (2.9) together, we obtain (2.1), which proves our conclusion.

Next we come to the proof of Theorem 1.1.

Proof of Theorem 1.1 From Theorem 1.2, we know that

$$(f,L_1 g)=(L_1 f,g) \qquad \text{for every } f,g\in\mathscr{D}(L_1)\cap \mathrm{L}^2(\mu).$$

On the other hand, the operator $L_0 := L - L_1$ is a diffusion operator, then, by (1.4) (1.9) and (1.7), L_0 is symmetric with respect to μ, i.e.

$$(f, L_0 g) = (L_0 f, g) \qquad \text{for every } f, g \in \mathscr{D}(L_0) \cap \mathrm{L}^2(\mu).$$

Therefore, for every $f, g \in \mathscr{D}(L) \cap \mathrm{L}^2(\mu)$, we have

$$(Lf, g) = (L_0 f, g) + (L_1, g) = (f, L_0 g) + (f, L_1 g) = (f, Lg),$$

which is exactly our conclusion.

Remark 2.1 In the proof of Theorem 1.2, we use the approximation method to deal with $2I_4 + I_5$, which is due to the fact that a Lévy measure only satisfies (2.3) in general. We must point out that using the above method, if the Lévy measure $\nu(x, dz)$ satisfies the following stronger condition

$$\int_{\{|z|\le 1\}} |z|\nu(x, dz) < +\infty \text{ for each } x \in \mathbb{R}^d,$$

then under the condition in Theorem 1.2, the operator (1.8) turns to

$$\begin{aligned} L_1 f(x) &= \int_{\mathbb{R}^d\setminus\{0\}} (f(x+z) - f(x)) j(x, x+z)\rho(x+z)dz \\ &= \int_{\mathbb{R}^d\setminus\{0\}} (f(y) - f(x)) j(x, y)\mu(dy), \end{aligned}$$

which reduces to the classic situation for q-processes, see [2] for details.

3 Regularity of the Dirichlet Form

From Section 2, we know that Theorem 1.2 is the key to get Theorem 1.1. Our method is based essentially on the general theory of Dirichlet form (cf. [4]). If we assume that the operator L_1 (1.8) is non-positive definite self-adjoint operator in $\mathrm{L}_2(\mu)$, and corresponds to a regular conservative Dirichlet form, then the connection between Dirichlet form $(D, \mathscr{D}(D))$ and the operator L_1 is given by

$$D(f, g) = -(f, L_1 g) \quad \text{for} \quad f \in \mathscr{D}(D),\ g \in D(L_1). \tag{3.1}$$

In terms of Beuring–Deny decomposition of a regular Dirichlet form, it is clear to see that under our situation $D(f)$ has the following form

$$D(f) = \frac{1}{2}\int\int ((f(x) - f(y))^2 j(x, y)\mu(dx)\mu(dy), \tag{3.2}$$

where $j(x, y)$ is a nonnegative symmetric function such that $j(x, x) = 0$ for each $x \in \mathbb{R}^d$. Therefore, putting (3.1) and (3.2) together, we get (2.1), which explains our main idea in the proof of Theorem 1.2.

The following proposition gives some suitable conditions to make sure that the operator L_1 corresponds to a regular conservative Dirichlet form.

Proposition 3.1 *Under the conditions in Theorem* 1.2, *if the following condition holds*

$$\int (1 \wedge |z|^2)\nu(x, dz) \in \mathrm{L}^1_{loc}(\mu), \tag{3.3}$$

then the operator L_1 determines a conservative Dirichlet form, which is defined as (3.2). *Therefore, without confusion, we still denote it as $(D, \mathscr{D}(D))$, then $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(D)$. Furthermore, let $\mathscr{F}$ be the closure of $C_0^\infty(\mathbb{R}^d)$ under $D_1^{1/2}$, where $D_1^{1/2}$ denotes D_1-norm,* i.e. *$D_1^{1/2}(f) = \sqrt{D(f) + ||f||^2}$ for each $f \in \mathscr{D}(D)$. Then $(D, \mathscr{D}(D))$ is a regular Dirichlet form.*

Proof The idea of the proof comes from Example 1.2.4 in [4]. From the proof of Theorem 1.2 and (2.1), we know that L_1 corresponds to a symmetric form D, which is defined as (3.2). Therefore, it suffices to prove that D is a Dirichlet forms $(D, \mathscr{D}(D))$, and $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(D)$. From Example 1.2.4 in [4], we only need to prove the following two assertions:

(1) For any $\epsilon > 0$, set $f_\epsilon(x) = \int_{\mathbb{R}^d \setminus U_\epsilon(x)} j(x, y)\mu(dy)$, then $f_\epsilon(x) \in L^1_{\mathrm{loc}}(R^1)$, where $U_\epsilon(x)$ is ϵ-neighborhood of x.

(2) $\int\int_{K\times K} |x-y|^2 j(x, y)\mu(dx)\mu(dy) < +\infty$ for any compact $K \subset \mathbb{R}^d$.

For (1), assume $\epsilon \in (0, 1)$. Then for any compact set $K \subset \mathbb{R}^d$, by (3.3), we have

$$\begin{aligned}
\int_K f_\epsilon(x)dx &= \int_K \int_{\{|y-x|>\epsilon\}} j(x, y)\mu(dy)\mu(dx) \\
&= \int_K \int_{\{1\geq|y-x|>\epsilon\}} j(x, y)\mu(dy)\mu(dx) + \int_K \int_{\{|y-x|>1\}} j(x, y)\mu(dy)\mu(dx) \\
&\leq \epsilon^{-2} \int_K \int_{\{1\geq|y-x|>\epsilon\}} |x-y|^2 j(x, y)\mu(dy)\mu(dx) + \int_K \int_{\{|y-x|>1\}} j(x, y)\mu(dy)\mu(dx) \\
&\leq \epsilon^{-2} \int_K \int (1 \wedge |z|^2)\nu(x, dz)\mu(dx) < +\infty.
\end{aligned}$$

Now, we turn to (2). For any compact set $K \subset \mathbb{R}^d$, set $d = \sup\{|x-y| : x, y \in K\}$. Then, using (3.3), we have

$$\begin{aligned}
&\int\int_{K\times K} |x-y|^2 j(x, y)\mu(dx)\mu(dy) \leq \int_K \int_{\{|y-x|\leq d\}} |x-y|^2 j(x, y)\mu(dx)\mu(dy) \\
&\leq \int_K \int_{\{|y-x|\leq 1\}} |x-y|^2 j(x, y)\mu(dx)\mu(dy) + \int_K \int_{\{1<|y-x|\leq d\vee 1\}} |x-y|^2 j(x, y)\mu(dx)\mu(dy) \\
&\leq \int_K \int_{\{|y-x|\leq 1\}} |x-y|^2 j(x, y)\mu(dx)\mu(dy) + (d\vee 1)^2 \int_K \int_{\{1<|y-x|\leq d\vee 1\}} j(x, y)\mu(dx)\mu(dy) \\
&\leq (d\vee 1)^2 \int_K \int \left(1 \wedge |z|^2\right)\nu(x, dz)\mu(dx) < +\infty.
\end{aligned}$$

Thus we have got the assertions (1) and (2). The other assertions are trivial.

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Added in Proof The author acknowledges Prof. R. L. Schilling for pointing out a different view for Theorem 1.2. That is, Theorem 2.2 in his paper [6] establishes the connection between jump-type Dirichlet from and pure Lévy type operator under similar conditions like (1.9). Note that this paper also indicates some properties about Lévy type operator.

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