Acta Mathematica Sinica, English Series
Sep., 2008, Vol. 24, No. 9, pp. 1409–1432
Published online: July 28, 2008
DOI: 10.1007/s10114-008-7090-7
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Oscillation Theorems for Second-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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Abstract By employing the generalized Riccati transformation technique, we will establish some new oscillation criteria and study the asymptotic behavior of the nonoscillatory solutions of the second-order nonlinear neutral delay dynamic equation

$$[r(t)[y(t)+p(t)y(\tau(t))]^{\Delta}]^{\Delta}+q(t)f(y(\delta(t)))=0,$$

on a time scale $\mathbb{T}$. The results improve some oscillation results for neutral delay dynamic equations and in the special case when $\mathbb{T}=\mathbb{R}$ our results cover and improve the oscillation results for second-order neutral delay differential equations established by Li and Liu [*Canad. J. Math.*, **48** (1996), 871–886]. When $\mathbb{T}=\mathbb{N}$, our results cover and improve the oscillation results for second order neutral delay difference equations established by Li and Yeh [*Comp. Math. Appl.*, **36** (1998), 123–132]. When $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=\{t : t=q^k,\ k\in\mathbb{N},\ q>1\}$, $\mathbb{T}=\mathbb{N}^2=\{t^2 : t\in\mathbb{N}\}$, $\mathbb{T}=\mathbb{T}_n=\{t_n=\sum_{k=1}^{n}\frac{1}{k},\ n\in\mathbb{N}_0\}$, $\mathbb{T}=\{t^2 : t\in\mathbb{N}\}$, $\mathbb{T}=\{\sqrt{n} : n\in\mathbb{N}_0\}$ and $\mathbb{T}=\{\sqrt[3]{n} : n\in\mathbb{N}_0\}$ our results are essentially new. Some examples illustrating our main results are given.

Keywords oscillation, neutral delay dynamic equation, generalized Riccati technique, time scales
MR(2000) Subject Classification 34B10, 39A10, 34K11, 34C10

1 Introduction

The study of dynamic equations on time scales, which goes back to Stefan Hilger [1], is an area of mathematics that has recently received a lot of attention. It has been created to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice — once for differential equations and once again for difference equations. The general idea is to prove a

Received February 28, 2007, Accepted May 21, 2008

result for a dynamic equation where the domain of the unknown function is a nonempty closed subset of the reals $\mathbb{R}$ which is called a time scale $\mathbb{T}$. In this way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [2] and the references cited therein.

The three most popular examples of calculus on time scales are differential calculus, difference calculus and quantum calculus (see Kac and Cheung [3]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [4]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [5] discusses several possible applications. The books on the subject of time scales by Bohner and Peterson [4] summarizes and organizes much of time scale calculus and some applications.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of neutral delay dynamic equations on time scales, see for example [6–14]. Following this trend, in this paper we are concerned with the oscillation of the second-order nonlinear neutral delay dynamic equation

$$[r(t)[y(t) + p(t)y(\tau(t))]^{\Delta}]^{\Delta} + q(t)f(y(\delta(t))) = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$. Throughout this paper we assume that the delay functions $\tau(t) \leq t$ and $\delta(t) \leq t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$, $r(t)$, $p(t)$ and $q(t)$ are real-valued rd-continuous positive functions defined on $\mathbb{T}$, and

(h_1) $0 \leq p(t) < 1$, $f(u) : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(u) > 0$ and $f(u)/u \geq K > 0$ for all $u \neq 0$.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper these assumptions will be supposed to hold. Let $\tau^*(t) = \min\{\tau(t), \delta(t)\}$ and let $T_0 = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : \tau^*(s) \leq t\}$ for $t \geq T_0$. Clearly $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists.

By a solution of (1.1) we mean a nontrivial real-valued function $y(t)$ which has the properties $[y(t) + p(t)y(\tau(t))] \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$, and $r(t)\,[y(t) + p(t)y(\tau(t))]^{\Delta} \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

In studying the oscillation of higher order differential or difference equations, there are two techniques used to reduce the higher order equation to a first order Riccati equation or inequality. The first one is called the Riccati transformation technique. This technique has been already extended and employed to different types of neutral delay dynamic equations, and several sufficient conditions for oscillation have been obtained in [6–14].

In 2004 Agarwal et al. [6] considered the second-order nonlinear neutral delay dynamic equation

$$[r(t)([y(t)+p(t)y(t-\tau)]^{\Delta})^{\gamma}]^{\Delta}+f(t,y(t-\delta))=0, \tag{1.2}$$

on a time scale $\mathbb{T}$; here $\gamma>0$ is a quotient of odd positive integers, τ and δ are positive constants such that the delay functions $\tau(t):=t-\tau<t$ and $\delta(t):=t-\delta<t$ satisfy $\tau(t):\mathbb{T}\to\mathbb{T}$ and $\delta(t):\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$, $r(t)$ and $p(t)$ are real-valued rd-continuous positive functions defined on $\mathbb{T}$, and the following conditions are satisfied:

(A_1) $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t=\infty$, $0\le p(t)<1$,

(A_2) $f(t,u):\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$, and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\ge q(t)\,|u^{\gamma}|$.

In [6] the authors considered the case when $\gamma>0$ is an odd positive integer and proved that the oscillation of (1.2) is equivalent to the oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation. Also they considered the case when $\gamma\ge 1$ and established some sufficient conditions for oscillation by employing the Riccati technique. The results were applied only in discrete time scales, i.e., when the graininess function $\mu(t)\neq 0$.

In 2006 Saker [7] considered (1.2) where $\gamma\ge 1$ is an odd positive integer, (A_1)–(A_2) hold and established some new sufficient conditions for oscillation of (1.2) by employing the Riccati transformation technique. However the results established in [6–7] can be applied only on the time scales $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, $\mathbb{T}=h\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}}=\{t:t=q^k,\ k\in\mathbb{N},\ q>1\}$, and cannot be applied on the time scales $\mathbb{T}=\mathbb{N}^2=\{t^2:t\in\mathbb{N}\}$, $\mathbb{T}_2=\{\sqrt{n}:n\in\mathbb{N}_0\}$, $\mathbb{T}_3=\{\sqrt[3]{n}:n\in\mathbb{N}_0\}$, and $\mathbb{T}=\mathbb{T}_n=\{t_n:n\in\mathbb{N}_0\}$ where $\{t_n\}$ is the set of harmonic numbers. This follows from the fact that when $t\in\mathbb{T}$, the functions $t-\tau$ and $t-\delta$ may be not belong to the time scales $\mathbb{T}=\mathbb{N}^2$, $\mathbb{T}=\mathbb{T}_2$, $\mathbb{T}=\mathbb{T}_3$ and $\mathbb{T}=\mathbb{T}_n$.

In 2006 Şahiner [7] considered the general equation

$$[r(t)([y(t)+p(t)y(\tau(t))]^{\Delta})^{\gamma}]^{\Delta}+f(t,y(\delta(t)))=0, \tag{1.3}$$

on a time scale $\mathbb{T}$ and followed the argument in [6–7] by reducing the oscillation of (1.3) to the oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation, when the following conditions are satisfied:

(B_1) δ, τ are positive rd-continuous functions, $\delta,\ \tau:\mathbb{T}\to\mathbb{T}$,

(B_2) $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t=\infty$, $\gamma\ge 1$, and $0\le p(t)<1$;

(B_3) $f(t,u):\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function with $uf(t,u)>0$ for all $u\neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\ge q(t)\,|u^{\gamma}|$.

However one can easily see that the two examples that are given in [8] to illustrate the main results are valid only when $\mathbb{T}=\mathbb{R}$ and cannot be applied when $\mathbb{T}=\mathbb{N}$ since the delay functions that are considered in this paper are given by $t/2$, $\sqrt{t}$ and $t/64$ which are not in $C_{rd}(\mathbb{T},\mathbb{T})$ for a general time scale $\mathbb{T}$. Also the results cannot give a sharp sufficient condition for oscillation of (1.3) when $q(t)=\gamma/t^2$.

In 2006 Wu et al. [9] considered also (1.3) on a time scale $\mathbb{T}$. They followed the argument in [7] by using the Riccati transformation technique and the Chain rule $(\omega \circ \nu)^{\Delta}(t) = (w^{\widetilde{\Delta}} \circ \nu)\nu^{\Delta}$, where $\widetilde{\Delta}$ is the delta derivative defined on $\widetilde{\mathbb{T}}$ and $\nu(t)$ is strictly increasing, and established some sufficient conditions for oscillation of (1.3), when the following conditions are satisfied:

(C_1) $\delta : \mathbb{R} \to \mathbb{R}$ is continuous $\delta : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} = \delta(\mathbb{T}) \subset \mathbb{T}$ is a time scale;

(C_2) $(\delta \circ \sigma)(t) = (\sigma \circ \delta)(t)$;

(C_3) $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty$, $\gamma \geq 1$, and $0 \leq p(t) < 1$;

(C_4) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t) |u^{\gamma}|$.

We note that the results in [9], which are based on the Chain rule, can only be applied if $\widetilde{\mathbb{T}}$ is a time scale and if $\tau(t) \leq t$ and $\delta(t) \geq \tau(\delta(t))$. The condition ($C_2$) also can be a restrictive condition, since on the time scale $\mathbb{T}=q^{\mathbb{N}}$ by choosing $\delta(t) = t - q^{n_0}$ one can easily see that $\delta(\sigma(t)) = \delta(qt) = qt - q^{n_0} \neq \sigma(\delta(t)) = q(t - q^{n_0}) = qt - q^{n_0+1}$, so the results in [9] cannot be applied on the time scale $\mathbb{T}=q^{\mathbb{N}}$ when $\delta(t) = t - q^{n_0}$. Also in the proof of the main results in ([9], Lemma 2.5) the authors used the Chain rule $(f(g(t)))^{\Delta} = f^{\Delta}(g(t))g^{\Delta}(t)$ which is not true on general time scales. Of course trivially $(x \circ \tau)^{\Delta} = (x^{\Delta} \circ \tau)\tau^{\Delta}$ if δ is a constant with $\tau(t) = t - \delta \in \mathbb{T}$ for $t \in \mathbb{T}$.

Agarwal, O'Regan and Saker [10] considered the general nonlinear neutral delay dynamic equation (1.3) where $\gamma \geq 1$ is an odd positive integer,

(D_1) $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(D_2) $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty$, $r^{\Delta}(t) \geq 0$, $0 \leq p(t) < 1$;

(D_3) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t) |u^{\gamma}|$,

and employed the Riccati technique and established some new oscillation criteria which can be applied on any time scale $\mathbb{T}$ and improved the results established in [6–9].

Recently Saker in [11–12] considered the linear version of (1.1) when $r(t) = 1$ and employed the Riccati transformation technique and established some new oscillation criteria of Hille and Nehari types and also established some alternative oscillation criteria when the Hille and Nehari types criteria fail to apply.

The other technique used in studying the oscillation of differential and difference equations is called the generalized Riccati technique. This technique can introduce some new sufficient conditions for oscillation and can be applied to different equations which cannot be covered by the results established by the Riccati technique.

In 1995 Li [15] employed this technique on the second order linear differential equation

$$(r(t)x')' + p(t)x(t) = 0, \tag{1.4}$$

and established some new sufficient conditions for oscillation and applied the results when $r(t) = 1/t$ and $q(t) = 1/t^3$ and showed that the results that had been established by the Riccati technique cannot be applied in this case.

In 1996 Li and Liu [16] considered the nonlinear neutral delay differential equation

$$[r(t)[y(t) + p(t)y(t-\tau)]']' + q(t)f(y(t-\delta)) = 0, \quad t \geq t_0, \tag{1.5}$$

where

(E_1) τ and δ are nonnegative constants;

(E_2) $\int_{t_0}^{\infty} \frac{1}{r(t)}dt = \infty$, $0 \leq p(t) < 1$;

(E_3) $f(u) : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(u) > 0$ and $f(u)/u \geq K > 0$ for all $u \neq 0$,

and employed the generalized Riccati technique and established some new sufficient conditions for the oscillation of (1.5). Li and Liu utilized the class of functions as follows: Suppose there exist continuous functions H, $h : \mathbb{D} \equiv \{(t, s) : t \geq s \geq t_0\} \to \mathbb{R}$ such that $H(t, t) = 0$, $t \geq t_0$, $H(t, s) > 0$, $t > s \geq t_0$, and H has a continuous and nonpositive partial derivative on $\mathbb{D}$ with respect to the second variable. Moreover, let $h : \mathbb{D} \to \mathbb{R}$ be a continuous function with

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad t,\ s \in \mathbb{D}. \tag{1.6}$$

They then proved that if there exists a positive function $g \in C^1[t_0, \infty), \mathbb{R}^+)$ such that

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\psi(s) - \frac{r(s-\delta)a(s)}{4}h^2(t,s)\right] ds = \infty, \tag{1.7}$$

where $a(s) = \exp\{-2\int_0^s g(u)du\}$ and $\psi(t) = a(t)[Kq(t)(1-p(t-\delta))+r(t-\delta)g^2(t)-[r(t-\delta)g(t)]']$, then every solution of (1.5) oscillates. Using the same approach they established alternative oscillation criteria when (1.7) fails.

In 1998 Li and Yeh [17] considered the nonlinear neutral delay difference equation

$$\Delta\left[r(n)\Delta\left[y(n) + p(n)y(n-\tau)\right]\right] + q(n)f(y(n-\delta)) = 0, \quad n \geq n_0, \tag{1.8}$$

where

(F_1) τ and δ are nonnegative integers;

(F_2) $\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty$, $0 \leq p(n) < 1$;

(F_3) $f(u) : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(u) > 0$ and $f(u)/u \geq K > 0$ for all $u \neq 0$,

and obtained the discrete analogue of the results established by Li and Liu [16]. Li and Yeh [17] utilized the class of double sequences as follows. Suppose that there exist two double sequences $H_{m,n}$ and $h_{m,n}$ for $m \geq n \geq 0$ such that $H_{m,m} = 0$ for $m \geq 0$, $H_{m,n} > 0$ for $m > n > 0$, $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n}$ and $h(m,n) = -\frac{\Delta_2 H_{m,n}}{\sqrt{H(m,n)}}$.

They then proved that if there exist two positive sequence $a(n)$ and $g(n)$ such that $g(n) = -\frac{\Delta a(n)}{2a(n)}$ and

$$\lim_{m\to\infty} \sup \frac{1}{H(m,n_0)} \sum_{n=n_0}^{m-1} \left[H(m,n)\psi(n) - \frac{a^2(n+1)r(n-\delta)}{4a(n)}h^2(m,n)\right] = \infty, \tag{1.9}$$

where $\psi(n) = a(n)[Kq(n)(1-p(n-\delta)) + r(n-\delta)g^2(n) - \Delta[r(n-1-\delta)g(n-1)]]$, then every solution of the difference equation (1.8) oscillates. Also they established alternative oscillation criteria when (1.9) fails.

We note that the results established by Li and Liu [16] and Li and Yeh [17] can be applied only when the delays are finite.

The following question arises. Can we obtain oscillation criteria on time scales which can be applied in the case of finite delays as well as in infinite delays and from which we are able to deduce the corresponding results for neutral delay differential and difference equations and in the special case, cover the criteria established by Li [15] and Li and Liu [16] Li and Yeh [17] and others?

The main aim of this paper is to give a positive answer to this question by extending the generalized Riccati transformation technique in the time scales setting to obtain some new oscillation criteria of Li-type for (1.1) when

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty.$$

The paper is organized as follows: In Section 2, we will employ the generalized Riccati technique to establish the main oscillation criteria and study the asymptotic behavior of the nonoscillatory solutions. In Section 3, we will apply the results in different types of time scales and establish some sufficient conditions for oscillation of different types of differential and difference equations. Section 4 contains some examples to illustrate the main results. Our results in the special case when $\mathbb{T} = \mathbb{R}$ extend and improve the results established by Li and Liu [16] for neutral delay differential equations when $\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty$. In the case when $\mathbb{T} = \mathbb{N}$ our results extend and improve the results established by Li and Yeh [17] for neutral delay difference equations when $\sum_{t_0}^{\infty} \frac{1}{r(t)} = \infty$. When $p(t) = 0$, and $\delta(t) = t$ the results include the results established by Li [15] for differential equations. The results improve the oscillation results established for neutral delay dynamic equations established in [2, 6–10] when $\gamma = 1$.

2 Main Results

In this Section, by using the generalized Riccati transformation technique, we will establish some new oscillation criteria for (1.1) and establish the upper bounds of the nonoscillatory solutions. In what follows and later, we assume that

$$r^{\Delta}(t) \geq 0 \text{ and } \int_{t_0}^{\infty} \delta(s)q(s)[1 - p(\delta(s))]\Delta s = \infty. \tag{2.1}$$

We start with the following Lemma which will play an important role in the proof of the main results.

Lemma 2.1 *Suppose that* (h_1) (2.1), (h_2) *hold and* $\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty$.

Assume that (1.1) *has a positive solution* $y(t)$ *on* $[t_0, \infty)_{\mathbb{T}}$ *and define*

$$x(t) := y(t) + p(t)y(\tau(t)). \tag{2.2}$$

Then there exists a $T \in [t_0,\infty)_{\mathbb{T}}$, *sufficiently large, so that*

(i) $x^{\Delta}(t) > 0$, $x(t) > tx^{\Delta}(t)$ *for* $t \in [T, \infty)_{\mathbb{T}}$,

(ii) $(x(t)/t)$ *is strictly decreasing on* $[T, \infty)_{\mathbb{T}}$.

Proof Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since $y(t)$ is a positive solution of (1.1), then $x(t)$ is also positive and satisfies

$$(r(t)x^{\Delta}(t))^{\Delta} = -q(t)f(y(\delta(t))) < 0, \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

Thus $r(t)x^{\Delta}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We claim that $r(t)x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r(t_2)x^{\Delta}(t_2) =: c < 0$. Then from (2.3), we have $r(t)x^{\Delta}(t) \le r(t_2)x^{\Delta}(t_2) = c$, for $t \in [t_2, \infty)_{\mathbb{T}}$, and therefore

$$x^{\Delta}(t) \le \frac{c}{r(t)}, \text{ for } \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.4}$$

Integrating the last inequality from t_2 to t, we find by (h_2) that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \le x(t_2) + c\int_{t_2}^{t} \frac{\Delta s}{r(s)} \to -\infty \text{ as } t \to \infty, \tag{2.5}$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and so $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$x(t) > 0, \; x^{\Delta}(t) > 0, \; (r(t)x^{\Delta}(t))^{\Delta} < 0, \text{ for } t \ge t_1. \tag{2.6}$$

This implies that

$$\begin{aligned} y(t) &= x(t) - p(t)y(\tau(t)) = x(t) - p(t)[x(\tau(t)) - p(\tau(t))y(\tau(\tau(t)))] \\ &\ge x(t) - p(t)x(\tau(t)) \ge (1 - p(t))x(t). \end{aligned} \tag{2.7}$$

Thus for $t \ge t_2 = \delta^{-1}(t_1)$, we have

$$y(\delta(t)) \ge (1 - p(\delta(t)))x(\delta(t)). \tag{2.8}$$

From (h_1), (2.3), and (2.8), we have

$$(r(t)x^{\Delta}(t))^{\Delta} + Kq(t)(1 - p(\delta(t)))x(\delta(t)) \le 0, \text{ for } t \ge t_2. \tag{2.9}$$

Since $(r(t)x^{\Delta}(t))^{\Delta} < 0$ on $[t_2, \infty)_{\mathbb{T}}$, we have

$$x^{\Delta\Delta}(t) < -\frac{r^{\Delta}(t)x^{\Delta}(t)}{r^{\sigma}(t)} \le 0, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.10}$$

Next let $U(t) := x(t) - tx^{\Delta}(t)$. From (2.10), we have $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$, and this implies that $U(t)$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. We claim that there is a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_3, \infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

which implies that $x(t)/t$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. Pick $t_3 \in [t_2, \infty)_{\mathbb{T}}$ so that $\delta(t) \ge \delta(t_2)$, for $t \ge t_3$. Then, since $x(t)/t$ is strictly increasing, we have $x(\delta(t))/\delta(t) \ge x(\delta(t_2))/\delta(t_2) =: d > 0$, so that $x(\delta(t)) \ge d\delta(t)$ for $t \ge t_3$. Now by integrating both sides of (2.9) from t_3 to t, we have

$$r(t)x^{\Delta}(t) - r(t_3)x^{\Delta}(t_3) + K\int_{t_3}^{t} q(s)[1 - p(\delta(s))]x(\delta(s))\Delta s \le 0, \tag{2.11}$$

where $Q_1(s) = Kq(s)[1 - p(\delta(s))]$. From (2.11), since $r(t)x^{\Delta}(t) > 0$, we have

$$r(t_3)\left(x^{\Delta}(t_3)\right) \ge K\int_{t_3}^{t} q(s)[1 - p(\delta(s))]x(\delta(s))\Delta s \ge dK\int_{t_3}^{t} q(s)\delta(s)[1 - p(\delta(s))]\Delta s,$$

which contradicts (2.1). Hence there is a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_3, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.12}$$

Then we have $x(t) > tx^{\Delta}(t)$, and $(\frac{x(t)}{t})$ is strictly decreasing on $[t_3, \infty)_{\mathbb{T}}$. The proof is complete.

Now, we state and prove the main oscillation results for equation (1.1). We say $H \in \Re$ provided $H : [t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ and satisfies:

(i) $H(\sigma(t), t) = 0$, $t \geq t_0$, $H(t, s) > 0$, $\sigma(t) > s \geq t_0$,

(ii) $H^{\Delta_s}(t, s) \leq 0$ for $t \geq s \geq t_0$, and for each fixed t, $H(t, s)$ is an rd-continuous function with respect to s.

Important examples of H when $\mathbb{T} = \mathbb{R}$ are $H(t, s) = (t - s)^m$ for $m \geq 1$. When $\mathbb{T} = \mathbb{Z}$, $H(t, s) = (t - s)^{\underline{k}}$, $k \in \mathbb{N}$, where $t^{\underline{k}} = t(t-1) \cdots (t - k + 1)$.

Theorem 2.1 *Assume that* (h_1), (h_2), (2.1) *hold and* $H \in \Re$. *Suppose that there are a function* $a(t)$ *and a positive, differentiable function* $\alpha(t)$, *such that for sufficiently large* t_1

$$\limsup_{t \to \infty} \frac{1}{H(\sigma(t), t_1)} \int_{t_1}^{t} \bar{H}(t, s) \alpha^{\sigma}(s) \left[\psi(s) - \phi(t, s)\right] \Delta s = \infty, \tag{2.13}$$

where $\bar{H}(t, s) := H(\sigma(t), \sigma(s))$,

$$\phi(t, s) := \frac{1}{4}\left(\frac{\alpha(s)}{\alpha^{\sigma}(s)}\right)^2 \frac{r(s)A^2(t, s)}{C(s)}, \; C(s) := \frac{s}{\sigma(s)}, \quad C_1(s) := \frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)} + 2\frac{sa(s)}{\sigma(s)},$$

$$A(t, s) := \frac{\alpha^{\sigma}(s) C_1(s)}{\alpha(s)} + \frac{H^{\Delta_s}(\sigma(t), s)}{\bar{H}(t, s)}, \quad \psi(s) := \frac{KQ(s)\delta(s)}{\sigma(s)} - (a(s)r(s))^{\Delta} + \frac{sr(s)a^2(s)}{\sigma(s)}.$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.1) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for all $t \geq t_1 > t_0$ sufficiently large. Let $x(t)$ be as defined by (2.2). Then from Lemma 2.1, we see that $x(t)$ is positive and there exists $t_2 \geq t_1$ such that (2.6) holds for $t \geq t_2$. Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) := \alpha(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)a(t)\right], \qquad \text{for } \; t \geq t_2. \tag{2.14}$$

Hence

$$\begin{aligned} w^{\Delta}(t) &= \alpha^{\Delta}(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)a(t)\right] + \alpha^{\sigma}\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)a(t)\right]^{\Delta} \\ &= \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma}(r(t)a(t))^{\Delta} + \alpha^{\sigma}\left[\frac{r(t)x^{\Delta}(t)}{x(t)}\right]^{\Delta} \\ &= \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma}\left(r(t)a(t)\right)^{\Delta} + \alpha^{\sigma}\left[\frac{x(t)(r(t)x^{\Delta}(t))^{\Delta} - r(t)\left(x^{\Delta}(t)\right)^2}{x(t)x^{\sigma}}\right] \\ &= \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma}\left(r(t)a(t)\right)^{\Delta} + \alpha^{\sigma}\frac{(r(t)x^{\Delta}(t))^{\Delta}}{x^{\sigma}} - \alpha^{\sigma}\frac{r(t)\left(x^{\Delta}(t)\right)^2}{x(t)x^{\sigma}}. \end{aligned} \tag{2.15}$$

Then from (2.9), (2.14) and (2.15), we have

$$\begin{aligned} w^{\Delta} &\le -\alpha^{\sigma} Q(t) \frac{f \circ x \circ \delta}{x^{\sigma}}(t) + \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma}\, (r(t)a(t))^{\Delta} - \alpha^{\sigma} r(t) \frac{(x^{\Delta}(t))^2}{x(t)x^{\sigma}} \\ &\le -\alpha^{\sigma} \frac{KQ(t)x(\delta(t))}{x^{\sigma}} + \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma}\, (r(t)a(t))^{\Delta} - \alpha^{\sigma} \frac{r(t)x(t)}{x^{\sigma}} \left(\frac{x^{\Delta}(t)}{x(t)}\right)^2. \quad (2.16) \end{aligned}$$

From the definition of $w(t)$, we have

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^2 = \left[\frac{w(t)}{r(t)\alpha(t)} - a(t)\right]^2 = \left[\frac{w(t)}{r(t)\alpha(t)}\right]^2 + a^2(t) - 2\frac{w(t)a(t)}{r(t)\alpha(t)}. \quad (2.17)$$

Also from Lemma 2.1, since $x(t)/t$ is strictly decreasing, we have

$$x(\delta(t))/x^{\sigma}(t) \ge \delta(t)/\sigma(t) \text{ and } x(t)/x^{\sigma}(t) \ge C(t). \quad (2.18)$$

Substituting (2.17) and (2.18) into (2.16), we obtain

$$\begin{aligned} w^{\Delta}(t) &\le -\alpha^{\sigma} \frac{KQ(t)\delta(t)}{\sigma(t)} + \frac{\alpha^{\Delta}(t)}{\alpha(t)} w(t) + \alpha^{\sigma} (ra)^{\Delta}(t) \\ &\quad - \alpha^{\sigma} r(t) C(t) \left[\left[\frac{w(t)}{r(t)\alpha(t)}\right]^2 + a^2(t) - 2\frac{a(t)w(t)}{r(t)\alpha(t)}\right] \\ &= -\alpha^{\sigma} \frac{KQ(t)\delta(t)}{\sigma(t)} + \frac{\alpha^{\Delta}}{\alpha(t)} w(t) + \alpha^{\sigma} (r(t)a(t))^{\Delta} - \frac{C\alpha^{\sigma} w^2(t)}{r(t)\alpha^2(t)} \\ &\quad - \alpha^{\sigma} r(t) C(t) a^2(t) + 2\frac{\alpha^{\sigma} C(t) a(t)}{\alpha(t)} w(t) \\ &= -\alpha^{\sigma} \psi(t) + \frac{\alpha^{\sigma}}{\alpha(t)} \left[\frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}} + 2a(t)C(t)\right] w(t) - \frac{C(t)\alpha^{\sigma}}{r(t)\alpha^2(t)} w^2(t). \end{aligned}$$

Thus

$$w^{\Delta}(t) \le -\alpha^{\sigma}(t)\psi(t) + \frac{\alpha^{\sigma} C_1(t)}{\alpha(t)} w(t) - \frac{\alpha^{\sigma} C(t)}{r(t)\alpha^2(t)} w^2(t). \quad (2.19)$$

Evaluating both sides of (2.19) at s, multiplying by $H(\sigma(t), \sigma(s))$ and integrating, we get

$$\begin{aligned} \int_{t_2}^{t} H(\sigma(t), \sigma(s)) \alpha^{\sigma}(s) \psi(s) \Delta s \le &- \int_{t_2}^{t} H(\sigma(t), \sigma(s)) w^{\Delta}(s) \Delta s \\ &+ \int_{t_2}^{t} \frac{\alpha^{\sigma}(s) H(\sigma(t), \sigma(s)) C_1(s)}{\alpha(s)} w(s) \Delta s \\ &- \int_{t_2}^{t} \frac{C(s) H(\sigma(t), \sigma(s)) \alpha^{\sigma}(s)}{r(s)\alpha^2(s)} w^2(s) \Delta s. \quad (2.20) \end{aligned}$$

Integrating by parts and using the fact that $H(\sigma(t), t) = 0$, we have

$$\int_{t_2}^{t} H(\sigma(t), \sigma(s)) w^{\Delta}(s) \Delta s = -H(\sigma(t), t_2) w(t_2) - \int_{t_2}^{t} H^{\Delta_s}(\sigma(t), s) w(s) \Delta s.$$

Substituting this into (2.20), we obtain

$$\begin{aligned} &\int_{t_2}^{t} H(\sigma(t), \sigma(s)) \alpha^{\sigma}(s) \psi(s) \Delta s \\ &\quad \le H(\sigma(t), t_2) w(t_2) - \int_{t_2}^{t} H(\sigma(t), \sigma(s)) \frac{C(s)\alpha^{\sigma}(s)}{r(s)\alpha^2(s)} w^2(s) \Delta s \\ &\qquad + \int_{t_2}^{t} H(\sigma(t), \sigma(s)) A(t, s) w(s) \Delta s. \quad (2.21) \end{aligned}$$

This implies, after completing the square, that

$$\int_{t_2}^{t} \bar{H}(t,s)\alpha^{\sigma}(s)\psi(s)\Delta s \le H(\sigma(t),t_2)w(t_2) + \int_{t_2}^{t} \bar{H}(t,s)\frac{r(s)\alpha^2(s)A^2(t,s)}{4C(s)\alpha^{\sigma}(s)}\Delta s. \tag{2.22}$$

However, then

$$\frac{1}{H(\sigma(t),t_2)}\int_{t_2}^{t} \bar{H}(t,s)\alpha^{\sigma}(s)\left[\psi(s) - \frac{1}{4}\left(\frac{\alpha(s)}{\alpha^{\sigma}(s)}\right)^2 \frac{r(s)A^2(t,s)}{4C(s)}\right]\Delta s \le w(t_2),$$

which contradicts (2.13). Therefore every solution of (1.1) oscillates on $[t_0,\infty)_{\mathbb{T}}$. The proof is complete.

From Theorem 2.1 by choosing the function $H(t,s)$, appropriately, we can obtain different sufficient conditions for oscillation of (1.1). For instance, if we define a function $h(t,s)$ by

$$h(t,s) = -\frac{H^{\Delta_s}(\sigma(t),s)}{\sqrt{H(\sigma(t),\sigma(s))}}, \tag{2.23}$$

we have the following oscillation result.

Corollary 2.1 *Assume that* (h_1), (h_2), (2.1) hold and $H \in \Re$. *Suppose that there is a differentiable function* $\alpha(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{t} \bar{H}(t,s)\alpha^{\sigma}(s)\left[\psi(s) - \phi(t,s)\right]\Delta s = \infty, \tag{2.24}$$

where $\bar{H}(t,s)$,$\psi(s)$, $C(s)$*and* $\phi(t,s)$*are as in Theorem* 2.1, *and* $A(t,s)$*simplifies to*

$$A(t,s) = \frac{\alpha^{\sigma}(s)C_1(s)}{\alpha(s)} - \frac{h(t,s)}{\sqrt{\bar{H}(t,s)}}. \tag{2.25}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

From Theorem 2.1 and Corollary 2.1, we can establish different sufficient conditions for the oscillation of (1.1) by using different choices of $\alpha(t)$ and $a(t)$. For instance, if we consider $\alpha(t) = t$, $a(t) = \frac{1}{t}$ and define $H(t,s)$ for $t_0 \le s \le \sigma(t)$ by $H(\sigma(t),t) = 0$ and $H(t,s) = 1$ otherwise, then we get the following oscillation result.

Corollary 2.2 *Assume that* (h_1), (h_2) *and* (2.1) *hold, and for sufficiently large* t_1

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[KQ(s)\delta(s) - \sigma(s)\left(\frac{r(s)}{s}\right)^{\Delta} - \frac{5r(s)}{4s}\right]\Delta s = \infty. \tag{2.26}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

If in Theorem 2.1, we choose $a(t)$ and $\alpha(t)$ such that

$$a(t) = -\frac{\sigma(t)\alpha^{\Delta}(t)}{2t\alpha^{\sigma}(t)}, \tag{2.27}$$

we have $C_1(t) = 0$ and from Corollary 2.1 we have the following oscillation result for (1.1).

Corollary 2.3 *Assume that* (h_1), (h_2), (2.1) *hold and* $H \in \Re$ *such that* (2.23) *holds. Suppose there exists a positive differentiable function* $\alpha(t)$ *such that for* t_1 *sufficiently large*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{t} \left[\bar{H}(t,s)\alpha^{\sigma}(s)\psi(s) - \frac{\sigma(s)\alpha^2(s)r(s)h^2(t,s)}{4s\alpha^{\sigma}(s)}\right]\Delta s = \infty. \tag{2.28}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

If we define $H(t,s)$ for $t_0 \le s \le \sigma(t)$ by $H(\sigma(t),t)=0$ and $H(t,s)=1$ otherwise and $a(t)$ and $\alpha(t)$ such that (2.24) holds, we have $C_1(t)=0,\ h(t,s)=0$ and from Corollary 2.3 we have the following oscillation result for (1.1).

Corollary 2.4 *Assume that* (h_1), (h_2), (2.1) *and* (2.27) *hold. Furthermore assume that for sufficiently large* t_1

$$\lim_{t\to\infty}\sup\int_{t_1}^{t}\alpha^{\sigma}(s)\psi(s)\Delta s=\infty, \tag{2.29}$$

where $\psi(s)$*reduces to*

$$\psi(s)=\frac{KQ(s)\delta(s)}{\sigma(s)}-(r(s)a(s))^{\Delta}+\frac{sr(s)a^2(s)}{\sigma(s)}. \tag{2.30}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

From Corollary 2.4, we can also establish different sufficient conditions for the oscillation of (1.1) by using different choices of $\alpha(t)$. For instance, if $\alpha(t)=t$ then $a(t)=-\frac{1}{2t}$ and if $\alpha(t)=t^2$, then $a(t)=-\frac{t+\sigma(t)}{2t\sigma(t)}$, and from Corollary 2.4 we have the following oscillation results.

Corollary 2.5 *Assume that* (h_1), (h_2) *and* (2.1) *hold. If, for sufficiently large* t_1,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\psi(s)\Delta s=\infty, \tag{2.31}$$

where in this case,

$$\psi(s)=\frac{KQ(s)\delta(s)}{\sigma(s)}+\frac{1}{2}\left(\frac{r(s)}{s}\right)^{\Delta}+\frac{r(s)}{4\sigma(s)s}, \tag{2.32}$$

then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 2.6 *Assume that* (h_1), (h_2) *and* (2.1) *hold. If, for sufficiently large* t_1,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma^2(s)\psi(s)\Delta s=\infty, \tag{2.33}$$

where, in this case,

$$\psi(s):=\frac{KQ(s)\delta(s)}{\sigma(s)}+\left(\frac{(s+\sigma(s))r(s)}{2s\sigma(s)}\right)^{\Delta}+\frac{r(s)(s+\sigma(s))^2}{4s(\sigma(s))^3}, \tag{2.34}$$

then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

The following two results provide alternative oscillation criteria when (2.13) fails. The results are essentially new for oscillation of neutral delay dynamic equations. The notations of Theorem 2.2 and its proof will be used.

We say $H\in\acute{\mathbf{R}}$ provided $H:[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}$ and satisfies:

(H_1) $H(\sigma(t),t)=0,\ t\ge t_0,\ H(t,s)>0,\ \sigma(t)>s\ge t_0,$

(H_2) $H^{\Delta_s}(t,s)\le 0$ for $t\ge s\ge t_0$, and for each fixed t, $H(t,s)$ is an rd-continuous function with respect to s,

(H_3) $h(t,s):=-\frac{H^{\Delta_s}(\sigma(t),s)}{\sqrt{\bar{H}(t,s)}}$, where $h(t,s)$ is define on $[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$,

(H_4) $0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(\sigma(t),s)}{H(\sigma(t),t_0)}\right]\le\infty.$

Theorem 2.2 *Assume that* (h_1), (h_2), (2.1) *hold and* $H \in \acute{\mathbf{R}}$. *Suppose that there exists a positive differentiable function* α *such that*

$$\lim_{t\to\infty} \sup \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{t} \frac{r(s)\,(\alpha(s))^2}{C(s)\alpha^\sigma(s)} \left(h(t,s) - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t,s)}\right)^2 \Delta s < \infty. \tag{2.35}$$

Let $\phi \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *such that, for* $t \geq t_0$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{C(s)\alpha^\sigma(s)\,(\phi^\sigma)^2}{r(s)\alpha^2(s)} \Delta s = \infty. \tag{2.36}$$

If, for any $T \in [t_0,\infty)_{\mathbb{T}}$,

$$\lim_{t\to\infty} \sup \frac{1}{H(\sigma(t),T)} \int_{T}^{t} \bar{H}(t,s) \left[\alpha^\sigma(s)\psi(s) - \frac{r(s)\alpha^2(s)}{4C(s)\alpha^\sigma(s)} B^2(t,s)\right] \Delta s \geq \phi(T), \tag{2.37}$$

where

$$B^2(t,s) = \frac{h(t,s)}{\sqrt{\bar{H}(t,s)}} - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)},$$

then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.1) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$. Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for all $t \geq t_1 > t_0$ sufficiently large. Let $x(t)$ be as defined by (2.2). Then from Lemma 2.1, we see that $x(t)$ is positive and there exists $T \geq t_1$ such that (2.6) and the results Lemma 2.1 hold for $t \geq T$. Define again the function $w(t)$ by the generalized Riccati substitution (2.14) and proceed as in the proof of Theorem 2.1 to obtain (2.21) with $t_0 = T$. Therefore,

$$\int_T^t \bar{H}(t,s)\alpha^\sigma(s)\psi(s)\Delta s \leq H(\sigma(t),t_2)w(t_2) - \int_T^t \bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}w^2(s)\Delta s + \int_T^t \bar{H}(t,s)A(t,s)w(s)\Delta s. \tag{2.38}$$

From (H_3) we see that $A(t,s)$ becomes

$$A(t,s) = \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)} - \frac{h(t,s)}{\sqrt{\bar{H}(t,s)}}. \tag{2.39}$$

Substituting (2.39) into (2.38), we get

$$\int_T^t \bar{H}(t,s)\alpha^\sigma(s)\psi(s)\Delta s \leq H(\sigma(t),T)w(T) - \int_T^t h(t,s)\sqrt{\bar{H}(t,s)}w(s)\Delta s + \int_T^t \bar{H}(t,s)\frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)}w(s)\Delta s - \int_T^t \bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}w^2(s)\Delta s. \tag{2.40}$$

Hence,

$$\int_T^t \bar{H}(t,s)\alpha^\sigma(s)\psi(s)\Delta s \leq H(\sigma(t),T)w(T) - \int_T^t D(t,s)w(s)\Delta s - \int_T^t \bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}(w(s))^2\Delta s, \tag{2.41}$$

where

$$D(t,s) = \left[h(t,s)\sqrt{\bar{H}(t,s)} - \frac{\bar{H}(t,s)\alpha^\sigma(s)C_1(s)}{\alpha(s)}\right].$$

Therefore,

$$\begin{aligned}
&\int_T^t \bar{H}(t,s)\alpha^\sigma(s)\psi(s)\Delta s \\
&\quad \le H(\sigma(t),T)w(T) - \int_T^t \left[\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}w(s) + \frac{D(t,s)}{2\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}}\right]^2 \Delta s \\
&\qquad + \int_T^t \bar{H}(t,s)\frac{r(s)\alpha^2(s)}{4C(s)\alpha^\sigma(s)}\left[h(t,s)/\sqrt{\bar{H}(t,s)} - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)}\right]^2 \Delta s. \qquad (2.42)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{H(\sigma(t),T)}\int_T^t \bar{H}(t,s)\alpha^\sigma(s)\psi(s)\Delta s \\
&\quad \le w(T) - \frac{1}{H(\sigma(t),T)}\int_T^t \left[\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}w(s) + \frac{D(t,s)}{2\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}}\right]^2 \Delta s \\
&\qquad + \frac{1}{H(\sigma(t),T)}\int_T^t \bar{H}(t,s)\frac{r(s)\alpha^2(s)}{4C(s)\alpha^\sigma(s)}\left[\frac{h(t,s)}{\sqrt{\bar{H}(t,s)}} - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)}\right]^2 \Delta s. \qquad (2.43)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{t\to\infty}\sup \frac{1}{H(\sigma(t),T)}\int_T^t \bar{H}(t,s)\left[\alpha^\sigma(s)\psi(s) - \frac{r(s)\alpha^2(s)}{4C(s)\alpha^\sigma(s)}B^2(t,s)\right]\Delta s \\
&\quad \le w(T) - \lim_{t\to\infty}\inf\frac{1}{H(\sigma(t),T)}\int_T^t \left[\sqrt{\frac{\bar{H}(t,s)C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}w(s) + \frac{D(t,s)}{2\sqrt{\frac{\bar{H}(t,s)C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}}\right]^2 \Delta s.
\end{aligned}$$

By (2.37), we have

$$\begin{aligned}
w(T) \ge \phi(T) &+ \lim_{t\to\infty}\inf\frac{1}{H(\sigma(t),T)} \\
&\times \int_T^t \left[\sqrt{\frac{\bar{H}(t,s)C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}w(s) + \frac{D(t,s)}{2\sqrt{\frac{\bar{H}(t,s)C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}}\right]^2 \Delta s.
\end{aligned}$$

Thus

$$w(T) \ge \phi(T) \quad \text{for all } T \ge t_0, \qquad (2.44)$$

and

$$\begin{aligned}
0 &\le \lim_{t\to\infty}\inf\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^t \left[\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}w(s) + \frac{D(t,s)}{2\sqrt{\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}}}\right]^2 \Delta s \\
&\le w(t_0) - \phi(t_0) < \infty.
\end{aligned}$$

Therefore, for all $t \ge t_0$, we obtain

$$\begin{aligned}
\lim_{t\to\infty}\inf\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^t &\left[\bar{H}(t,s)\frac{C(s)\alpha^\sigma(s)}{r(s)\alpha^2(s)}w^2(s)\right. \\
&\left. + \sqrt{\bar{H}(t,s)}\left(h(t,s) - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t,s)}\right)w(s)\right]\Delta s < \infty. \qquad (2.45)
\end{aligned}$$

Define F and G by

$$F(t) := \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} \bar{H}(t,s) \frac{U(s)}{\alpha^2(s)} w^2(s)\Delta s,$$
$$G(t) := \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} \sqrt{\bar{H}(t,s)} \left(h(t,s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)} \sqrt{\bar{H}(t,s)} \right) w\Delta s,$$

where $U(s) := \frac{C(s)\alpha^\sigma(s)}{r(s)}$. It follows from (2.45) that

$$\liminf_{t\to\infty} [F(t) + G(t)] < \infty. \tag{2.46}$$

Now, we claim that

$$\int_{t_0}^{\infty} \frac{U(s)}{\alpha^2(s)} w^2(s)\Delta s < \infty. \tag{2.47}$$

Suppose to the contrary that

$$\int_{t_0}^{\infty} \frac{U(s)}{\alpha^2(s)} w^2(s)\Delta s = \infty. \tag{2.48}$$

By (H_4), we see that there exists a positive constant ζ such that

$$\inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(\sigma(t), s)}{H(\sigma(t), t_0)} \right] > \zeta. \tag{2.49}$$

Let λ be any arbitrary positive number. Then by (2.48) there exists $t_2 \geq t_1$ such that

$$\int_{t_0}^{t} \frac{U(s)}{\alpha^2(s)} w^2(s)\Delta s \geq \frac{\lambda}{\zeta}, \qquad t \geq t_2.$$

Now since $\frac{U(s)}{\alpha^2(s)} w^2(s)$ is the derivative of

$$\left[\int_{t_0}^{s} \frac{U(u)w^2(u)}{\alpha^2(u)} \Delta u \right]^\Delta,$$

we have after integration by parts, for $t \geq t_2$, that

$$\begin{aligned}
F(t) &= \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} \bar{H}(t,s) \left[\int_{t_0}^{s} \frac{U(u)w^2(u)}{\alpha^2(u)} \Delta u \right]^\Delta \Delta s \\
&= \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} H(\sigma(t), \sigma(s)) \left[\int_{t_0}^{s} \frac{U(u)w^2(u)}{\alpha^2(u)} \Delta u \right]^\Delta \Delta s \\
&= \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{t} [-H^{\Delta_s}(\sigma(t), s)] \left[\int_{t_0}^{s} \frac{U(u)w^2(u)}{\alpha^2(u)} \Delta u \right] \Delta s \\
&\geq \frac{1}{H(\sigma(t), t_0)} \int_{t_2}^{t} [-H^{\Delta_s}(\sigma(t), s)] \left[\int_{t_0}^{s} \frac{U(u)w^2(u)}{\alpha^2(u)} \Delta u \right] \Delta s \\
&\geq \frac{\lambda}{\zeta} \frac{1}{H(\sigma(t), t_0)} \int_{t_2}^{t} [-H^{\Delta_s}(\sigma(t), s)] \Delta s \\
&= \frac{\lambda}{\zeta} \frac{H(\sigma(t), t_2)}{H(\sigma(t), t_0)}.
\end{aligned} \tag{2.50}$$

By (2.49), there exists $t_3 \geq t_2$, such that

$$\frac{H(\sigma(t), t_2)}{H(\sigma(t), t_0)} \geq \zeta, \qquad t \geq t_3.$$

This implies that $F(t) \geq \lambda$ for all $t \geq t_3$. As λ is arbitrary, we have

$$\lim_{t\to\infty} F(t) = \infty. \tag{2.51}$$

Next, consider the sequence $\{t_n\}$ in the interval $[t_2, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} [F(t_n) + G(t_n)] = \liminf_{t\to\infty} [F(t) + G(t)]. \tag{2.52}$$

In view of (2.46), there exists a constant ν such that

$$F(t_n) + G(t_n) \leq \nu, \qquad n = 1, 2, \dots .$$

In view of (2.51), we conclude that

$$\lim_{n\to\infty} F(t_n) = \infty, \tag{2.53}$$

and hence (2.52) gives that

$$\lim_{n\to\infty} G(t_n) = -\infty. \tag{2.54}$$

By (2.52), (2.53), given $\epsilon \in (0,1)$ for n large enough, we have

$$1 + \frac{G(t_n)}{F(t_n)} \leq \frac{\nu}{F(t_n)} < \epsilon.$$

Then

$$\frac{G(t_n)}{F(t_n)} < \epsilon - 1 < 0. \tag{2.55}$$

In view of (2.54), we see that

$$\lim_{n\to\infty} \frac{G(t_n)}{F(t_n)} G(t_n) = \infty. \tag{2.56}$$

On the other hand, by employing Schwarz's inequality (cf. [4]),

$$\left[\int_a^b |f(t)g(t)|\,\Delta t\right]^2 \leq \left[\int_a^b |f(t)|^2\,\Delta t\right]\left[\int_a^b |g(t)|^2\,\Delta t\right],$$

we have

$$\begin{aligned}
G^2(t_n) &= \left[\frac{1}{H(\sigma(t_n), t_0)} \int_{t_0}^{t_n} w(s)\sqrt{\bar{H}(t_n, s)} \times \left(h(t_n, s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t_n, s)}\right)\Delta s\right]^2 \\
&= \left[\frac{1}{H(\sigma(t_n), t_0)} \int_{t_0}^{t_n} \frac{\sqrt{\bar{H}(t_n, s) U(s)}}{\alpha(s)} w(s)\right. \\
&\quad \left.\times \frac{\alpha(s)}{\sqrt{U(s)}}\left(h(t_n, s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t_n, s)}\right)\Delta s\right]^2. \\
&\leq \frac{1}{H(\sigma(t_n), t_0)}\left[\int_{t_0}^{t_n} \frac{\bar{H}(t_n, s) U(s)}{\alpha^2(s)} w^2(s)\Delta s\right] \times \\
&\quad \frac{1}{H(\sigma(t_n), t_0)}\left[\int_{t_0}^{t_n} \frac{\alpha^2(s)}{U(s)}\left(h(t_n, s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t_n, s)}\right)^2 \Delta s\right] \\
&= \frac{F(t_n)}{H(\sigma(t_n), t_0)}\left[\int_{t_0}^{t_n} \frac{\alpha^2(s)}{U(s)}\left(h(t_n, s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)}\sqrt{\bar{H}(t_n, s)}\right)^2 \Delta s\right].
\end{aligned}$$

Hence

$$\frac{G^2(t_n)}{F(t_n)} \leq \frac{1}{H(\sigma(t_n), t_0)} \int_{t_0}^{t_n} \frac{\alpha^2(s)}{U(s)}\left(h(t_n, s) - \frac{\alpha^\sigma(s) C_1(s)}{\alpha(s)}\sqrt{H(t_n, s)}\right)^2 \Delta s.$$

It follows from (2.56) that

$$\lim_{n\to\infty} \frac{1}{H(\sigma(t_n),t_0)} \int_{t_0}^{t_n} \frac{\alpha^2(s)}{U(s)} \left(h(t_n,s) - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)} \sqrt{\bar{H}(t_n,s)} \right)^2 \Delta s = \infty.$$

Hence

$$\lim_{t\to\infty} \sup \frac{1}{H(\sigma(t_n),t_0)} \int_{t_0}^{t} \frac{\alpha^2(s)}{U(s)} \left(h(t,s) - \frac{\alpha^\sigma(s)C_1(s)}{\alpha(s)} \sqrt{\bar{H}(t,s)} \right)^2 \Delta s = \infty, \tag{2.57}$$

which contradicts (2.35). Therefore we have proved that (2.48) fails, so the inequality (2.47) holds. Furthermore from (2.44), we have

$$\int_{t_0}^{\infty} \frac{U(s)\left(\phi^\sigma\right)^2}{\alpha^2(s)} \Delta s \le \int_{t_0}^{\infty} \frac{U(s)w^2(s)}{\alpha^2(s)} \Delta s < \infty,$$

which contradicts (2.36). The proof is complete.

In the following, we give an estimation on the upper bounds of the nonoscillatory solutions of (1.1). The results will be established by using the extension of Gronwall's inequality that has been given in [18]. We note here that by using different extensions of this inequality we can obtain different results. We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd-continuous and regressive by $\mathscr{R}$. If $p \in \mathscr{R}$, then we can define the exponential function (see [2, 4]) by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternately, for $p \in \mathscr{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = 1.$$

Theorem 2.3 *Assume that* (h$_1$), (h$_2$), (2.1) *hold. Let $y(t)$ be a positive solution of* (1.1) *and let $x(t)$ be as defined by* (2.2). *Then $x(t)$ satisfies*

$$x^\sigma < \frac{a(t)}{r^\sigma} + \frac{b(t)}{r^\sigma} \int_{t_0}^{t} a(s)Q_1(s)e_{bQ_1}(t,\sigma(s))\Delta s, \tag{2.58}$$

where $a(t) = C\sigma(t)b(t)$, $b(t) = r^\sigma(r^\sigma - \mu(t)r^\Delta(t))^{-1}$,

$$Q_1(s) = \frac{1}{r^\sigma}\left[r^\Delta(s) + Q(s)\sigma(t)\right], \ Q(s) = Kq(s)(1 - p(\delta(s))),$$

and $C = (rx)^\sigma(t_2) + (r(t_2)x^\Delta(t_2))$ is a positive constant.

Proof Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for all $t \ge t_1 > t_0$ sufficiently large. From Lemma 2.1, we see that $x(t)$ is positive and there exists $t_2 \ge t_1$ such that (2.6) and (2.9) hold for $t \geq t_2$. Integrating (2.9) from t_2 to s, we obtain

$$\begin{aligned}(r(s)x^{\Delta}(s)) &\leq (r(t_2)x^{\Delta}(t_2)) - K\int_{t_2}^{s} q(u)(1-p(\delta(u)))x(\delta(u))\Delta u \\ &< M + \int_{t_2}^{s} Q(u)x(\delta(u))\Delta u \leq M + \int_{t_2}^{s} Q(u)x(u)\Delta u,\end{aligned}$$

where $M = (r(t_2)x^{\Delta}(t_2))$. Thus, we have

$$(r(s)x(s))^{\Delta} < M + \frac{r^{\Delta}(s)}{r^{\sigma}}(rx)^{\sigma} + \int_{t_2}^{s} \frac{Q(u)}{r^{\sigma}}(rx)^{\sigma}\Delta u. \tag{2.59}$$

Integrating (2.59) again from t_2 to $\sigma(t)$, and using Lemma 3 in [19], we have

$$\begin{aligned}(rx)^{\sigma} &< (rx)^{\sigma}(t_2) + M(\sigma(t)-t_2) + \int_{t_2}^{\sigma(t)} \frac{r^{\Delta}(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s \\ &\quad + \int_{t_2}^{\sigma(t)}\int_{t_2}^{s} \frac{Q(u)}{r^{\sigma}}(rx)^{\sigma}\Delta u\Delta s \\ &\leq C\sigma(t) + \int_{t_2}^{\sigma(t)} \frac{r^{\Delta}(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s + \int_{t_2}^{t}(\sigma(t)-\sigma(s))\frac{Q(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s \\ &= C\sigma(t) + \int_{t_2}^{t} \frac{r^{\Delta}(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s + \int_{t}^{\sigma(t)} \frac{r^{\Delta}(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s \\ &\quad + \int_{t_2}^{t}(\sigma(t)-\sigma(s))\frac{Q(s)}{r^{\sigma}}(rx)^{\sigma}\Delta s.\end{aligned}$$

Using the formula (cf. [5])

$$\int_{t}^{\sigma(t)} f(s)\Delta s = \mu(t)f(t),$$

we have

$$(rx)^{\sigma} < C\sigma(t) + \mu(t)\frac{r^{\Delta}(t)}{r^{\sigma}}(rx)^{\sigma} + \int_{t_2}^{t} Q_1(s)(rx)^{\sigma}\Delta s.$$

Thus

$$(rx)^{\sigma} < a(t) + b(t)\int_{t_2}^{t} Q_1(s)(rx)^{\sigma}\Delta s.$$

Putting $z(t) = (rx)^{\sigma}$, we get

$$z(t) < a(t) + b(t)\int_{t_2}^{t} Q_1(s)z(s)\Delta s.$$

Now, by applying the extension of the Gronwall's inequality ([18] Theorem 3.1]), we have

$$z(t) < a(t) + b(t)\int_{t_2}^{t} a(s)Q_1(s)e_{bQ_1}(t,\sigma(s)\Delta s.$$

This implies that

$$x^{\sigma} < \frac{a(t)}{r^{\sigma}} + \frac{b(t)}{r^{\sigma}}\int_{t_2}^{t} a(s)Q_1(s)e_{bQ_1}(t,\sigma(s))\Delta s.$$

The proof is complete.

3 Applications

In this Section, we apply the oscillation results to different types of time scales. In the case when $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$, $\mu(t)=0$, $y^{\Delta}(t)=y'(t)$ and (1.1) becomes the second-order neutral delay differential equation

$$[r(t)(y(t)+p(t)y(\tau(t)))']'+q(t)f(y(\delta(t)))=0. \tag{3.1}$$

When $\mathbb{T}=\mathbb{N}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, $y^{\Delta}(t)=\Delta y(t)=y(t+1)-y(t)$ and (1.1) becomes the second-order neutral delay difference equation

$$\Delta\left[r(t)\Delta(y(t)+p(t)y(\tau(t)))\right]+q(t)f(y(\delta(t))=0. \tag{3.2}$$

When $\mathbb{T}=h\mathbb{N}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$,

$$y^{\Delta}(t)=\Delta_h y(t)=\frac{y(t+h)-y(t)}{h},$$

and (1.1) becomes the generalized second-order neutral delay difference equation

$$\Delta_h\left[r(t)\Delta_h(y(t)+p(t)y(\tau(t)))\right]+q(t)f(y(\delta(t)))=0. \tag{3.3}$$

When $\mathbb{T}=\{t:t=q^n,\ n\in\mathbb{N},\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$y^{\Delta}(t)=\Delta_q y(t)=\frac{y(qt)-y(t)}{(q-1)\,t},$$

and (1.1) becomes the second-order q-neutral delay difference equation

$$\Delta_q\left[r(t)\Delta_q(y(t)+p(t)y(\tau(t)))\right]+q(t)f(y(\delta(t)))=0. \tag{3.4}$$

When $\mathbb{T}=\mathbb{N}^2=\{t^2:t\in\mathbb{N}\}$, we have $\sigma(t)=(\sqrt{t}+1)^2$ and $\mu(t)=1+2\sqrt{t}$,

$$y^{\Delta}(t)=\Delta_N y(t)=\frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}},$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_N\left[r(t)\Delta_N(y(t)+p(t)y(\tau(t)))\right]+q(t)f(y(\delta(t)))=0. \tag{3.5}$$

When $\mathbb{T}=\mathbb{T}_n=\{t_n:n\in\mathbb{N}\}$ where t_n are the so-called harmonic numbers defined by

$$t_0=0,\quad t_n=\sum_{k=1}^{n}\frac{1}{k},\ n\in\mathbb{N}_0,$$

we have $\sigma(t_n)=t_{n+1}$, $\mu(t_n)=\frac{1}{n+1}$, $y^{\Delta}(t)=\Delta_{t_n}y(t_n)=(n+1)\Delta y(t_n)$ and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_{t_n}\left[r(t_n)\Delta_{t_n}\left(y(t_n)+p(t_n)y(\tau(t_n))\right)\right]+q(t_n)f(y(\delta(t_n)))=0. \tag{3.6}$$

When $\mathbb{T}=\mathbb{T}_2=\{\sqrt{n}:n\in\mathbb{N}_0\}$, we have $\sigma(t)=\sqrt{t^2+1}$ and $\mu(t)=\sqrt{t^2+1}-t, x^{\Delta}(t)=\Delta_2 x(t)=(x(\sqrt{t^2+1})-x(t))/\sqrt{t^2+1}-t$, and (1.1) becomes the second-order difference equation

$$\Delta_2\left[r(t)\Delta_2\left(y(t)+p(t)y(\tau(t))\right)\right]+q(t)f(y(\delta(t)))=0. \tag{3.7}$$

When $\mathbb{T}=\mathbb{T}_3=\{\sqrt[3]{n}:n\in\mathbb{N}_0\}$, we have $\sigma(t)=\sqrt[3]{t^3+1}$ and $\mu(t)=\sqrt[3]{t^3+1}-t, x^{\Delta}(t)=\Delta_3 x(t)=(x(\sqrt[3]{t^3+1})-x(t))/\sqrt[3]{t^3+1}-t$, and (1.1) becomes the second-order perturbed delay difference equation

$$\Delta_3(r(t)(\Delta_3\left[y(t)+p(t)y(\tau(t))\right])+q(t)f(y(\delta(t)))=0. \tag{3.8}$$

We start with the case when $\mathbb{T}=\mathbb{R}$, then we have from Theorem 2.1 the following oscillation results for the neutral delay differential equation (3.1). Note that when $\mathbb{T}=\mathbb{R}$, we have $H(\sigma(t),\sigma(s))=H(t,s)$.

Theorem 3.1 *Assume that* (h_1) *holds, and*

$$r'(t)\geq 0,\ \int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty,\ \int_{t_0}^{\infty}\delta(s)Q(s)ds=\infty. \tag{3.9}$$

Furthermore, assume that there exists a differentiable function $a(t)$ *and a positive differentiable function* $\alpha(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\alpha(s)\left[\psi(s)-\frac{1}{4}r(s)A^2(t,s)\right]ds=\infty, \tag{3.10}$$

where

$$A(t,s)=C_1(s)+\frac{\frac{\partial H(t,s)}{\partial s}}{H(t,s)},\quad C_1(s)=\frac{\alpha'(s)}{\alpha(s)}+2a(s),$$

and

$$\psi(s)=\frac{KQ(s)\delta(s)}{s}-(a(s)r(s))'+r(s)a^2(s).$$

Then every solution of (3.1) *is oscillatory.*

From Theorem 3.1 by choosing $h(t,s)$ so that $\frac{\partial}{\partial s}H(t,s)=-h(t,s)\sqrt{H(t,s)}$, and taking $a(t)=-\frac{\alpha'(t)}{2\alpha(t)}$, we get the following result.

Corollary 3.1 *Assume that* (h_1) *and* (3.9) *hold and let* $H\in\Re$ *and define* $h(t,s)$ *by* $\frac{\partial}{\partial s}H(t,s)=-h(t,s)\sqrt{H(t,s)}$. *Assume there is a differentiable function* $\alpha(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\alpha(s)\psi(s)-\frac{1}{4}\alpha(s)r(s)h^2(t,s)\right]ds=\infty, \tag{3.11}$$

where

$$\psi(s)=\frac{KQ(s)\delta(s)}{s}+\frac{1}{2}\left(\frac{r(s)\alpha'(s)}{\alpha(s)}\right)'+r(s)\left(\frac{\alpha'(s)}{2\alpha(s)}\right)^2.$$

Then every solution of (3.1) *oscillates.*

If we let $\delta(t)=\tau(t)=t$ and $p(t)=0$ in Corollary 3.1 we get a result of Li [15].

Letting $H(t,s)=1$ for $t_0\leq s<t$ and $H(t,t)=0$ in Corollary 3.1 we get the following result.

Corollary 3.2 *Assume that* (h_1) *and* (3.9) *hold and there exists a differentiable function* $\alpha(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty}\int_{t_1}^{t}\alpha(s)\left[\frac{KQ(s)\delta(s)}{s}+\frac{1}{2}\left(\frac{r(s)\alpha'(s)}{\alpha(s)}\right)'+r(s)\left(\frac{\alpha'(s)}{2\alpha(s)}\right)^2\right]ds=\infty.$$

Then every solution of the delay differential equation (3.1) *is oscillatory.*

Letting $\alpha(t)=t$ in Corollary 3.2 we get the following result.

Corollary 3.3 *Assume that* (h_1) *and* (3.9) *hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[KQ(s)\delta(s)+\frac{s}{2}\left(\frac{r(s)}{s}\right)'+\frac{r(s)}{4s}\right]ds=\infty, \tag{3.12}$$

then every solution of (3.1) *is oscillatory.*

Letting $\alpha(t) = t^2$ in Corollary 3.2 we get the following result.

Corollary 3.4 *Assume that* (h_1) *and* (3.9)*hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[KsQ(s)\delta(s) + s^2 \left(\frac{r(s)}{s} \right)' + r(s) \right] ds = \infty, \tag{3.13}$$

then every solution of (3.1) *is oscillatory.*

Now, we apply our results in the time scale $\mathbb{T} = \mathbb{N}$ and establish some oscillation criteria for the delay difference equation (3.2). From Theorem 2.1 we get the following result. Note that when $\mathbb{T} = \mathbb{N}$, we have $H(\sigma(t), \sigma(s)) = H(t+1, s+1)$.

Theorem 3.2 *Assume that* (h_1) *holds and*

$$\Delta r(t) \geq 0, \ \sum_{t=t_0}^{\infty} \frac{1}{r(t)} = \infty, \ \sum_{t=t_0}^{\infty} \delta(t)Q(t) = \infty. \tag{3.14}$$

Furthermore, assume that there exist a sequence $a(t)$*and a positive sequence* $\alpha(t)$ *such that for sufficiently large integers* t_1

$$\limsup_{t\to\infty} \frac{1}{H(t+1,t_1)} \sum_{s=t_1}^{t-1} H(t+1,s+1)\alpha(s+1) \left[\psi(s) - \phi(t,s) \right] = \infty, \tag{3.15}$$

where

$$\psi(s) = \frac{KQ(s)\delta(s)}{s+1} - \Delta\left(a(s)r(s)\right) + \frac{sr(s)a^2(s)}{s+1}, \ \phi(t,s) = \frac{(s+1)r(s)\alpha^2(s)A^2(t,s)}{4s\alpha^2(s+1)},$$

and

$$A(t,s) = \frac{\alpha(s+1)C_1(s)}{\alpha(s)} + \frac{H(t+1,s+1) - H(t+1,s)}{H(t+1,s+1)}, \ C_1(s) = \frac{\Delta\alpha(s)}{\alpha(s+1)} + 2\frac{sa(s)}{s+1}.$$

Then every solution of (3.2) *is oscillatory.*

From Theorem 3.2 by choosing $h(t,s)$ so that

$$\Delta_s H(t+1,s) = -h(t,s)\sqrt{H(t+1,s+1)}, \tag{3.16}$$

we have the following result.

Corollary 3.5 *Assume that* (h_1) *and* (3.14) *hold and let* $H \in \Re$ *such that* (3.16) *holds. If there exist a sequence* $a(t)$ *and a positive sequence* $\delta(t)$ *such that for sufficiently large integers* t_1, (3.15) *holds where* ψ, ϕ, *and* C_1*are as in Theorem* 3.2, *where*

$$A(t,s) := \frac{\delta(s+1)C_1(s)}{\delta(s)} - \frac{h(t,s)}{\sqrt{H(t+1,s+1)}},$$

then every solution of (3.2) *is oscillatory.*

From Theorem 3.2 and Corollary 3.5, we can establish different sufficient conditions for the oscillation of (1.1) by using different choices of $H(t,s)$, $\alpha(t)$ and $a(t)$. For instance, if we let $\alpha(t) = t$, $a(t) = \frac{1}{t}$ and $H(t,s) = 1$ for $t > s \geq t_0$, and $H(t+1,t) = 0$ in Theorem 3.2, we get the following oscillation result.

Corollary 3.6 *Assume that* (h_1) *and* (3.14) *hold. Furthermore, assume that, for sufficiently large integer* t_1,

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[KQ(s)\delta(s)-(s+1)\Delta\left(\frac{r(s)}{s}\right)-\frac{5}{4}\frac{r(s)}{s}\right]=\infty. \tag{3.17}$$

Then every solution is oscillatory.

If we choose $a(t)$ and $\alpha(t)$ such that

$$a(t)=-\frac{(t+1)\Delta\alpha(t)}{2t\alpha(t+1)}, \tag{3.18}$$

we have $C_1(t)=0$ and we obtain the following result from Corollary 3.5.

Corollary 3.7 *Assume that* (h_1), (3.14) *and* (3.18) *hold and let* $H\in\Re$ *such that* (3.16) *holds. If, for* t_1 *sufficiently large,*

$$\limsup_{t\to\infty}\frac{1}{H(t+1,t_1)}\sum_{s=t_1}^{t-1}H(t+1,s+1)\delta(s+1)\left[\psi(s)-\phi(t,s)\right]=\infty,$$

where

$$\phi(t,s)=\frac{1}{4}\left(\frac{\delta(s)}{\delta(s+1)}\right)^2\frac{(s+1)r(s)h^2(t,s)}{sH(t+1,s+1)},$$

then every solution of (3.2) *is oscillatory.*

From Corollary 3.7 we have, taking $H(t,s)=1$ for $t_0\le s\le t$ and $H(t+1,t)=0$, the following oscillation result for the delay difference equation (3.2).

Corollary 3.8 *Assume that* (h_1) *and* (3.14) *hold and there are a sequence* $a(t)$ *and a positive sequence* $\alpha(t)$ *such that* (3.18) *holds. If, for sufficiently large integers* t_1,

$$\lim_{t\to\infty}\sup\sum_{s=t_1}^{t-1}\alpha(s+1)\left[\frac{KQ(s)\delta(s)}{s+1}-\Delta\left(r(s)a(s)\right)+\frac{sr(s)a^2(s)}{s+1}\right]=\infty,$$

then every solution of (3.2) *is oscillatory.*

Letting $\alpha(t)=t$ and $a(t)=-\frac{1}{2t}$ in Corollary 3.8, we get the following result.

Corollary 3.9 *Assume that* (h_1) *and* (3.14) *hold. Furthermore assume that, for sufficiently large integers* t_1,

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left[KQ(s)\delta(s)+\frac{s+1}{2}\Delta\left(\frac{r(s)}{s}\right)+\frac{r(s)}{4s}\right]=\infty. \tag{3.19}$$

Then every solution of (3.2) *is oscillatory.*

Letting $\alpha(t)=t^2$ and $a(t)=-\frac{2t+1}{2t(t+1)}$ in Corollary 3.8, we get the following result.

Corollary 3.10 *Assume that* (h_1) *and* (3.14) *hold. Furthermore assume that, for sufficiently large integers* t_1,

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}(s+1)^2\left[\frac{Kp(s)\delta(s)}{s+1}+\Delta\left(\frac{(2s+1)r(s)}{2s(s+1)}\right)+\frac{r(s)(2s+1)^2}{4s(s+1)^3}\right]=\infty. \tag{3.20}$$

Then every solution (1.3) *is oscillatory.*

Next we give an oscillation result for the neutral delay q-difference equation (3.3). This result follows easily by applying Theorem 2.1 for the time scale $\mathbb{T} = [q^{n_0}, \infty)_{q^{\mathbb{N}_0}}$, $q > 1$.

Theorem 3.3 *Assume that* (h_1) *holds and*

$$\Delta_q r(t) \geq 0, \ \sum_{k=n_0}^{\infty} \mu(q^k)\frac{1}{r(q^k)} = \infty, \ \sum_{k=k_0}^{\infty} \mu(q^k)\delta(q^k)Q(q^k) = \infty. \tag{3.21}$$

Furthermore, assume that there exists a positive sequence $\alpha(t)$ *such that, for sufficiently large integers* n_1,

$$\limsup_{n\to\infty} \frac{1}{H(q^{n+1}, q^{n_1})} \sum_{k=n_1}^{n-1} q^n H(q^{n+1}, q^{k+1})\alpha(q^{k+1}) \left[\psi(q^k) - \phi(q^n, q^k)\right] = \infty, \tag{3.22}$$

where

$$\phi(t,s) = \frac{qr(s)\alpha^2(s)A^2(t,s)}{4\alpha^2(qs)},$$
$$A(t,s) = \frac{\alpha(qs)C_1(s)}{\alpha(s)} + \frac{H^{\Delta_s}(qt,s)}{H(t,qs)}, \ C_1(s) := \frac{\alpha^{\Delta_s}(s)}{\alpha(qs)} + 2\frac{a(s)}{q},$$

and

$$\psi(s) := \frac{KQ(s)\delta(s)}{qs} - \Delta_q\left(r(s)a(s)\right) + \frac{r(s)}{q}a^2(s).$$

Then every solution of (3.3) *is oscillatory.*

4 Examples

In this Section, we give some examples to illustrate the main results.

Example 4.1 Consider the following second-order neutral delay dynamic equation

$$\left[y(t) + \frac{1}{\delta^{-1}(t)}y(\tau(t))\right]^{\Delta\Delta} + \frac{\beta}{t\sigma(t)}y(\delta(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \ t_0 > 0, \tag{4.1}$$

such that

$$\beta \geq k \limsup_{t\to\infty} \frac{\sigma(t)}{\delta(t)}, \quad \text{where} \quad k > \frac{1}{4}.$$

Here $r(t) = 1$, $q(t) = \frac{\beta}{t\sigma(t)}$, and $\tau(t)$ and $\delta(t)$ are delay functions satisfying $\tau(t) = \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$, $\tau(t) \leq t$, $\delta(t) \leq t$ and $f(u) = u$, so that $K = 1$. It is clear that (h_1), (h_2) and (2.1) hold. To apply Corollary 2.5, it remains to prove that the condition (2.31) holds. In this case (2.31) reads

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^{t} \sigma(s)\left[\frac{\beta}{s\sigma(s)}\frac{\delta(s)}{\sigma(s)}\left(1 - \frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s}\right)^{\Delta} + \frac{1}{4\sigma(s)s}\right]\Delta s \\
&\quad = \limsup_{t\to\infty} \int_{t_0}^{t} \sigma(s)\left[\frac{\beta}{s\sigma(s)}\frac{\delta(s)}{\sigma(s)} - \frac{1}{4}\frac{1}{\sigma(s)s} - \frac{\beta}{s\sigma(s)}\frac{\delta(s)}{\sigma(s)}\frac{1}{s}\right]\Delta s \\
&\quad = \limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\beta\frac{\delta(s)}{\sigma(s)} - \frac{1}{4}}{s} - \frac{\beta}{s}\frac{\delta(s)}{\sigma(s)}\frac{1}{s}\right]\Delta s
\end{aligned}$$

$$\geq \limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{\beta\frac{\delta(s)}{\sigma(s)}-\frac{1}{4}}{s}-\frac{\beta}{s}\frac{1}{\sigma(s)}\right]\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\beta\frac{\delta(s)}{\sigma(s)}-\frac{1}{4}}{s}+\beta\left(\frac{1}{s}\right)^{\Delta}\right]\Delta s=\infty,$$

provided that $\beta \geq k\limsup_{t\to\infty}\frac{\sigma(t)}{\delta(t)}$, where $k>\frac{1}{4}$, and hence equation (4.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$. In particular, if $\mathbb{T}=\mathbb{R}$, $t_0>0$, and the delay is the constant delay $\delta(t)=t-\alpha$, where $\alpha>0$, then if $\beta>\frac{1}{4}$, equation (4.1) is oscillatory; if $\mathbb{T}=\mathbb{Z}$, $t_0=1$, and the delay is the constant delay $\delta(t)=t-n$, n a positive integer, then if $\beta>\frac{1}{4}$, equation (4.1) is oscillatory; and finally if $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$, $t_0=1$, and the delay function is $\delta(t)=\rho^n(t)=\frac{t}{q^n}$, where n is a positive integer, then if $\beta>\frac{1}{4}q^{n+1}$, equation (4.1) is oscillatory.

Example 4.2 Consider the following second-order neutral delay dynamic equation

$$\left[y(t)+\frac{\delta^{-1}(t)-1}{\delta^{-1}(t)}y(\tau(t))\right]^{\Delta\Delta}+\frac{\beta}{\delta(t)}y(\delta(t))(1+y^2(\delta(t)))=0,\ t\in[t_0,\infty)_{\mathbb{T}},\ t_0>0, \tag{4.2}$$

where β is a positive constant, and $\tau(t)\leq t$ and $\delta(t)\leq t$ are the delay functions defined on $\mathbb{T}$. In (4.2), $r(t)=1$, $p(t)=\frac{\delta^{-1}(t)-1}{\delta^{-1}(t)}$, $q(t)=\frac{\beta}{t\delta(t)}$ and $f(u)=u(1+u^2)\geq u$. It is easy to see that the assumptions (h_1) and (h_2) hold with $K=1$ and $q(t)(1-p(\delta(t)))=\frac{\beta}{\delta(t)}(\frac{1}{t})$. We will apply Corollary 2.5. Note, for $t_0>0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\sigma(s)\left[\frac{\delta(s)}{\sigma(s)}\frac{\beta}{\delta(s)}(\frac{1}{s})+\frac{1}{2}\left(\frac{1}{s}\right)^{\Delta}+\frac{1}{4\sigma(s)s}\right]\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}\sigma(s)\left[\frac{\beta}{s}\frac{1}{\sigma(s)}-\frac{1}{\sigma(s)}\frac{\beta}{s}\frac{1}{s}-\frac{1}{4}\frac{1}{\sigma(s)s}\right]\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\beta-\frac{1}{4}}{s}-\frac{\beta}{s}\frac{1}{s}\right]\Delta s=\infty,$$

provided that $\beta>\frac{1}{4}$ since $\int_{t_0}^{\infty}\frac{1}{s^{\gamma}}\Delta s<\infty$ for $\gamma>1$ (cf. [4]). Hence every solution of (4.2) oscillates if $\beta>\frac{1}{4}$.

Remark 4.1 We note that the results established by Agarwal et al. [5], Wu et al. [9] Li and Liu [16] and Li and Yeh [17], Şahiner [8] and Saker [7] cannot be applied to equations (4.1)–(4.2) in their general form with infinite delays. Thus our results are new and improve the results in the above mentioned papers.

References

[1] Hilger, S.: Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.,* **18**, 18–56 (1990)

[2] Agarwal, R. P., Bohner, M. O'Regan, D., Peterson, A.: Dynamic equations on time scales: A survey. *J. Comp. Appl. Math., Special Issue on Dynamic Equations on Time Scales*, edited by R. P. Agarwal, M. Bohner, and D. O'Regan, (Preprint in Ulmer Seminare 5), **141**(1–2), 1–26 (2002)

[3] Kac, V., Cheung, P.: Quantum Calculus, Springer, New York, 2001

[4] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston, 2001

[5] Spedding, V.: Taming Nature's Numbers. *New Scientist.* **19**, 28–31 (2003)

[6] Agarwal, R. P., O'Regan, D., Saker, S. H.: Oscillation criteria for second-order nonlinear neutral delay dynamic equations. *J. Math. Anal. and Appl.,* **300**, 203–217 (2004)

[7] Saker, S. H.: Oscillation of second-order nonlinear neutral delay dynamic equations on time scales. *J. Comp. Appl. Math.*, **187**, 123–141 (2006)

[8] Şahiner, Y.: Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales. *Adv. Difference Eqns.,* **2006**, 1–9 (2006)

[9] Wu, H., Wu, Zhuang, R. K., Mathsen, R. M.: Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations. *Appl. Math. Comp.*, **178**, 321–331 (2006)

[10] Agarwal, R. P., O'Regan, D., Saker, S. H.: Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales. *Appl. Analysis,* **86**, 1–17 (2007)

[11] Saker, S. H.: Hille and Nehari types oscillation criteria for second-order neutral delay dynamic equations. *Dyn. Cont. Disc. Imp. Sys*, (accepted)

[12] Saker, S. H.: Oscillation of second-order delay and neutral delay dynamic equations on time scales. *Dyn. Syst.* & *Appl.*, **16**, 345–360 (2007)

[13] Mathsen, R. M., Wang, Q. R., Wu, H. W.: Oscillation for neutral dynamic functional equations on time scales. *J. Diff. Eqns. Appl.*, **10**, 651–659 (2004)

[14] Saker, S. H.: Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type. *Dyn. Syst.* & *Appl.,* **15**, 629–644 (2006)

[15] Li, H. J.: Oscillation criteria for second order linear differential equations. *J. Math. Anal. Appl.,* **194**, 312–321 (1995)

[16] Li, H. J., Liu, W. L.: Oscillation criteria for second order neutral differential equations. *Canad. J. Math.,* **48**, 871–886 (1996)

[17] Li, H. J., Yeh, C. C.: Oscillation criteria for second-order neutral delay difference equations. *Comp. Math. Appl.*, **36**, 123–132 (1998)

[18] Bohner, E. Akin, Bohner, M., Akin, F.: Pachpatte inequalities on time scales. *JIPAM. J. Ineq. Pure Appl. Math.*, **6**, 1–23 (2005)

[19] Bohner, M., Stević, S.: Asymptotic behavior of second-order dynamic equations. *Appl. Math. Comp.*, **188**, 1503–1512 (2007)