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The Existence of Global Attractors for 2D Navier–Stokes Equations in H^k Spaces

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Abstract In this paper, we prove that the 2D Navier–Stokes equations possess a global attractor in $H^k(\Omega, R^2)$ for any $k \geq 1$, which attracts any bounded set of $H^k(\Omega, R^2)$ in the H^k-norm. The result is established by means of an iteration technique and regularity estimates for the linear semigroup of operator, together with a classical existence theorem of global attractor. This extends Ma, Wang and Zhong's conclusion.

Keywords Navier–Stokes equations, semigroup of operator, global attractor, regularity of attractor

MR(2000) Subject Classification 35B40, 35B41

1 Introduction

In infinite-dimensional dynamical systems, one of the most important problems is to prove the existence of global attractors. The dynamic properties of the Navier–Stokes equations, i.e., the global asymptotic behaviors of solutions and existence of global attractors, play an important role in the study of fluid mechanics. Because they determine the stability of the flow and provide a mathematical foundation for the study of viscous incompressible fluids. Many scholars have paid attention to this problem for quite a long time, especially in the past decades. For the classical results we refer to the monographs [1–4].

The main purpose of this paper is to prove the existence of global attractors in Sobolev space $H^k(\Omega, R^2)$ with any $k \geq 1$ for the following 2D Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = \Delta u - \nabla p + f(x), \\ \mathrm{div} u = 0, \quad u|_{\partial\Omega} = 0, \\ u(x, 0) = \varphi(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the velocity vector, p is the pressure, f represents the external volume forces applied to the fluid. $\Omega \subset R^2$ is a C^∞ bounded domain.

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The existence of global attractors for 2D Navier–Stokes equations in L^2 space was first proved by Foias and Temam [1], and in H_0^1 space, it was derived by Ma, Wang, and Zhong et al. [5–6]. Here, we will use the iteration technique for regularity estimates to derive the existence of global attractors in H^k for any $k \geq 1$.

The basic idea employed in this paper follows from Ma and Wang's recent books (cf. [7–8]). We briefly introduce the method in the following.

Firstly, the equations (1.1) can be written in the following abstract form

$$\begin{cases} \dfrac{du}{dt} = Lu + G(u), \\ u(0) = \varphi, \end{cases} \tag{1.2}$$

where $L : H_1 \longrightarrow H$ is a sectorial operator, which is associated with the Laplace operator Δ, and $G : H_1 \longrightarrow H$ a nonlinear operator. The sectorial operator L can induce the fractional power operators $\mathscr{L}^\alpha (\mathscr{L} = -L)$ and the fractional order spaces H_α for $\alpha \in R^1$. H and H_1 are both Hilbert spaces, and H_1 is compact and dense in H.

Secondly, the solution of (1.2) can be expressed as

$$u(t, \varphi) = \mathrm{e}^{tL}\varphi + \int_0^t \mathrm{e}^{(t-\tau)L} G(u) d\tau. \tag{1.3}$$

For the case $\varphi \in H_{\frac{1}{2}}$, it is well known that the solution $u(t, \varphi)$ of (1.2) is uniformly bounded for $0 \leq t < \infty$ in $H_{\frac{1}{2}}$, i.e., there is a constant $C > 0$ such that

$$||u(t, \varphi)||_{H_{\frac{1}{2}}} < C, \quad \forall\, t \geq 0. \tag{1.4}$$

On the other hand, the nonlinear operator

$$G : H_{\frac{1}{2}} \longrightarrow H_{-\theta} \tag{1.5}$$

is bounded for $\theta > 0$, hence we have the estimates

$$||u(t, \varphi)||_{H_\alpha} \leq ||\mathrm{e}^{tL}\varphi||_{H_\alpha} + \int_0^t ||\mathscr{L}^{\alpha+\theta}\mathrm{e}^{(t-\tau)L}|| \cdot ||\mathscr{L}^{-\theta} G||_H d\tau.$$

By(1.4) and (1.5), $||\mathscr{L}^{-\theta} G(u(t, \varphi))||_H \leq C$, thus

$$||u(t, \varphi)||_{H_\alpha} \leq C + C \int_0^t \tau^{-(\alpha+\theta)} \mathrm{e}^{-\delta\tau} d\tau,$$

which implies that, when $\alpha < 1$,

$$u(t, \varphi) \in H_\alpha, \ \forall\, t \geq 0, \ \varphi \in H_\alpha,$$

and $u(t, \varphi)$ is uniformly bounded in H_α, i.e.

$$||u(t, \varphi)||_{H_\alpha} < C, \quad \forall\, t \geq 0, \ \alpha < \frac{3}{4}. \tag{1.6}$$

By iteration, from (1.6) we can deduce that $u(t, \varphi) \in H_k$ for all $k \geq 0$, and is unifomly bounded in H_k. Finally, in the same fashion, for any $k \geq 0$, we can prove that (1.2) has a bounded absorbing set in H_k. Thus, the desired result is obtained.

This paper is organized as follows: In Section 2, we recall some properties of semigroups and prove some lemma which will be used later. Then, in Section 3, we give some mathematical setting. Finally, in Section 4, we present detailed proofs of the main existence theorem.

2 Preliminaries

Let H and H_1 be Hilbert spaces, H_1 is compact and dense in H. We consider the problem given by (1.2), where $L : H_1 \longrightarrow H$ is a linear operator, and $G : H_1 \longrightarrow H$ is a nonlinear operator.

A family of operators $S(t) : H \longrightarrow H$ $(t \geq 0)$ is called as an semigroup of operator generated by (1.2) if $S(t)$ enjoys the usual semigroup properties:

(1) $S(t) : H \longrightarrow H$ is continuous mapping for any $t \geq 0$,

(2) $S(0) = id : H \longrightarrow H$ an identity,

(3) $S(t + s) = S(t) \cdot S(s)$, $\forall\, t, s \geq 0$, and the solution of (1.2) can be expressed as $u(t, \varphi) = S(t)\varphi$.

In the following, we introduce the concepts and definitions of invariant set, global attractor, ω-limit set for the semigroup $S(t)$.

Definition 2.1 *Let $S(t)$ be an semigroup of operator defined on H. A set $\Sigma \subset H$ is called as an invariant set for the semigroup $S(t)$ if*

$$S(t)\Sigma = \Sigma, \ \forall\ t \geq 0.$$

An invariant set Σ is an attractor for the semigroup $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that, for any $\varphi \in U$,

$$\mathrm{dist}(S(t)\varphi, \Sigma) = \inf_{v \in \Sigma} ||S(t)\varphi - v||_H \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of H, then Σ is called as a global attractor of $S(t)$.

For a set $D \subset H$, we define the ω-limit set of D by

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D},$$

where the closure is taken in the H-norm.

The following is the classical existence theorem of global attractor, which can be found in Temam [4].

Theorem 2.1 *Let $S(t) : H \to H$ be a semigroup of operator generated by* (1.2). *If the following conditions are satisfied*:

(1) *$S(t)$ has a bounded absorbing set $B \subset H$, i.e., for any bounded set $A \subset H$, there exists a time $t_A \geq 0$ such that $S(t)\varphi \in B$, $\forall\ \varphi \in A$ and $t > t_A$;*

(2) *$S(t)$ is uniformly compact, i.e., for any bounded set $U \subset H$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in H; then, the ω-limit set $\mathscr{A} = \omega(B)$ of B is a global attractor of* (1.2), *and $\mathscr{A}$ is connected providing B is connected.*

Hereafter, we always assume that the linear operator $L : H_1 \longrightarrow H$ in (1.2) *is a sectorial operator, which generates an analytic semigroup e^{tL}, and L can induce the fractional power operators and fractional order spaces as follows*:

$$\mathscr{L}^\alpha = (-L)^\alpha : \ H_\alpha \longrightarrow H, \ \alpha \in R^1,$$

where $H_\alpha = D(\mathscr{L}^\alpha)$, the domain of $\mathscr{L}^\alpha$, are the Hilbert spaces endowed with the norm

$$||x||_{H_\alpha} = ||\mathscr{L}^\alpha x||_H, \quad \alpha \in R^1. \tag{2.1}$$

It is well known that, for any $\alpha > \beta$, $H_\alpha \subset H_\beta$ is compact and dense inclusion.

Thus, Theorem 2.1 can be equivalently expressed in the following form, which can be used later.

Theorem 2.2 *Let $u(t, \varphi) = S(t)\varphi$ $(\varphi \in H,\ t \geq 0)$ be a solution of* (1.2), *$S(t)$ the semigroup of operator generated by* (1.2). *If*

(1) *for some* $\alpha \geq 0$, *there is a bounded set* $B \subset H_\alpha$, *as* $\varphi \in H_\alpha$, *there exists* $t_\varphi > 0$ *such that* $u(t,\varphi) \in B,\ \forall\ t > t_\varphi$;

(2) *there is a* $\beta > \alpha$, *for any bounded set* $U \subset H_\beta$, *there exist* $T > 0$ *and* $C > 0$ *such that*

$$||u(t,\varphi)||_{H_\beta} \leq C,\ \forall\ t \geq T,\ \varphi \in U,$$

then, the problem (1.2) *possesses a global attractor* $\mathscr{A} \subset H_\alpha$, *and* $\mathscr{A}$ *attracts any bounded set of* H_α *in the* H_α*-norm.*

Inaddition, for sectorial operators we need the following properties in later discussion, which can be found in Pazy [9].

Theorem 2.3 *Let* $L : H_1 \longrightarrow H$ *be a sectorial operator, which generates an analytic semigroup* $T(t) = \mathrm{e}^{tL}$. *If all eigenvalues* λ *of* L *satisfy that* $\mathrm{Re}\lambda < -\lambda_0$ *for some* $\lambda_0 > 0$, *then we have*

(1) $T(t) : H \longrightarrow H_\alpha$ *is bounded for all* $\alpha \in R^1$, $t > 0$, *and*

$$||T(t)\varphi||_{H_\alpha} \leq C\mathrm{e}^{-\delta t}||\varphi||_{H_\alpha},\quad \text{for some}\quad \delta > 0,$$

(2) $T(t)(-L)^\alpha\varphi = (-L)^\alpha T(t)\varphi,\quad \forall\ \varphi \in H_\alpha,\quad t \geq 0,$

(3) *for every* $t > 0$, $(-L)^\alpha T(t) : H \longrightarrow H$ *is bounded, and*

$$||(-L)^\alpha T(t)|| \leq C_\alpha t^{-\alpha}\mathrm{e}^{-\delta t},$$

for some $\delta > 0$, *where* $C_\alpha > 0$ *is a constant only depending on* α.

The following lemma is due to Ma and Wang (cf. [7]), which will be used later.

Lemma 2.1 *Let* $\varphi \in H$, *and* $u(t,\varphi) \in L^1((0,T),H)$ *be a weak solution of* (1.2), *i.e.,* $u(t,\varphi)$ *satisfies*

$$\langle u, v\rangle_H = \int_0^t (\langle u, L^*v\rangle_H + \langle G(u), v\rangle_H)dt + \langle\varphi, v\rangle_H, \tag{2.2}$$

for all $v \in H_1$ *and* $0 \leq t \leq T$. *If* $G(u(\cdot,\varphi)) \in L^1((0,T),H)$, *then the weak solution* $u(t,\varphi)$ *of* (1.2) *can be expressed by* (1.3).

Proof For the weak solution $u(t,\varphi)$ of (1.2), we note that

$$v = \mathrm{e}^{tL}\varphi + \int_0^t \mathrm{e}^{(t-\tau)L}G(u(\tau,\varphi))d\tau. \tag{2.3}$$

By (2.2), for all $\tilde{u} \in H_1$, $u(t,\varphi)$ satisfies

$$\langle u, \tilde{u}\rangle = \int_0^t (\langle u, L^*\tilde{u}\rangle_H + \langle G(u), \tilde{u}\rangle_H)dt + \langle\varphi, \tilde{u}\rangle_H. \tag{2.4}$$

From (2.3) we deduce that

$$\langle v, \tilde{u}\rangle_H = \langle \mathrm{e}^{tL}\varphi, \tilde{u}\rangle_H + \int_0^t \langle L^{-1}\mathrm{e}^{(t-\tau)L}G(u), L^*\tilde{u}\rangle_H d\tau,\ \forall\ \tilde{u} \in H_1. \tag{2.5}$$

It is easy to see that

$$\frac{d}{dt}\int_0^t \langle L^{-1}\mathrm{e}^{(t-\tau)L}G(u), L^*\tilde{u}\rangle_H d\tau = \langle G(u), \tilde{u}\rangle_H + \int_0^t \langle \mathrm{e}^{(t-\tau)L}G(u), L^*\tilde{u}\rangle_H d\tau,$$

which exists for all $\tilde{u} \in H_1$. Hence, (2.5) is differentiable for $t \geq 0$, and

$$\begin{aligned}\frac{d}{dt}\langle v, \tilde{u}\rangle_H &= \langle L\mathrm{e}^{tL}\varphi, \tilde{u}\rangle_H + \langle G(u), \tilde{u}\rangle_H \\ &\quad + \left\langle \int_0^t \mathrm{e}^{(t-\tau)L}G(u)d\tau, L^*\tilde{u}\right\rangle_H \\ &= \langle v, L^*\tilde{u}\rangle_H + \langle G(u), \tilde{u}\rangle_H.\end{aligned} \tag{2.6}$$

Integrating (2.6) for $t > 0$, from (2.3) we get

$$\langle v, \tilde{u}\rangle_H = \int_0^t (\langle v, L^*\tilde{u}\rangle_H + \langle G(u), \tilde{u}\rangle_H)dt + \langle \varphi, \tilde{u}\rangle_H. \tag{2.7}$$

Let $w = u - v$, from (2.4) and (2.7), it follows that

$$\langle w, \tilde{u}\rangle_H = \int_0^t \langle w, L^*\tilde{u}\rangle_H dt, \ \forall\ \tilde{u} \in H_1. \tag{2.8}$$

We recall the Ball's theorem (cf. Ball [10] and Pazy [9]), which amounts to saying that, for each $\varphi \in H$, there exists a unique solution $u \in L^2((0, T), H)$ of the following equation

$$\langle u, \tilde{u}\rangle_H = \langle \varphi, \tilde{u}\rangle_H + \int_0^t (\langle u, L^*\tilde{u}\rangle_H + \langle g(\tau), \tilde{u}\rangle_H)d\tau, \quad \forall\ \tilde{u} \in H_1,$$

if and only if L generates a strongly continuous semigroup on H.

Thus, by the Ball's theorem, it follows from (2.8) that $w = 0$ for all $t \geq 0$, i.e., $v = u$. Hence, this lemma is proved.

3 Mathematical Setting

For the 2D Navier–Stokes equations (1.1), we introduce the spaces as follows:

$$H = \{u \in L^2(\Omega, R^2)|\ \ \mathrm{div} u = 0, \quad u \cdot n|_{\partial\Omega} = 0\},$$
$$H_1 = \{u \in H^2(\Omega, R^2) \cap H|\ \ u|_{\partial\Omega} = 0\},$$

and define the operators $L : H_1 \longrightarrow H$ and $G : H_1 \longrightarrow H$ by

$$\begin{cases} Lu = P[\Delta u], \\ Gu = P[f(x) - (u \cdot \nabla)u], \end{cases} \tag{3.1}$$

where $P : L^2(\Omega, R^2) \longrightarrow H$ is a Leray projection.

It is well known that the operator $L : H_1 \longrightarrow H$ defined by (3.1) is a sectorial operator, and the associated space $H_{\frac{1}{2}}$ is

$$H_{\frac{1}{2}} = \{u \in H^1(\Omega, R^2) \cap H|\ \ u|_{\partial\Omega} = 0\}.$$

Under the definitions of operators L and G in (3.1), the 2D Navier–Stokes equations (1.1) are referred to the abstract form of (1.2). It is a classical result that the (1.2) associated with (1.1) generates semigroups of operators:

$$S(t) : H_\alpha \longrightarrow H_\alpha, \ \ \forall\ 0 \leq \alpha \leq \frac{1}{2},$$

see Temam [4].

We recall some known results related to existence of global attractors for the 2D Navier–Stokes equations (see Temam [4] and Ma et al. [5–6]):

Theorem 3.1 *The semigroups $S(t) : H_\alpha \longrightarrow H_\alpha (0 \leq \alpha \leq \frac{1}{2})$ associated with* (1.1) *possess a global attractor $\mathscr{A} \subset H_\alpha$, and it attracts all bounded set of H_α in the H_α-norm.*

4 Main Theorem

We are now in a position to state and prove the main theorem in this paper, which provides the existence of global attractor in spaces H^k for any k-th differentiable functions.

Theorem 4.1 *Let $f \in C^\infty(\overline{\Omega}, R^2)$ be a C^∞ function. Then for any $k \geq 1$, there exists an attractor attractor $\mathscr{A} \subset H^k(\Omega, R^2)$ for the equations* (1.1), *and $\mathscr{A}$ attracts all bounded sets of $H^k(\Omega, R^2) \cap H_{\frac{1}{2}}$ in the H^k-norm.*

Proof By Lemma 2.1, for any $\varphi \in H$, the solution $u(t,\varphi)$ of (1.1) can be expressed by

$$u(t,\varphi) = \mathrm{e}^{tL}\varphi + \int_0^t \mathrm{e}^{(t-\tau)L} G(u) d\tau. \tag{4.1}$$

First, we prove that the mapping

$$G : H_{\frac{1}{2}} \longrightarrow H_{-\theta} \ \ (\theta > 0) \ \text{ is bounded.} \tag{4.2}$$

It is known that, for any $\alpha \in R^1$, H_α and $H_{-\alpha}$ are dual, and if, for all $v \in H_\alpha$, we have

$$\langle u, v\rangle_H \leq C||v||_\alpha,$$

then, $u \in H_{-\alpha}$ and $||u||_{H_{-\alpha}} \leq C$. We note that $G(u) = P[f - (u \cdot \nabla)u]$, hence,

$$\begin{aligned} \langle G(u), v\rangle_H &= \int_\Omega (fv - (u\cdot\nabla)u \cdot v)dx \\ &\leq ||f||_{C^0} \cdot ||v||_{L^1} + C\int_\Omega |u|\cdot|Du|\cdot|v|dx \\ &\leq ||f||_{C^0}||v||_{L^1} + C||u||_{H^1}||u||_{L^{2q}}||v||_{L^{2p}}, \end{aligned} \tag{4.3}$$

where $p > 1$ is arbitrary, $q = p/(p-1)$. By the embedding theorems for the fractional order spaces (see Pazy [9]), we have

$$||v||_{L^{2p}} \leq C||v||_{H_\theta}, \quad \forall\, \theta > \frac{p-1}{2p}. \tag{4.4}$$

Hence, from(4.3) and (4.4) we derive that

$$\langle G(u), v\rangle_H \leq C[||f||_{C^0} + ||u||^2_{H_{\frac{1}{2}}}]\cdot||v||_\theta, \ \ \forall\, \theta > 0,$$

which implies that (4.2) holds.

Next, we prove that the solution $u(t,\varphi)$ of (1.1) is uniformly bounded in H_α for all $\alpha < 1$, i.e.,

$$||u(t,\varphi)||_{H_\alpha} \leq C, \quad \forall\, t \geq 0,\ \varphi \in H_\alpha, \alpha < 1. \tag{4.5}$$

By Theorem 3.1, for any $\varphi \in H_\alpha(\alpha \geq \frac{1}{2})$ the solution of (1.1) is uniformly bounded in $H_{\frac{1}{2}}$, i.e., (4.5) is valid for $\alpha = \frac{1}{2}$. Hence, by (4.2) we have

$$||\mathscr{L}^{-\theta}G(u(t,\varphi))||_H \leq C, \ \ \forall\, t \geq 0,\ \varphi \in H_\alpha \left(\alpha \geq \frac{1}{2}\right),\ \theta > 0, \tag{4.6}$$

where $\mathscr{L} = -L$. From (4.1) we get

$$\begin{aligned} ||u(t,\varphi)||_{H_\alpha} &\leq ||\mathrm{e}^{tL}\varphi||_{H_\alpha} + \int_0^t ||\mathscr{L}^{\alpha+\theta}\mathrm{e}^{(t-\tau)L}||\cdot||\mathscr{L}^{-\theta}G||_H d\tau \\ &\leq (\text{by Theorem 2.2 and (4.6)}) \\ &\leq C + C\int_0^t \tau^{-(\alpha+\theta)}\mathrm{e}^{-\delta\tau}d\tau, \qquad \alpha+\theta < 1. \end{aligned}$$

Therefore, (4.5) holds true.

In the following, we need to prove that for any α with $\frac{1}{2} < \alpha \leq \frac{k}{2}$ $(k \geq 2)$ there is a number β with $0 < \beta \leq \frac{k-1}{2}$, such that

$$G : H_\alpha \longrightarrow H_\beta \ \text{ bounded , and } \beta \to \frac{k-1}{2} \ \text{ as } \ \alpha \to \frac{k}{2}. \tag{4.7}$$

Since f is a C^∞ function, $f \in H_\alpha, \forall\, \alpha \in R^1$. We note that $G(u) = P[f - (u\cdot\nabla)u]$. Hence, it suffices to prove (4.7) only for the bilinear operator

$$B(u,v) = P[(u\cdot\nabla)v] : H_{\alpha_1} \times H_{\alpha_2} \longrightarrow H_\beta. \tag{4.8}$$

It is known that the spaces $H_\alpha(\frac{k-1}{2}\leq\alpha\leq\frac{k}{2})$ are interpolations between $H_{\frac{k}{2}}$ and $H_{\frac{k-1}{2}}$ (see Temam [4]):

$$\begin{aligned}&H_\alpha=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta,\quad \alpha=\frac{k-1+\theta}{2},\quad \theta\in[0,1],\\ &H_{\frac{k-1}{2}}=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_0,\\ &H_{\frac{k}{2}}=[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_1.\end{aligned}$$

Moreover, if $L:H_{\frac{k}{2}}\longrightarrow H_{\frac{k-1}{2}}$ and $L:H_{\frac{k-1}{2}}\longrightarrow H_{\frac{k-2}{2}}$ are linear bounded, then L is also linear bounded on $[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta$:

$$L:[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta\to[H_{\frac{k-1}{2}},H_{\frac{k-2}{2}}]_\theta,\quad \forall\,\theta\in[0,1]. \tag{4.9}$$

See Temam [4].

For the bilinear operators (4.8), it is readily to verify that for any bounded set $\Omega\subset H_{\frac{k-1}{2}}$, there is a constant $C>0$, such that for all $u\in\Omega$, we have

$$\begin{cases}B(u,\cdot):H_{\frac{k}{2}}\to H_{\frac{k-1}{2}}\text{ linear bounded }(k\geq 2),\\ B(u,\cdot):H_{\frac{1}{2}}\to H_{-\gamma}\text{ linear bounded, }\forall\,\gamma>0,\\ ||B(u,\cdot)||\leq C,\quad\forall\,u\in\Omega\subset H_{\frac{k}{2}}\ (k\geq 1).\end{cases} \tag{4.10}$$

By the interpolation relations (4.9) for linear bounded operators, we infer from (4.10) that

$$\begin{cases}B(u,\cdot):[H_{\frac{k}{2}},H_{\frac{k-1}{2}}]_\theta\to[H_{\frac{k-1}{2}},H_{\frac{k-2}{2}}]_\theta\ (k\geq 3),\\ B(u,\cdot):[H_1,H_{\frac{1}{2}}]_\theta\to[H_{\frac{1}{2}},H_{-\gamma}]_\theta,\ \gamma>0,\\ ||B(u,\cdot)||\leq C,\quad\forall\,u\in\Omega\subset H_{\frac{k-1}{2}}\ (k\geq 3),\text{ or }\ u\in\Omega\subset[H_1,H_{\frac{1}{2}}]_\theta.\end{cases} \tag{4.11}$$

We denote

$$\alpha=\frac{k-1+\theta}{2}(k\geq 2),\quad \beta=\begin{cases}\dfrac{k-2+\theta}{2}, & k\geq 3,\\ -\gamma+\left(\dfrac{1}{2}+\gamma\right)\theta, & k=2.\end{cases}$$

Then, from (4.11) we derive that

$$B:H_\alpha\longrightarrow H_\beta\ \text{ bounded, and }\beta\to\frac{k-1}{2}\ \text{ as }\ \alpha\to\frac{k}{2}, \tag{4.12}$$

where $B(u)=B(u,u)$ is as in (4.8). Thus, (4.7) follows from (4.12).

In the same fashion as in the proof of (4.5), by iteration we can prove that for all $\alpha\geq 0$, the solution $u(t,\varphi)$ is uniformly bounded in H_α, i.e., for any bounded set $U\subset H_\alpha$, there is a constant $C>0$ such that

$$||u(t,\varphi)||_{H_\alpha}\leq C,\quad\forall\ \ t\geq 0,\ \varphi\in U\subset H_\alpha,\ \alpha\geq 0. \tag{4.13}$$

Finally, we shall verify that for any $\alpha\geq 0$, (1.1) has a bounded absorbing set in H_α. By Theorem 3.1, for the case $0\leq\alpha\leq\frac{1}{2}$ the above conclusion is valid. We proceed only for the case $\alpha>\frac{1}{2}$. By (4.1) we have

$$u(t,\varphi)=\mathrm{e}^{(t-T)L}u(T,\varphi)+\int_T^t\mathrm{e}^{(t-\tau)L}G(u)d\tau. \tag{4.14}$$

Let $D\subset H_{\frac{1}{2}}$ be the bounded absorbing set of (1.1) in $H_{\frac{1}{2}}$, and $T_0>0$ the time such that

$$u(t,\varphi)\in D,\quad\forall\ \ t>T_0,\ \varphi\in U\subset H_\alpha\ \left(\alpha\geq\frac{1}{2}\right). \tag{4.15}$$

By Assertion (1) of Theorem 2.3, for all $\varphi\in U$ and a given $T>0$, we have that

$$\lim_{t\to\infty}||\mathrm{e}^{(t-T)L}u(T,\varphi)||_{H_\alpha}\to 0. \tag{4.16}$$

By Assertion (3) of Theorem 2.3, it follows from (4.14) and (4.16) that, for any $\alpha < 1$ and $T > T_0$, we have

$$\begin{aligned} ||u(t,\varphi)||_{H_\alpha} &\le ||\mathrm{e}^{(t-T)L}u(T_0,\varphi)||_{H_\alpha} + \int_T^t ||\mathscr{L}^{\alpha+\theta}\mathrm{e}^{(t-\tau)L}\mathscr{L}^{-\theta}G||_H d\tau \\ &\le C + C\int_T^t (t-\tau)^{-(\alpha+\theta)}\mathrm{e}^{-\delta(t-\tau)}||\mathscr{L}^{-\theta}G(u)||_H d\tau \\ &\le (\text{by (4.2) and (4.15)}) \\ &\le C + C\int_0^{t-T} \tau^{-(\alpha+\theta)}\mathrm{e}^{-\delta\tau}d\tau, \end{aligned} \tag{4.17}$$

where $C > 0$ is independent of $\varphi \in U$. For any $\alpha < 1$ we can take $\theta > 0$ such that $\alpha + \theta < 1$. Hence, it follows from (4.17) that, for any $\alpha < 1$ and $U \subset H_\alpha$, there is a $T > 0$, such that

$$||u(t,\varphi)||_{H_\alpha} \le C, \quad \forall\, t \ge T,\ \varphi \in U, \tag{4.18}$$

where $C > 0$ is independent of φ.

By iteration, we can derive (4.18) for all $\alpha \ge 0$. Hence, (1.1) has a bounded absorbing set in H_α for all $\alpha \ge 0$.

Thus, this theorem follows from (4.13), (4.18) and Theorem 2.2. The proof is complete.

Remark 4.1 The attractors $\mathscr{A}_\alpha \subset H_\alpha$ in Theorem 4.1 are the same for all $\alpha \ge 0$, i. e. $\mathscr{A}_\alpha = \mathscr{A}$, $\forall\ \alpha \ge 0$. Hence, $\mathscr{A} \subset C^\infty(\Omega, R^2)$. It is known that, for $\varphi \in H$ and $t > 0$, the solution $u(t,\varphi)$ of (1.1) is analytic in t with values in $C^\infty(\Omega, R^2)$ providing $f \in C^\infty(\Omega, R^2)$ and $\Omega \subset R^2$ being C^∞, see Temam [4]. Hence Theorem 4.1 implies that for any $\varphi \in H$, $u(t,\varphi)$ satisfies

$$\lim_{t\to\infty} \inf_{v\in\mathscr{A}} ||u(t,\varphi) - v||_{C^k} = 0, \quad \forall\, k \ge 1.$$

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