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Integral Formula of Minkowski Type and New Characterization of the Wulff Shape

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Abstract Given a positive function F on S^n which satisfies a convexity condition, we introduce the r-th anisotropic mean curvature M_r for hypersurfaces in $\mathbb{R}^{n+1}$ which is a generalization of the usual r-th mean curvature H_r. We get integral formulas of Minkowski type for compact hypersurfaces in R^{n+1}. We give some new characterizations of the Wulff shape by the use of our integral formulas of Minkowski type, in case $F = 1$ which reduces to some well-known results.

Keywords Wulff shape, F-Weingarten operator, anisotropic principal curvature, r-th anisotropic mean curvature, integral formula of Minkowski type

MR(2000) Subject Classification 53C42, 53A30, 53B25

1 Introduction

Let $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies the following convexity condition:

$$(D^2F + F1)_x > 0, \quad \forall\, x \in S^n, \tag{1}$$

where D^2F denotes the intrinsic Hessian of F on S^n and 1 denotes the identity on T_xS^n, > 0 means that the matrix is positive definite. We consider the map

$$\phi\colon S^n \to \mathbb{R}^{n+1},$$
$$x \to F(x)x + (\mathrm{grad}_{S^n} F)_x,$$

its image $W_F = \phi(S^n)$ is a smooth, convex hypersurface in $\mathbb{R}^{n+1}$ called the Wulff shape of F (see [1–5]).

Now let $X\colon M \to \mathbb{R}^{n+1}$ be a smooth immersion of a compact, orientable hypersurface without boundary. Let $\nu\colon M \to S^n$ denotes its Gauss map, that is, ν is a unit inner normal vector of M.

Let $A_F = D^2F + F1$, $S_F = -A_F \circ \mathrm{d}\nu$. S_F is called the F-Weingarten operator, and the eigenvalues of S_F are called anisotropic principal curvatures. Let σ_r be the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1, \lambda_2, \dots, \lambda_n$:

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \quad (1 \le r \le n).$$

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We set $\sigma_0 = 1$. The r-anisotropic mean curvature M_r is defined by $M_r = \sigma_r/C_n^r$, which was introduced by Reilly in [6].

In this paper we first give the following integral formulas of Minkowski type for compact hypersurfaces in $\mathbb{R}^{n+1}$.

Theorem 1.1 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*. Then we have the following integral formulas of Minkowski type hold:*

$$\int_M (FM_r + M_{r+1}\langle X, \nu\rangle)\mathrm{d}A_X = 0, \quad r = 0, 1, \ldots, n-1. \tag{2}$$

By the use of the above integral formulas of Minkowski type, we prove the following new characterizations of the Wulff shape:

Theorem 1.2 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*, and $M_1 = \mathrm{const}$ and $\langle X, \nu\rangle$ has fixed sign. Then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Theorem 1.3 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*. If $M_1 = \mathrm{const}$ and $M_r = \mathrm{const}$ for some r, $2 \le r \le n$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Theorem 1.4 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*. If $\frac{M_r}{M_k} = \mathrm{const}$ for some k and r, with $0 \le k < r \le n$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Theorem 1.5 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*. If $\frac{M_k}{M_n} = \mathrm{const}$ for some k, with $0 \le k \le n-1$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Choosing $k = 0$ in Theorem 1.4, we get

Corollary 1.1 *Let $X\colon M \to \mathbb{R}^{n+1}$ be an n-dimensional compact convex hypersurface, $F\colon S^n \to \mathbb{R}^+$ be a smooth function which satisfies* (1)*, and for a fixed r with $1 \le r \le n$, $M_r = \mathrm{const}$. Then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Remark 1.1 When $F = 1$, Wulff shape is just the round sphere and $M_r = H_r$, formula (2) reduces to the classical Minkowski integral formula (see [7] or [8]). Theorem 1.2 reduces to the classical Theorem given by Süss [4], Corollary 1.1 reduces to Theorem of Yano [9], Theorem 1.3 reduces to Theorem of Choe [10]. We also note that in [11], the authors proved the integral formula of Minkowski type for compact spacelike hypersurfaces in de Sitter space.

2 Preliminaries

Let $\{E_1, \ldots, E_n\}$ is a local orthogonal frame on S^n, let $e_i = E_i \circ \nu$, where $i = 1, \ldots, n$. Then $\{e_1, \ldots, e_n\}$ is a local orthogonal frame of $X\colon M \to \mathbb{R}^{n+1}$.

The structure equation of S^n is:

$$\begin{cases} \mathrm{d}x = \sum_i \theta_i E_i, \\ \mathrm{d}E_i = \sum_j \theta_{ij} E_j - \theta_i x, \\ \mathrm{d}\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \\ \mathrm{d}\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = -\frac{1}{2} \sum_{kl} \tilde{R}_{ijkl} \theta_k \wedge \theta_l = -\theta_i \wedge \theta_j, \end{cases} \tag{3}$$

where $\theta_{ij} + \theta_{ji} = 0$ and

$$\tilde{R}_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{4}$$

The structure equation of X is (see [12], [13]):

$$\begin{cases} \mathrm{d}X = \sum_i \omega_i e_i, \\ \mathrm{d}\nu = -\sum_{ij} h_{ij}\omega_j e_i, \\ \mathrm{d}e_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j \nu, \\ \mathrm{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \\ \mathrm{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}\sum_{kl} R_{ijkl}\theta_k \wedge \theta_l, \end{cases} \tag{5}$$

where $\omega_{ij} + \omega_{ji} = 0$, $R_{ijkl} + R_{ijlk} = 0$, and R_{ijkl} are the components of the Riemannian curvature tensor of M with respect to the induced metric $dX \cdot dX$.

From $\mathrm{d}e_i = \mathrm{d}(E_i \circ \nu) = \nu^* \mathrm{d}E_i = \sum_j \nu^*\theta_{ij} e_j - \nu^*\theta_i \nu$, we get

$$\begin{cases} \omega_{ij} = \nu^*\theta_{ij}, \\ \nu^*\theta_i = -\sum_j h_{ij}\omega_j, \end{cases} \tag{6}$$

where $\omega_{ij} + \omega_{ji} = 0$, $h_{ij} = h_{ji}$.

Let $F\colon S^n \to \mathbb{R}^+$ be a smooth function. We denote the coefficients of covariant differential of F, $\mathrm{grad}_{\mathrm{S^n}}\mathrm{F}$, D^2F with respect to $\{E_i\}_{i=1,\dots,n}$ by F_i, F_{ij}, F_{ijk} respectively.

From Ricci identity and (4), we have

$$F_{ijk} - F_{ikj} = \sum_m F_m \tilde{R}_{mijk} = \delta_{ik}F_j - \delta_{ij}F_k, \tag{7}$$

where F_{ijk} denote the coefficients of the covariant differential of F_{ij} on S^n.

So, if we denote the coefficients of A_F by A_{ij}, then we have from (7)

$$A_{ijk} = A_{jik} = A_{ikj}, \tag{8}$$

where A_{ijk} denote the coefficients of the covariant differential of A_F on S^n.

Let $s_{ij} = \sum_k (A_{ik} \circ \nu) h_{kj}$, $S_F = -A_F \circ \mathrm{d}\nu$. Then we have $S_F(e_j) = \sum_i s_{ij} e_i$. We call S_F the F-Weingarten operator. From the positive definiteness of (A_{ij}) and the symmetry of (h_{ij}), we know the eigenvalues of (s_{ij}) are all real (in fact, because $A = (A_{ij})$ is positive definite, there exists a non-singular matrix C such that $A = C^tC$, we have $S = (s_{ij}) = AB$ has the same eigenvalues with the real symmetric matrix CBC^T, which follows from $|\lambda I - S| = |\lambda I - AB| = |\lambda I - C^tCB| = |\lambda I - CBC^t|$, where $B = (h_{ij})$). We call them anisotropic principal curvatures, and denote them by $\lambda_1, \dots, \lambda_n$.

We have n invariants, and the elementary symmetric function σ_r of the anisotropic principal curvatures:

$$\sigma_r = \sum_{i_1 < \cdots i_r} \lambda_{i_1} \cdots \lambda_{i_n} \quad (1 \le r \le n). \tag{9}$$

For convenience, we set $\sigma_0 = 1$. The r-anisotropic mean curvature M_r is defined by

$$M_r = \sigma_r / C_n^r, \quad C_n^r = \frac{n!}{r!(n-r)!}. \tag{10}$$

Using the characteristic polynomial of S_F, σ_r is defined by

$$\det(tI - S_F) = \sum_{r=0}^{n} (-1)^r \sigma_r t^{n-r}. \tag{11}$$

So, we have

$$\sigma_r = \frac{1}{r!} \sum_{i_1,\dots,i_r; j_1,\dots,j_r} \delta_{i_1 \cdots i_r}^{j_1 \cdots j_r} s_{i_1 j_1} \cdots s_{i_r j_r}, \tag{12}$$

where $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ is the usual generalized Kronecker symbol, i.e., $\delta^{j_1\cdots j_r}_{i_1\cdots i_r}$ equals $+1$ (resp. -1) if $i_1\cdots i_r$ are distinct and $(j_1\cdots j_r)$ is an even (resp. odd) permutation of $(i_1\cdots i_r)$ and in other cases it equals zero.

We define $(F\circ\nu)_i, (F_i\circ\nu)_j, (A_{ij}\circ\nu)_k$ by

$$\mathrm{d}(F\circ\nu)=\sum_i (F\circ\nu)_i\omega_i, \tag{13}$$

$$\mathrm{d}(F_i\circ\nu)+\sum_j (F_j\circ\nu)\omega_{ji}=\sum_j (F_i\circ\nu)_j\omega_j, \tag{14}$$

$$\mathrm{d}(A_{ij}\circ\nu)+\sum (A_{kj}\circ\nu)\omega_{ki}+\sum_k (A_{ik}\circ\nu)\omega_{kj}=\sum_k (A_{ij}\circ\nu)_k\omega_k. \tag{15}$$

By the use of (3), (5) and (6), we have by a direct calculation

$$\begin{cases} (F\circ\nu)_i=-\sum\limits_j h_{ij}F_j\circ\nu, \\ (F_i\circ\nu)_j=-\sum\limits_k h_{jk}F_{ik}\circ\nu, \\ (A_{ij}\circ\nu)_k=-\sum\limits_l h_{kl}A_{ijl}\circ\nu. \end{cases} \tag{16}$$

3 Some Lemmas

We introduce an important operator P_r (also see Reilly [6]) by

$$P_r=\sigma_r I-\sigma_{r-1}S_F+\cdots+(-1)^r S_F^r, \quad r=0,1,\ldots,n. \tag{17}$$

We have the following lemmas:

Lemma 3.1 $(S_FA_F)^t=S_FA_F$, $(\mathrm{d}\nu\circ S_F)^t=\mathrm{d}\nu\circ S_F$, $s_{ijk}=s_{ikj}$, $\sum_l h_{il}s_{lk}=\sum_l h_{kl}s_{li}$, $\sum_l h_{kl}(P_r)_{lj}=\sum_l h_{jl}(P_r)_{lk}$, *where s_{ijk} are the components of the covariant derivarive of s_{ij}.*

Proof Since $S_F=-A_F\circ\mathrm{d}\nu$, and A_F, $\mathrm{d}\nu$ are symmetric operators, the first two identities are obvious. From the symmetry property (8) of A_{ijk}, $h_{ij}=h_{ji}$ and Codazzi equation $h_{ijk}=h_{ikj}$, we have, by the use of (16),

$$\begin{aligned} s_{ijk}&=\left(\sum_l A_{il}h_{lj}\right)_k=\sum_l (A_{il}\circ\nu)_k h_{lj}+\sum_l A_{il}h_{ljk} \\ &=-\sum_{l,m}(A_{ilm}\circ\nu)h_{lj}h_{km}+\sum_l A_{il}h_{ljk} \\ &=\sum_m (A_{im}\circ\nu)_j h_{mk}+\sum_l A_{il}h_{lkj}=\left(\sum_l A_{il}h_{lk}\right)_j=s_{ikj}. \end{aligned} \tag{18}$$

$$\sum_l h_{il}s_{lk}=\sum_{l,m}h_{il}A_{lm}h_{mk}=\sum_{l,m}h_{km}A_{ml}h_{li}=\sum_l h_{kl}s_{li}.$$

By the use of the above formula and the definition of P_r, we get the last identity in Lemma 3.1.

Lemma 3.2 *The matrix of P_r is given by*:

$$(P_r)_{ij}=\frac{1}{r!}\sum_{i_1,\ldots,i_r;j_1,\ldots,j_r}\delta^{j_1\cdots j_r i}_{i_1\cdots i_r j}s_{i_1j_1}\cdots s_{i_rj_r}. \tag{19}$$

Proof We prove Lemma 3.2 inductively. For $r=0$, it is easy to check that (19) is true.

We can check directly

$$\delta^{j_1\cdots j_q}_{i_1\cdots i_q}=\begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{q-1}}_{i_1} & \delta^{j_q}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{q-1}}_{i_2} & \delta^{j_q}_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{j_1}_{i_{q-1}} & \delta^{j_2}_{i_{q-1}} & \cdots & \delta^{j_{q-1}}_{i_{q-1}} & \delta^{j_q}_{i_{q-1}} \\ \delta^{j_1}_{i_q} & \delta^{j_2}_{i_q} & \cdots & \delta^{j_{q-1}}_{i_q} & \delta^{j_q}_{i_q} \end{vmatrix}. \tag{20}$$

Assume that (19) is true for $r = k$, we only need to show that it is also true for $r = k+1$. For $r = k+1$, using (12) and (20), we have

$$\begin{aligned}
RHS \text{ of } (19) &= \frac{1}{(k+1)!} \sum_{i_1,\dots,i_{k+1};j_1,\dots,j_{k+1}} \delta^{j_1\cdots j_{k+1}i}_{i_1\cdots i_{k+1}j} s_{i_1j_1}\cdots s_{i_{k+1}j_{k+1}} \\
&= \frac{1}{(k+1)!} \sum \begin{vmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_{k+1}}_{i_1} & \delta^{i}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_{k+1}}_{i_2} & \delta^{i}_{i_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^{j_1}_{i_{k+1}} & \delta^{j_2}_{i_{k+1}} & \cdots & \delta^{j_{k+1}}_{i_{k+1}} & \delta^{i}_{i_{k+1}} \\ \delta^{j_1}_{j} & \delta^{j_2}_{j} & \cdots & \delta^{j_{k+1}}_{j} & \delta^{i}_{j} \end{vmatrix} s_{i_1j_1}\cdots s_{i_{k+1}j_{k+1}} \\
&= \frac{1}{(k+1)!} \sum (\delta^i_j \delta^{j_1\cdots j_{k+1}}_{i_1\cdots i_{k+1}} - \delta^{j_{k+1}}_j \delta^{j_1\cdots j_k i}_{i_1\cdots i_k i_{k+1}} + \cdots) s_{i_1j_1}\cdots s_{i_{k+1}j_{k+1}} \\
&= \sigma_{k+1}\delta_{ij} - \frac{1}{(k+1)!} \sum \delta^{j_{k+1}}_j \delta^{j_1\cdots j_k i}_{i_1\cdots i_k i_{k+1}} s_{i_1j_1}\cdots s_{i_{k+1}j_{k+1}} + \cdots \\
&= \sigma_{k+1}\delta_{ij} - \textstyle\sum (P_k)_{i i_{k+1}} s_{i_{k+1}j} \\
&= (P_{k+1})_{ij}.
\end{aligned}$$

Lemma 3.3 *For each r, we have*

(i) $\sum_j (P_r)_{jij} = 0$,

(ii) $\mathrm{tr}(P_r S_F) = (r+1)\sigma_{r+1}$,

(iii) $\mathrm{tr}(P_r) = (n-r)\sigma_r$.

Proof (i) Noting (j, j_r) is skew-symmetric in $\delta^{j_1\cdots j_r j}_{i_1\cdots i_r i}$ and (j, j_r) is symmetric in $s_{i_1j_1}\cdots s_{i_rj_rj}$ (from Lemma 3.1), we have

$$\sum_j (P_r)_{jij} = \frac{1}{(r-1)!} \sum_{i_1,\cdots,i_r;j_1,\cdots,j_r;j} \delta^{j_1\cdots j_r j}_{i_1\cdots i_r i} s_{i_1j_1}\cdots s_{i_rj_rj} = 0.$$

(ii) Using (19) and (12), we have

$$\begin{aligned}
\mathrm{tr}(P_r S_F) &= \sum_{ij} (P_r)_{ij} s_{ji} \\
&= \frac{1}{r!} \sum_{i_1,\dots,i_r;j_1,\dots,j_r;i,j} \delta^{j_1\cdots j_r i}_{i_1\cdots i_r j} s_{i_1j_1}\cdots s_{i_rj_r} s_{ji} \\
&= (r+1)\sigma_{r+1}.
\end{aligned}$$

(iii) Using (ii) and the definition of P_r, we have

$$\mathrm{tr}(P_r) = \mathrm{tr}(\sigma_r I) - \mathrm{tr}(P_{r-1}S_F) = n\sigma_r - r\sigma_r = (n-r)\sigma_r.$$

Remark 3.1 When $F = 1$, Lemma 3.3 is a well-known result (for example, see Barbosa–Colares [14]).

Lemma 3.4 *If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \mathrm{const} \neq 0$, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Proof Choose a local orthogonal frame $e_1, e_2, \dots, e_n$ such that A_F is diagonalized:

$$A_F = \mathrm{diag}(\mu_1, \dots, \mu_n), \tag{21}$$

where $\mu_i > 0$ for $i = 1, \dots, n$ by the convexity condition. Then we have $S_{ij} = \mu_i h_{ij}$. From (10) and (12), we get

$$0 = M_1^2 - M_2 = \left(\frac{1}{n}\sum_i \mu_i h_{ii}\right)^2 - \frac{2}{n(n-1)} \sum_{i<j} \mu_i\mu_j (h_{ii}h_{jj} - h_{ij}^2)$$

$$= \frac{1}{n^2(n-1)}\Big\{(n-1)\Big(\sum_i \mu_i h_{ii}\Big)^2 - 2n\sum_{i<j}\mu_i\mu_j(h_{ii}h_{jj}-h_{ij}^2)\Big\}$$
$$= \frac{1}{n^2(n-1)}\sum_{i<j}\{(\mu_i h_{ii}-\mu_j h_{jj})^2 + 2n\mu_i\mu_j h_{ij}^2\},$$

so, $\mu_1 h_{11} = \mu_2 h_{22} = \cdots = \mu_n h_{nn}$ and $h_{ij} = 0$ when $i \neq j$. Then, from [1] or [3], [15], up to translations and homotheties, $X(M)$ is the Wulff shape.

4 Proofs of Theorem 1.1–Theorem 1.5

Proof of Theorem 1.1 By the use of (5), we have

$$\langle X,\nu\rangle_i = -\sum_j h_{ij}\langle X,e_j\rangle, \quad \langle X,e_j\rangle_i = \delta_{ij} + h_{ij}\langle X,\nu\rangle, \tag{22}$$

so, from (16), Lemma 3.1 and (i), (ii), (iii) of Lemma 3.3, we have the following calculation

$$\begin{aligned}
&\mathrm{div}\{P_r(\langle X,\nu\rangle \mathrm{grad}_{S^n}\mathrm{F} - \mathrm{F}\mathrm{grad}|\mathrm{X}|^2/2)\}\\
&\quad= \sum_{ij}\{(P_r)_{ij}(\langle X,\nu\rangle F_j - F\langle X,e_j\rangle)\}_i\\
&\quad= \sum_{ij}(P_r)_{ij}\Big\{-\sum_k h_{ik}(\langle X,e_k\rangle F_j + \langle X,\nu\rangle F_{jk} - F_k\langle X,e_j\rangle) - F\delta_{ij} - Fh_{ij}\langle X,\nu\rangle\Big\}\\
&\quad= -\sum_{ijk} h_{ki}(P_r)_{ij}\langle X,e_k\rangle F_j + \sum_{ijk} h_{ki}(P_r)_{ij}\langle X,e_j\rangle F_k\\
&\qquad - \langle X,\nu\rangle\sum_{ijk}(P_r)_{ij}(F_{jk}+F\delta_{jk})h_{ki} - F\sum_i (P_r)_{ii}\\
&\quad= -\sum_{ijk} h_{ki}(P_r)_{ij}\langle X,e_k\rangle F_j + \sum_{ijk} h_{ji}(P_r)_{ik}\langle X,e_k\rangle F_j\\
&\qquad - \langle X,\nu\rangle\sum_{ijk}(P_r)_{ij}A_{jk}h_{ki} - F\sum_i (P_r)_{ii}\\
&\quad= -\langle X,\nu\rangle\sum_{ij}(P_r)_{ij}s_{ji} - F\sum_i (P_r)_{ii}\\
&\quad= -\langle X,\nu\rangle \mathrm{tr}(P_r S_F) - F\mathrm{tr}(P_r)\\
&\quad= -\langle X,\nu\rangle (r+1)\sigma_{r+1} - F(n-r)\sigma_r\\
&\quad= -(n-r)C_n^r(FM_r + M_{r+1}\langle X,\nu\rangle).
\end{aligned}$$

Integrating the above formula over M, we get (2) by the use of Stokes Theorem.

Proof of Theorem 1.2 From (2), we have

$$\int_M (F + M_1\langle X,\nu\rangle)\mathrm{d}A_X = 0, \tag{23}$$
$$\int_M (FM_1 + M_2\langle X,\nu\rangle)\mathrm{d}A_X = 0. \tag{24}$$

By the assumption $M_1 = \mathrm{const}$, we get from (23) and (24)

$$\int_M \langle X,\nu\rangle(M_1^2 - M_2)\mathrm{d}A_X = 0. \tag{25}$$

On the other hand,

$$M_1^2 - M_2 = \frac{1}{n^2(n-1)}\sum_{j<i}(\lambda_i - \lambda_j)^2 \geq 0. \tag{26}$$

Thus, if $\langle X, \nu \rangle$ has fixed sign, then $M_1^2 - M_2 = 0$, so

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n.$$

Thus, from Lemma 3.4, up to translations and homotheties, $X(M)$ is the Wulff shape.

Proof of Theorem 1.3 We have the fact that if M is compact and $M_r > 0$ then

$$M_{r-1} \geq M_r^{(r-1)/r}, \quad 2 \leq r \leq n \tag{27}$$

with equality holding if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ on M (cf. [10], [16]). Indeed (27) holds if $M_r \equiv \text{const}$, since M is compact, there exists a point p_0 on M such that all principal curvatures are positive at p_0, so all anisotropic principal curvatures are positive at p_0. Applying (27) inductively, one sees that if $M_r \equiv \text{const}$, then

$$M_r \leq M_1^r, \tag{28}$$

here again equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$.

Integrating $FM_r^{(r-1)/r} \leq FM_{r-1}$ over M, using (2) and $M_r = \text{const}$, we get

$$M_r^{(r-1)/r} \int_M F \mathrm{d}A_X \leq \int_M FM_{r-1} \mathrm{d}A_X = -M_r \int_M \langle X, \nu \rangle \mathrm{d}A_X. \tag{29}$$

On the other hand, our assumption $M_1 = \text{const}$ (thus $M_1 > 0$) and (23) implies

$$\int_M \langle X, \nu \rangle \mathrm{d}A_X = -\frac{1}{M_1} \int_M F dA_X. \tag{30}$$

Putting (30) into (29), we get

$$M_1^r \leq M_r. \tag{31}$$

Therefore equality holds in (28) and $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ on M. Thus, from Lemma 3.4, up to translations and homotheties, $X(M)$ is the Wulff shape.

Proof of Theorem 1.4 From (2), we have

$$\int_M (FM_k + M_{k+1} \langle X, \nu \rangle) \mathrm{d}A_X = 0, \tag{32}$$

$$\int_M (FM_r + M_{r+1} \langle X, \nu \rangle) \mathrm{d}A_X = 0. \tag{33}$$

From the assumptions $\frac{M_r}{M_k} = \text{const}$, $\frac{M_r}{M_k} \times (32) - (33)$ implies

$$\int_M \langle X, \nu \rangle (M_{r+1} - \frac{M_r}{M_k} M_{k+1}) \mathrm{d}A_X = 0. \tag{34}$$

From the convexity of M, all the principal curvatures of M are positive, so all the anisotropic principal curvature are positive, we have $M_l > 0$, $0 \leq l \leq n$ on M. From (see [17])

$$M_k \cdot M_{k+2} \leq M_{k+1}^2, \quad \ldots, \quad M_{r-1} M_{r+1} \leq M_r^2, \tag{35}$$

where equality holds in one of (35) if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, we can check

$$M_k M_{r+1} \leq M_{k+1} M_r,$$

that is,

$$M_{r+1} - \frac{M_r}{M_k} M_{k+1} \leq 0. \tag{36}$$

On the other hand, we can choose the position of origin O such that $\langle X, \nu \rangle$ has fixed sign. Thus, from (34) and (36), $M_k M_{r+1} = M_{k+1} M_r$, so $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Thus, from Lemma 3.4, up to translations and homotheties, $X(M)$ is the Wulff shape.

Proof of Theorem 1.5 From the proof of Theorem 1.3, we know that there exists a point $p_0 \in M$ such that the anisotropic principal curvature $\lambda_i(p_0) > 0$, $1 \leq i \leq n$. From $\frac{M_k}{M_n} =$ constant, we have $\frac{M_k}{M_n} = \frac{M_k}{M_n}(p_0) > 0$. Thus $M_n \neq 0$ on M, by the continuity of λ_i, we have $\lambda_i > 0$, $1 \leq i \leq n$, on M. Therefore, all principal curvatures of M are positive on M, and M is convex. Theorem 1.5 follows from Theorem 1.4.

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