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The Dichotomy Between Traces on d-sets Γ in $\mathbb{R}^n$ and the Density of $D(\mathbb{R}^n\backslash\Gamma)$ in Function Spaces

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Abstract A space $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ and $s \in \mathbb{R}$, $0 < p, q < \infty$ either has a trace in $L_p(\Gamma)$, where Γ is a compact d-set in $\mathbb{R}^n$ with $0 < d < n$, or $D(\mathbb{R}^n\backslash\Gamma)$ is dense in it. Related dichotomy numbers are introduced and calculated.

Keywords function spaces, d-sets, fractals, traces

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1 Introduction and Main Results

The nowadays well-known two scales of space

$$A^s_{pq}(\mathbb{R}^n), \quad \text{where} \quad s \in \mathbb{R}, \quad 0 < p < \infty, \quad 0 < q < \infty, \tag{1.1}$$

with $A = B$ and $A = F$ cover as special cases (classical and fractional) Sobolev spaces, classical Besov spaces and (inhomogeneous) Hardy spaces. Recall that $D(\mathbb{R}^n)$ is dense in all these spaces (excluding $p = \infty$ and/or $q = \infty$). For $0 < d < n$, we furnish compact d-sets Γ in $\mathbb{R}^n$ with the restriction of the Hausdorff measure $\mathscr{H}^d$ in $\mathbb{R}^n$ to Γ, denoted by $\mathscr{H}^d_\Gamma$. One may ask the following two mutually exclusive questions:

1 In which of the above spaces $A^s_{pq}(\mathbb{R}^n)$ is $D(\mathbb{R}^n\backslash\Gamma)$ dense?

2 For which of the above spaces $A^s_{pq}(\mathbb{R}^n)$ does there exist a linear and bounded trace operator

$$\mathrm{tr}_\mu : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu),$$

where $\mu = \mathscr{H}^d_\Gamma$?

It comes out that the above spaces divide sharply in these two contrasting classes (dichotomy). The well-known inclusion properties of these spaces suggest the following formulation. Let $D_\Gamma = D(\mathbb{R}^n\backslash\Gamma)$ be as usual the collection of all (complex-valued) C^∞ functions in $\mathbb{R}^n$ with compact support in in $\mathbb{R}^n\backslash\Gamma$.

Definition 1 *Let $0 < p < \infty$ and let*

$$A_p(\mathbb{R}^n) = \left\{A^s_{pq}(\mathbb{R}^n) : \ 0 < q < \infty, \ s \in \mathbb{R}\right\}, \tag{1.2}$$

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where $A^s_{pq}(\mathbb{R}^n)$ *are the above spaces with* $A = B$ *or* $A = F$. *Let* $0 < d < n$ *and let* $\Gamma = \operatorname{supp}\mu$ *be a compact* d*-set in* $\mathbb{R}^n$ *with* $\mu = \mathscr{H}^d_\Gamma$. *Let* $\sigma \in \mathbb{R}$. *Then*

$$\mathbb{D}(A_p(\mathbb{R}^n), L_p(\Gamma,\mu)) = (\sigma, u) \quad \textit{with} \quad 0 < u < \infty$$

is called the dichotomy of $\{A_p(\mathbb{R}^n), L_p(\Gamma,\mu)\}$ *if*

$$\operatorname{tr}_\mu \ \textit{exists for} \quad \begin{cases} s > \sigma, & 0 < q < \infty, \\ s = \sigma, & 0 < q \le u, \end{cases} \tag{1.3}$$

and

$$D_\Gamma \ \textit{is dense in} \ A^s_{pq}(\mathbb{R}^n) \ \textit{for} \quad \begin{cases} s = \sigma, & u < q < \infty, \\ s < \sigma, & 0 < q < \infty. \end{cases} \tag{1.4}$$

Furthermore,

$$\mathbb{D}(A_p(\mathbb{R}^n), L_p(\Gamma,\mu)) = (\sigma, 0)$$

means that

$$\begin{cases} \operatorname{tr}_\mu \ \textit{exists for} \ s > \sigma, \ 0 < q < \infty, \\ D_\Gamma \ \textit{is dense in} \ A^s_{pq}(\mathbb{R}^n) \ \textit{for} \ s \le \sigma, \ 0 < q < \infty; \end{cases}$$

and

$$\mathbb{D}(A_p(\mathbb{R}^n), L_p(\Gamma,\mu)) = (\sigma, \infty)$$

means that

$$\begin{cases} \operatorname{tr}_\mu \ \textit{exists for} \ s \ge \sigma, \ 0 < q < \infty, \\ D_\Gamma \ \textit{is dense in} \ A^s_{pq}(\mathbb{R}^n) \ \textit{for} \ s < \sigma, \ 0 < q < \infty. \end{cases}$$

Remark 2 It is the main aim of this paper to prove the existence of such sharp breaking points (σ, u) and to calculate them with the following outcome.

Theorem 3 *Let* $0 < d < n$ *and* $0 < p < \infty$. *Let* Γ *be a compact* d*-set in* $\mathbb{R}^n$ *and* $\mu = \mathscr{H}^d_\Gamma$. *Then*

$$\mathbb{D}\Big(B_p(\mathbb{R}^n), L_p(\Gamma,\mu)\Big) = \begin{cases} \left(\dfrac{n-d}{p}, 1\right), & \textit{if } p > 1, \\ \left(\dfrac{n-d}{p}, p\right), & \textit{if } p \le 1, \end{cases} \tag{1.5}$$

and

$$\mathbb{D}\Big(F_p(\mathbb{R}^n), L_p(\Gamma,\mu)\Big) = \begin{cases} \left(\dfrac{n-d}{p}, 0\right), & \textit{if } p > 1, \\ \left(\dfrac{n-d}{p}, \infty\right), & \textit{if } p \le 1. \end{cases} \tag{1.6}$$

Remark 4 It is convenient but not really necessary to assume that the above d-set is compact. But instead of discussing what happens at infinity we complement the above assertions by the most distinguished closed but not compact d-set,

$$\Gamma = \mathbb{R}^d, \quad n > d \in \mathbb{N},$$

interpreted as a d-dimensional hyper-plane in $\mathbb{R}^n$. Then $\mu = \mathscr{H}^d_\Gamma$ is the d-dimensional Lebesgue measure and we write $L_p(\mathbb{R}^d)$ instead of $L_p(\Gamma,\mu)$. Although the outcome is rather obvious it is worth being formulated, also for historical reasons.

Corollary 5 *Let $n > d \in \mathbb{N}$. Then*

$$\mathbb{D}(B_p(\mathbb{R}^n), L_p(\mathbb{R}^d)) = \begin{cases} \left(\frac{n-d}{p}, 1\right), & \text{if } p > 1, \\ \left(\frac{n-d}{p}, p\right), & \text{if } p \le 1, \end{cases}$$

and

$$\mathbb{D}(F_p(\mathbb{R}^n), L_p(\mathbb{R}^d)) = \begin{cases} \left(\frac{n-d}{p}, 0\right), & \text{if } p > 1, \\ \left(\frac{n-d}{p}, \infty\right), & \text{if } p \le 1. \end{cases}$$

Remark 6 Density and trace assertions for the *classical Besov spaces*

$$B^s_{pq}(\mathbb{R}^n), \qquad s > 0, \quad 1 < p < \infty, \quad 1 \le q < \infty, \tag{1.7}$$

and the (fractional) *Sobolev spaces* (or Bessel-potential spaces)

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad s > 0, \tag{1.8}$$

as covered by Corollary 5 have been known since the late 1960s and the early 1970s. Corresponding formulations and detailed references may be found in [1, Section 2.9.4, pp. 223–226] with [2] as our own contribution at this time. In Section 2.3 we collect further known results related to Theorem 3 and Corollary 5 on which we rely afterwards.

In connection with Theorem 3 one can ask several questions. Any $0 < s < \frac{n}{p}$ may serve as a breaking point $s = \frac{n-d}{p}$. What about the spaces with $s \le 0$? The following observation sheds some light on the corresponding situation.

Proposition 7 *Let Γ be a compact set in $\mathbb{R}^n$ with Lebesgue measure $|\Gamma| = 0$. Then D_Γ is dense in all spaces*

$$A^s_{pq}(\mathbb{R}^n), \qquad s < 0, \quad 0 < p < \infty, \quad 0 < q < \infty. \tag{1.9}$$

Secondly we put forward the somewhat mysterious claim that traces in $L_p(\Gamma, \mathscr{H}^d_\Gamma)$ or $L_p(\mathbb{R}^d)$ make sense even for the spaces

$$A^s_{pq}(\mathbb{R}^n), \qquad 0 < p < 1, \quad 0 < q < \infty, \quad \frac{n-d}{p} < s < n\left(\frac{1}{p} - 1\right), \tag{1.10}$$

containing singular distributions as elements; for example the δ-distribution belongs to all these spaces. Then $\mathrm{tr}_\mu \delta \in L_p(\Gamma, \mathscr{H}^d_\Gamma)$ makes sense and one gets

$$\mathrm{tr}_\mu \delta = 0 \quad \text{in} \quad L_p(\Gamma, \mathscr{H}^d_\Gamma). \tag{1.11}$$

One can replace $L_p(\Gamma, \mathscr{H}^d_\Gamma)$ by $L_p(\mathbb{R}^d)$ where even $d = n$ may be admitted. All these require some explanations which will be given below.

The plan of the paper is the following. In Section 2, we introduce the spaces according to (1.1) and collect a few useful properties. Furthermore we discuss in detail what is meant by traces of spaces on sets. Finally we deal with some curiosities connected with (1.10), (1.11). In Section 3, we prove Theorem 3, Corollary 5, Proposition 7 and some other complementary assertions formulated in Section 2. The proofs are mainly based on atomic decompositions of the spaces in (1.1). Furthermore we need the remarkable interplay between the porosity of sets in $\mathbb{R}^n$ and the spaces $F^s_{pq}(\mathbb{R}^n)$. For the convenience of the reader we list these ingredients in an Appendix (Section A).

2 Preliminaries and Further Assertions

2.1 Definitions and Density Properties

We assume that the reader is familiar with the theory of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ as it may be found in [3–5]. We restrict ourselves to the bare minimum. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. Let $D(\mathbb{R}^n)$ be the collection of all complex-valued C^∞ functions in $\mathbb{R}^n$ with compact support. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p < \infty$ is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

The Fourier transform $\widehat{\varphi} = F\varphi$ and its inverse $\varphi^\vee = F^{-1}\varphi$ have the usual canonical meaning on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$. Let $\varphi_0 \in D(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{2.1}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Since $1 = \sum_{j=0}^\infty \varphi_j(x)$ for $x \in \mathbb{R}^n$ the φ_j form a dyadic resolution of unity. Recall that $(\varphi_j \widehat{f})^\vee(x)$ makes sense for any $f \in S'(\mathbb{R}^n)$ since $(\varphi_j \widehat{f})^\vee$ is an entire analytic function in $\mathbb{R}^n$.

Definition 8 *Let $0 < p < \infty$, $0 < q < \infty$, $s \in \mathbb{R}$, and let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above resolution of unity. Let*

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{2.2}$$

and

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^\infty 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\|. \tag{2.3}$$

Then

$$A^s_{pq}(\mathbb{R}^n) = \left\{f \in S'(\mathbb{R}^n) : \ \|f\,|A^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty\right\}, \tag{2.4}$$

where either $A = B$ or $A = F$.

Remark 9 These quasi-Banach spaces (Banach spaces if $p \ge 1$, $q \ge 1$) are independent of φ (equivalent quasi-norms) which will be omitted on the left-hand sides of (2.2), (2.3) and in (2.4) in what follows. The possible extensions of these definitions to $p = \infty$ and/or $q = \infty$ will not be considered in connection with dichotomy. Special cases have been mentioned in (1.7) and (1.8) with the *classical Sobolev spaces*

$$W^m_p(\mathbb{R}^n) = H^m_p(\mathbb{R}^n), \quad m \in \mathbb{N}, \quad 1 < p < \infty$$

as sub-cases. The following assertion is well known.

Proposition 10 *Both $D(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are dense in $A^s_{pq}(\mathbb{R}^n)$ with $0 < p < \infty$, $0 < q < \infty$, $s \in \mathbb{R}$.*

Remark 11 For our later considerations about traces it is useful to outline an explicit approximation procedure for $f \in A^s_{pq}(\mathbb{R}^n)$. With φ_0 as in (2.1) one gets

$$f_j = (\varphi_0(2^{-j}\cdot)\widehat{f})^\vee \to f \quad \text{in } A^s_{pq}(\mathbb{R}^n) \text{ if } j \to \infty$$

as a consequence of Fourier multiplier assertions and (in case of the F-spaces) Lebesgue's dominated convergence theorem, [6, p. 37]. Recall that f_j is an entire analytic function. Let $\psi \in S(\mathbb{R}^n)$ with $\psi(0) = 1$ such that $\widehat{\psi}$ has a compact support. Then

$$f^l_j = \psi(2^{-l}\cdot) f_j \in S(\mathbb{R}^n) \quad \text{and} \quad \operatorname{supp} \widehat{f^l_j} \subset \{y \in \mathbb{R}^n : \ |y| \le c\, 2^j\}$$

for some $c > 0$ (all $j \in \mathbb{N}$) and all $l \in \mathbb{N}$ where $f^l_j \in S(\mathbb{R}^n)$ follows from the Paley–Wiener–Schwartz theorem. One gets

$$f^l_j \to f_j \quad \text{in} \quad A^s_{pq}(\mathbb{R}^n) \quad \text{if} \quad l \to \infty$$

as a consequence of

$$\| f^l_j - f_j \,|A^s_{pq}(\mathbb{R}^n)\| \le c_j \, \|(1 - \psi(2^{-l}\cdot)\, f_j \,|L_p(\mathbb{R}^n)\| \to 0$$

using again Lebesgue's dominated convergence theorem. Hence choosing $l = l(j)$ suitably one obtains that

$$f^j = f^{l(j)}_j \in S(\mathbb{R}^n) \quad \text{and} \quad f^j \to f \quad \text{in} \quad A^s_{pq}(\mathbb{R}^n). \tag{2.5}$$

As for the density of $D(\mathbb{R}^n)$ one has to multiply afterwards with an appropriate sequence of C^∞ cut-off functions. As said the explicit construction of f^j in (2.5) will be of some use for us later on. This is the only reason for inserting the above arguments for this otherwise well-known basic observation. As for further details one may consult [3, Theorem pp. 48/49, Theorem, p. 22] and the related proofs.

2.2 Measures and Traces

We assume that the reader is familiar with basic measure and integration theory, especially in connection with fractal geometry and fractal analysis. Short descriptions may be found in [7, pp. 7–13], [8, pp. 1–6] and [9, pp. 1–2]. Let μ be a non-trivial finite compactly supported *Radon measure* in $\mathbb{R}^n$. Hence

$$0 < \mu(\mathbb{R}^n) < \infty, \qquad \Gamma = \operatorname{supp} \mu \quad \text{compact}, \tag{2.6}$$

where supp μ must be understood as the uniquely determined identification of μ as an element of $S'(\mathbb{R}^n)$. One may consult [5, Section 1.12.2, pp. 80/81] about this point. Otherwise $L_r(\Gamma, \mu)$ with $0 < r < \infty$ is the usual complex-valued quasi-Banach space, quasi-normed by

$$\|g \,|L_r(\Gamma,\mu)\| = \left(\int_{\mathbb{R}^n} |g(\gamma)|^r \, \mu(\mathrm{d}\gamma) \right)^{1/r} = \left(\int_\Gamma |g(\gamma)|^r \, \mu(\mathrm{d}\gamma) \right)^{1/r}. \tag{2.7}$$

Recall that a compact set Γ in $\mathbb{R}^n$ is called a d-set, where $0 < d < n$, if one finds a Radon measure μ according to (2.6) and two numbers $c_1 > 0$, $c_2 > 0$ such that

$$c_1\, \varrho^d \le (B(\gamma,\varrho)) \le c_2\, \varrho^d, \qquad \gamma \in \Gamma, \quad 0 < \varrho < 1, \tag{2.8}$$

where $B(\gamma,\varrho)$ is the ball in $\mathbb{R}^n$ centred at γ and of radius ϱ. Any two Radon measures with (2.6), (2.8) are equivalent to each other and, hence, one may choose $\mu = \mathscr{H}^d_\Gamma = \mathscr{H}^d|\Gamma$, the restriciton of the Hausdorff measure $\mathscr{H}^d$ in $\mathbb{R}^n$ to Γ. Details may be found in [9, p. 5]. Otherwise d-sets (especially self-similar d-sets) are a favourable subject of fractal geometry and fractal analysis, [8, 7, 10]. Their relations to function spaces have been studied in detail in [9, 11, 5]. This will be continued later on in this paper. But first we adopt a more general point of view.

Definition 12 *Let μ be a Radon measure in $\mathbb{R}^n$ with* (2.6) *and let $0 < r < \infty$. Let $A^s_{pq}(\mathbb{R}^n)$ be a space according to Definition* 8 *and let for some $c > 0$,*

$$\|\varphi\,|L_r(\Gamma,\mu)\| \le c\,\|\varphi\,|A^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all } \varphi \in S(\mathbb{R}^n). \tag{2.9}$$

Then the trace operator tr_μ,

$$\mathrm{tr}_\mu : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_r(\Gamma,\mu), \tag{2.10}$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu\varphi)(\gamma) = \varphi(\gamma)$ *with* $\varphi \in S(\mathbb{R}^n)$ *and* $\gamma \in \Gamma$.

Remark 13 It follows from Proposition 10 and standard arguments that this definition makes sense and that the outcome

$$\mathrm{tr}_\mu f \in L_r(\Gamma,\mu), \qquad f \in A^s_{pq}(\mathbb{R}^n),$$

is independent of the chosen approximating sequence

$$\varphi_j \to f \quad \text{in} \quad A^s_{pq}(\mathbb{R}^n), \qquad \varphi_j \in S(\mathbb{R}^n),$$

on the source side. On the target side one has

$$\mathrm{tr}_\mu\varphi_j \to \mathrm{tr}_\mu f \quad \text{in} \quad L_r(\Gamma,\mu). \tag{2.11}$$

By standard arguments of measure theory, [6, Theorem 5.2.7, p. 23 and Lemma 9.3.1, p. 51], it follows that there is a subsequence of $\{\varphi_j\}$ which converges μ-a.e. to $\mathrm{tr}_\mu f$. Identifying this subsequence with $\{\varphi_j\}$ we may assume in addition to (2.11) that

$$(\mathrm{tr}_\mu\varphi_j)(\gamma) \to (\mathrm{tr}_\mu f)(\gamma) \quad \mu\text{-a.e.} \quad \text{on} \quad \Gamma.$$

One gets that $\mathrm{tr}_\mu f$ is μ-a.e. independent of r as long as one has (2.9). This applies in particular to $0 < \widetilde{r} \le r$ with (2.9) as a consequence of $\mu(\Gamma) < \infty$ and Hölder's inequality. There is a similar situation on the source side assuming that one has (2.9) for $A^{s_1}_{p_1q_1}(\mathbb{R}^n)$ and $A^{s_2}_{p_2q_2}(\mathbb{R}^n)$ on the right-hand side and that

$$f \in A^{s_1}_{p_1q_1}(\mathbb{R}^n) \cap A^{s_2}_{p_2q_2}(\mathbb{R}^n). \tag{2.12}$$

Then one can rely on the distinguished approximating sequences according to (2.5) where one may assume that they apply to both spaces in (2.12). Hence one has a common approximating sequence and $\mathrm{tr}_\mu f$ is μ-a.e. the same in both cases. If (2.9) holds both for $L_{r_1}(\Gamma_1,\mu_1)$ and $L_{r_2}(\Gamma_2,\mu_2)$ then it is also valid for $L_r(\Gamma,\mu)$ with $\mu = \mu_1 + \mu_2$ and $r = \min(r_1,r_2)$. By the above considerations tr_{μ_1} and tr_{μ_2} are restrictions of tr_μ. In other words for *individual elements f the traces are independent of the source spaces and of the target spaces as long as one has* (2.9) *and whenever comparison makes sense.*

Remark 14 If $f \in A^s_{pq}(\mathbb{R}^n)$ is a regular distribution, hence locally Lebesgue-integrable in $\mathbb{R}^n$, then there is an alternative method to define traces on, say, compact sets Γ in $\mathbb{R}^n$ based on the Lebesgue points of f in $\mathbb{R}^n$ related to the $A^s_{pq}(\mathbb{R}^n)$-capacity of Γ. This approach has some history. The most elaborated treatments may be found in [12–13]. In [11, pp. 260/261], we compared these two methods with the expected outcome that they coincide at least for d-sets and some spaces $A^s_{pq}(\mathbb{R}^n)$. However the necessary restriction

$$A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n), \quad \text{hence} \quad s \ge n\left(\frac{1}{\min(1,p)} - 1\right)$$

does not fit in our concept.

2.3 The Dichotomy Problem and Traces on d-sets

As above $D_\Gamma = D(\mathbb{R}^n \backslash \Gamma)$ is the collection of all complex-valued C^∞ functions in $\mathbb{R}^n$ with compact support in $\mathbb{R}^n\backslash\Gamma$ where Γ is a compact set (or a hyper-plane) in $\mathbb{R}^n$. We always give preference of (Radon) measures μ to sets taking (2.6) as the starting point. Let $A_p(\mathbb{R}^n)$ be as in (1.2) and let $L_r(\Gamma,\mu)$ be the spaces introduced in Section 2.2 based on (2.6), (2.7). In generalisation of Definition 1 the *dichotomy problem* related to $A_p(\mathbb{R}^n)$ and $L_r(\Gamma,\mu)$ is the search for a couple

$$\mathbb{D}(A_p(\mathbb{R}^n), L_r(\Gamma,\mu)) = (\sigma, u), \qquad \sigma \in \mathbb{R}, \quad 0 \le u \le \infty, \tag{2.13}$$

with (1.3), now based on Definition 12, and (1.4) (with the indicated interpretations if $u = 0$ or $u = \infty$). The question makes sense since one has for fixed $A = B$ or $A = F$ and $0 < p < \infty$ the continuous embeddings,

$$A^{s_1}_{pq_1}(\mathbb{R}^n) \hookrightarrow A^{s_2}_{pq_2}(\mathbb{R}^n) \quad \text{for} \quad s_2 < s_1 \quad \text{and} \quad 0 < q_1, q_2 < \infty \tag{2.14}$$

and

$$A^{s}_{pq_1}(\mathbb{R}^n) \hookrightarrow A^{s}_{pq_2}(\mathbb{R}^n) \quad \text{for} \quad 0 < q_1 \le q_2 < \infty, \tag{2.15}$$

together with Proposition 10, Definition 12 and the following simple observation.

Proposition 15 *Let $L_r(\Gamma,\mu)$ be as in Definition* 12 *and let D_Γ be dense in $A^s_{pq}(\mathbb{R}^n)$. Then there is no $c > 0$ with* (2.9).

Proof We assume that there is a constant $c > 0$ with (2.9). Then one gets a contradiction if one approximates a function $\varphi \in S(\mathbb{R}^n)$ which is identically 1 near Γ in $A^s_{pq}(\mathbb{R}^n)$ by D_Γ-functions.

Remark 16 Hence, the question makes sense, but in general a sharp breaking point as in (2.13) cannot be expected. Let, for example, μ_1 be the Lebesgue measure on a d-dimensional hyper-plane $\mathbb{R}^d$ (or a compact ball in $\mathbb{R}^d$ to be consistent with the above set-up), $n > d \in \mathbb{N}$, and let μ_2 be an atomic measure, say, $\mu_2(\{0\}) = 1$ and $\mu_2(\mathbb{R}^n\backslash\{0\}) = 0$. Then the existence of a trace for $\mu = \mu_1 + \mu_2$ requires

$$A^s_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n), \quad \text{hence} \quad s \ge n/p,$$

whereas one needs $s \le \frac{n-d}{p}$ for the density assertion. This follows from Remark 6 and the references given there. Hence there is a gap. But this makes also clear that the above dichotomy is a global question and that related measures must respect that the spaces $A^s_{pq}(\mathbb{R}^n)$ are isotropic and translation-invariant. This results naturally in *isotropic measures* μ with (2.6) and

$$c_1\, h(\varrho) \le \mu(B(\gamma,\varrho)) \le c_2\, h(\varrho), \quad \gamma \in \Gamma, \quad 0 < \varrho < 1,$$

for some $c_1 > 0$, $c_2 > 0$, in generalisation of (2.8). These are the so-called h-sets, where h is a continuous strictly increasing function on the interval $[0,1]$, say with $h(0) = 0$ and $h(1) = 1$. The tricky problem which functions h generate such a measure had been solved in [14–16] and may also be found in [5, Theorem 1.555, p. 97]. In the present paper we restrict ourselves to the most prominent examples which are the d-sets according to (2.8) where $h(\varrho) = \varrho^d$.

Proposition 17 *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ furnished with the Radon measure $\mu = \mathscr{H}^d_\Gamma = \mathscr{H}^d|\Gamma$ as introduced at the beginning of Section 2.2. Then*

$$\mathrm{tr}_\mu B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma,\mu), \quad \textit{if} \quad 0 < p < \infty \quad \textit{and} \quad 0 < q \le \min(1,p) \tag{2.16}$$

and

$$\mathrm{tr}_\mu F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma,\mu), \quad \text{if} \quad 0 < p \le 1 \quad \text{and} \quad 0 < q < \infty. \tag{2.17}$$

Remark 18 Of course, (2.16) with (2.17) means that the linear and bounded trace operator tr_μ according to (2.10) exists and that it is a map **onto** $L_p(\Gamma,\mu)$. The above assertion has a substantial history. First we remark that Γ can be replaced by $\mathbb{R}^d$ with $n > d \in \mathbb{N}$ and the d-dimensional Lebesgue measure, interpreted as an d-dimensional hyperplane in $\mathbb{R}^n$. Hence, in an obvious notation,

$$\mathrm{tr}\, B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^d), \quad \text{if} \quad 0 < p < \infty \quad \text{and} \quad 0 < q \le \min(1,p), \tag{2.18}$$

and

$$\mathrm{tr}\, F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^d), \quad \text{if} \quad 0 < p \le 1 \quad \text{and} \quad 0 < q < \infty. \tag{2.19}$$

The first proof of (2.18) for $1 \le p < \infty$ (and $d = n-1$) goes back to [17–19] in the late 1970s. This had been extended to all $0 < p < \infty$ in [20]. The F-counterpart, hence (2.19) (with $d = n-1$) is due to [21–22]. Further details may be found in [4, Section 4.4.3, pp. 220/221]. The step from $\mathbb{R}^d$ to d-sets requires new technical instruments, especially atomic decompositions. Proposition 17 coincides with [5, Proposition 1.172]. But it goes back to [9, Corollary 18.12, p. 142] and the later observation that any d-set is porous (called ball condition in [9]) according to Proposition 27 below with a reference to [11, Remark 9.19, pp. 140/141] (what we overlooked in [9]). However we wish to mention that one finds some assertions of this type also in an earlier paper by Netrusov [23, Corollary, p. 193].

2.4 Curiosities

As mentioned above with a reference to [5, Section 1.12.2, pp. 80/81] any Radon measure μ according to (2.6) can be identified with the tempered distribution generated by μ, also written as $\mu \in S'(\mathbb{R}^n)$. This applies also to

$$g \in L_p(\Gamma,\mu) \subset L_1(\Gamma,\mu), \qquad 1 \le p < \infty,$$

since g can be interpreted as a complex Radon measure. If $p < 1$ then the situation is totally different and the spaces $S'(\mathbb{R}^n)$ and $L_p(\Gamma, \mu)$ have nothing in common. On the other hand, Definition 12 makes sense and we have Proposition 17. But the situation has some curious consequences. Nevertheless one may ask for traces

$$\mathrm{tr}_\mu : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu), \quad 0 < p < 1, \quad 0 < s < n \left(\frac{1}{p} - 1 \right), \tag{2.20}$$

on compact d-sets Γ with $0 < d < n$ and $\mu = \mathscr{H}^d_\Gamma$ and also for traces

$$\mathrm{tr} : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^m), \quad 0 < p < 1, \quad 0 < s < n \left(\frac{1}{p} - 1 \right), \tag{2.21}$$

on hyper-planes $\mathbb{R}^m \subset \mathbb{R}^n$ with $n \geq m \in \mathbb{N}$, admitting $m = n$, where (2.18), (2.19) might be considered as limiting cases. Recall that

$$\mathscr{H}^{d'}_\Gamma \in B^{(n-d')(\frac{1}{p}-1)}_{p\infty}(\mathbb{R}^n), \quad |\Gamma| = 0, \quad 0 < p < \infty, \tag{2.22}$$

for compact d'-sets in $\mathbb{R}^n$ with $0 \leq d' < n$ (admitting $d' = 0$ for the δ-distribution $\mu = \delta$ with $\Gamma = \{0\}$). Here the spaces $B^s_{p\infty}(\mathbb{R}^n)$ are defined according to (2.2), (2.4) extended to $q = \infty$. As for (2.22) we refer to [5, pp. 96/97, Proposition 7.32 and its proof, pp. 315/316]. In particular one finds in all spaces $A^s_{pq}(\mathbb{R}^n)$ according to (2.20), (2.21) singular distributions f with $|\mathrm{supp}\, f| = 0$.

Proposition 19 (i) *If $s > 0$, $0 < p < \infty$, $0 < q < \infty$, then*

$$\mathrm{tr} : \quad A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \tag{2.23}$$

exists. If, in addition, $0 < p < 1$, $0 < s \leq n(\frac{1}{p} - 1)$ then

$$\mathrm{tr}\, f = 0 \text{ in } L_p(\mathbb{R}^n) \text{ for any } f \in A^s_{pq}(\mathbb{R}^n) \text{ with } |\mathrm{supp}\, f| = 0. \tag{2.24}$$

(ii) *Let $0 < p < 1$. Let Γ be a compact d-set, $\mu = \mathscr{H}^d_\Gamma$ and Γ' be a compact d'-set, $\mu' = \mathscr{H}^{d'}_{\Gamma'}$ with*

$$0 \leq d' < d < n, \qquad d - d' > (n - d')p. \tag{2.25}$$

Then $\mu' \in B^{\frac{n-d}{p}}_{pp}(\mathbb{R}^n)$,

$$\mathrm{tr}_\mu \mu' \in L_p(\Gamma, \mu) \quad \text{exists and} \quad \mathrm{tr}_\mu \mu' = 0$$

in $L_p(\Gamma, \mu)$.

Remark 20 One gets $\mathrm{tr}\, \delta = 0$ as a special case of (2.24). We shift the proof of this proposition to Section 3.2. We only mention that one gets by (2.22), (2.25) and

$$(n - d') \left(\frac{1}{p} - 1 \right) > \frac{n - d}{p}$$

that

$$\mu' \in B^{(n-d')(\frac{1}{p}-1)}_{p\infty}(\mathbb{R}^n) \hookrightarrow B^{\frac{n-d}{p}}_{pp}(\mathbb{R}^n).$$

Then (2.25) ensures that $\mathrm{tr}_\mu \mu' \in L_p(\Gamma, \mu)$ makes sense.

3 Proofs

We have to prove Theorem 3 (our main result), Corollary 5 and the Propositions 7, 19. We rely on atomic decompositions and the remarkable interplay between F-spaces and porous sets in $\mathbb{R}^n$. For the convenience of the reader we collect what we need in an Appendix (Section A below).

3.1 Proof of Theorem 3 and Corollary 5

Step 1 We prove Theorem 3. By the discussions at the beginning of Section 2.3, especially in connection with (2.13)–(2.15) and Proposition 15 we have to show that the breaking points (σ, u) exist and that they coincide with the right-hand sides of (1.5), (1.6). By (2.16) it remains to prove that

$$D_\Gamma \text{ is dense in } B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \text{ if } 0 < p < \infty,\ q > \min(p,1), \tag{3.1}$$

in case of the B-spaces, hence (1.5). If $p \le 1$ then we have (2.17). On the other hand, it follows from

$$B^{\frac{n-d}{p}}_{p\tilde{q}}(\mathbb{R}^n) \hookrightarrow F^{\frac{n-d}{p}-\varepsilon}_{pq}(\mathbb{R}^n), \qquad p \le 1, \tag{3.2}$$

for $\varepsilon > 0$, $0 < q < \infty$, $\tilde{q} > p$, the density of $D(\mathbb{R}^n)$ in both spaces in (3.2), and (3.1) (with $\tilde{q}$) that D_Γ is dense in all spaces on the right-hand side of (3.2). This proves the lower line of (1.6), hence $p \le 1$. As for the upper line, hence $1 < p < \infty$, we first remark that for any $\varepsilon > 0$, $0 < q < \infty$,

$$F^{\frac{n-d}{p}+\varepsilon}_{pq}(\mathbb{R}^n) \hookrightarrow B^{\frac{n-d}{p}}_{p,1}(\mathbb{R}^n), \qquad p > 1. \tag{3.3}$$

By (2.16) all spaces on the left-hand side of (3.3) have traces. It remains to prove in case of the F-spaces that

$$D_\Gamma \text{ is dense in } F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \text{ if } 1 < p < \infty,\ 0 < q < \infty. \tag{3.4}$$

Step 2 We begin with a preparation by constructing a sequence $\{\varphi^J\}_{J=1}^\infty \subset D(\mathbb{R}^n)$ with

$$\varphi^J(x) = 1 \text{ in an open neighbourhood of } \Gamma$$

(depending on J), and

$$\varphi^J \to 0 \text{ in } B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \text{ if } J \to \infty, \quad p \ge 1, \quad q > 1. \tag{3.5}$$

For given $j \in \mathbb{N}$ we cover a neighbourhood of Γ with balls $B_{j,m}$ centred at Γ and of radius 2^{-j}, where $m = 1, \ldots, M_j$ and $M_j \sim 2^{jd}$ such that there is a resolution of unity,

$$\sum_{m=1}^{M_j} \varphi_{j,m}(x) = 1 \text{ near } \Gamma, \quad 0 \le \varphi_{j,m} \in D(B_{j,m}), \tag{3.6}$$

with the usual properties,

$$|D^\gamma \varphi_{j,m}(x)| \le c_\gamma\, 2^{j|\gamma|}, \qquad \gamma \in \mathbb{N}_0^n. \tag{3.7}$$

For $2 \le J \in \mathbb{N}$, let $J' \in \mathbb{N}$ such that

$$\sum_{j=J}^{J'+1} r_j = 1 \quad \text{with } r_j = j^{-1} \text{ if } J \le j \le J' \text{ and } 0 < r_{J'+1} \le (J'+1)^{-1}.$$

Then

$$\varphi^J(x) = \sum_{j=J}^{J'+1} r_j \, 2^{-\frac{jd}{p}} \sum_{m=1}^{M_j} 2^{\frac{jd}{p}} \, \varphi_{j,m}(x), \qquad x \in \mathbb{R}^n, \tag{3.8}$$

is an atomic decomposition in $B^s_{pq}(\mathbb{R}^n)$ according to Proposition 23(i) below for $s = \frac{n-d}{p}$, $p \geq 1$, $q > 1$, with $L = 0$ (no moment conditions). We used $s - \frac{n}{p} = -\frac{d}{p}$. By (A.6), one gets

$$\begin{aligned} \|\varphi^J \,|B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|^q &\leq c \sum_{j=J}^{J'+1} r_j^q \, 2^{-j\frac{dq}{p}} \Big(\sum_{m=1}^{M_j} 1 \Big)^{q/p} \\ &\leq c' \sum_{j=J}^{\infty} j^{-q} \sim J^{1-q}. \end{aligned} \tag{3.9}$$

This proves (3.5).

Step 3 We prove (3.1) for $p > 1$, $q > 1$. It is sufficient to approximate $f \in D(\mathbb{R}^n)$ in $B^s_{pq}(\mathbb{R}^n)$, $s = \frac{n-d}{p}$, by functions $f^J \in D_\Gamma$. Let φ^J be the above functions and

$$f = f_J + f^J \text{ with } f_J = \varphi^J f \text{ and } f^J = (1 - \varphi^J) f \in D_\Gamma.$$

By the pointwise multiplier theorem in [4, Corollary, p. 205], one has for $\varrho > \frac{n-d}{p}$, some $c > 0$, all $f \in D(\mathbb{R}^n)$ and all φ^J that

$$\big\| f_J \,|B^s_{pq}(\mathbb{R}^n)\big\| \leq c \, \|f \,|\mathscr{C}^\varrho(\mathbb{R}^n)\| \cdot \big\|\varphi^J \,|B^s_{pq}(\mathbb{R}^n)\big\| \to 0$$

if $J \to \infty$, where we used (3.5). This proves (3.1) for $p > 1$, $q > 1$.

Step 4 We prove (3.4). By Proposition 27 in Appendix, Γ is porous. Then one can apply the arguments from [11, pp. 142/143] (one may also consult [5, pp. 393/394, (9.90)]) with the outcome that at least for $p > 1$, $q \geq 1$,

$$\|\varphi^J \,|F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\| \sim \|\varphi^J \,|B^{\frac{n-d}{p}}_{pp}(\mathbb{R}^n)\| \to 0 \quad \text{if } J \to \infty, \tag{3.10}$$

where we used (3.5) and that we can choose $L = 0$ in (A.7) (hence no moment conditions in Proposition 23(ii) are needed). If $0 < q < 1$ then it may happen that (3.8) is no longer an atomic decomposition in $F^s_{pq}(\mathbb{R}^n)$ since some moment conditions in (A.7), (A.3) are required. However since Γ is porous one can complement $\varphi_{j,m}$ outside of Γ in an appropriate way such that one gets the needed moment conditions. Details may be found in [9, p. 143] with a reference to [24]. After this modification one obtains (3.10) now for all $1 < p < \infty$ and $0 < q < \infty$. Then one gets (3.4) by the same arguments as in Step 3.

Step 5 We prove (3.1) for $p < q$. This covers in particular the remaining cases with $p \leq 1$. We begin with a preparation, covering Γ, say with $\mu(\Gamma) = 1$, for given $L \in \mathbb{N}$ by d-sets Γ_l such that

$$\Gamma = \bigcup_{l=L}^{L'} \Gamma_l, \qquad \mu(\Gamma_l) \sim l^{-1}, \qquad \sum_{l=L}^{L'} \mu(\Gamma_l) \sim \mu(\Gamma) = 1,$$

where $L' \in \mathbb{N}$ with $L' > L$ is appropriately chosen. This can be done as follows. For given $l \in \mathbb{N}$ and appropriately chosen large $k \in \mathbb{N}$ (in dependence of l) one finds $\sim l^{-1} \, 2^{kd}$ balls centred at Γ, of radius $\sim 2^{-k}$ and having pairwise distance of at least $\sim 2^{-k}$, such that the intersection of Γ with the union of these balls is a sub-d-set Γ^l of Γ with $\mu(\Gamma^l) \sim l^{-1}$. Now one can start for the given $L \in \mathbb{N}$ with $\Gamma_L = \Gamma^L$ and applies afterwards the above procedure to $\overline{\Gamma \backslash \Gamma_L}$ and $l = L+1$. Iteration gives the desired decomposition. This can be done in such a way that there are functions $\psi_l \in D(\mathbb{R}^n)$, $\psi_l \geq 0$,

$$\sum_{l=L}^{L'} \psi_l(\gamma) = 1 \text{ if } \gamma \in \Gamma, \quad \Gamma_l \subset \text{supp } \psi_l \subset \{y \in \mathbb{R}^n : \text{ dist } (y, \Gamma_l) < \varepsilon_l\}$$

for some $\varepsilon_l > 0$. Let for given $l \in \mathbb{N}$ (between L and L') and appropriately chosen $j(l) \in \mathbb{N}$,

$$\sum_{m=1}^{M_{j(l)}} \varphi_{j(l),m}(x) = 1 \text{ near } \Gamma, \quad 0 \leq \varphi_{j(l),m} \in D\left(B_{j(l),m}\right)$$

as in (3.6) with the counterpart of (3.7) and $M_{j(l)} \sim 2^{j(l)d}$. With

$$j(L) < \cdots < j(l) < j(l+1) < \cdots < j(L'),$$

we put in analogy to (3.8)

$$\varphi^L(x) = \sum_{l=L}^{L'} \psi_l(x)\, 2^{-\frac{j(l)d}{p}} \sum_{m=1}^{M_{j(l)}} 2^{\frac{j(l)d}{p}} \varphi_{j(l),m}(x), \quad x \in \mathbb{R}^n. \tag{3.11}$$

First, we assume that we do not need moment conditions in Proposition 23(i) (hence $\frac{n-d}{p} = s > \sigma_p$). Then (3.11) with large $j(l)$ is an atomic decomposition which can be written as

$$\varphi^L(x) = \sum_{l=L}^{L'} 2^{-\frac{j(l)d}{p}} \sum_{m=1}^{M'_{j(l)}} \widetilde{\varphi}_{j(l),m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$M'_{j(l)} \sim \mu(\Gamma_l)\, 2^{j(l)d} \sim l^{-1}\, 2^{j(l)d}$$

counting only non-vanishing terms, where the equivalence constants are independent of l. We have $\varphi^L(x) = 1$ near Γ. Then one gets by Proposition 23(i) for $q > p$,

$$\|\varphi^L \,|B_{pq}^{\frac{n-d}{p}}(\mathbb{R}^n)\|^q \leq c \sum_{l=L}^{L'} 2^{-j(l)\frac{dq}{p}} \left(\sum_{m=1}^{M'_{j(l)}} 1 \right)^{q/p} \leq c' \sum_{l=L}^{\infty} l^{-q/p} \sim L^{1-\frac{q}{p}}.$$

This is the counterpart of (3.9). Now we can argue in the same way as in Step 3. This proves (3.1) for $p \leq 1$ and $q > p$ provided that one does not need moment conditions in Proposition 23(i). But otherwise one can rely on the same arguments and references as in Step 4. This proves (3.1) also in the remaining cases and completes the proof of Theorem 3.

Step 6 Corollary 5 is a by-product of the above arguments. On the one hand one has (2.18), (2.19) as the direct (and easier) counterpart of Proposition 17 (and the starting point of these considerations). On the other hand the approximation of $f \in D(\mathbb{R}^n)$ by functions from $D_\Gamma = D(\mathbb{R}^n \backslash \Gamma)$ with $\Gamma = \mathbb{R}^d$ is a local matter covered by the above arguments.

3.2 Proof of Propositions 7 and 19

Step 1 We prove Proposition 7. Obviously, D_Γ is dense in $L_p(\mathbb{R}^n)$ with $1 < p < \infty$. Then it follows from Proposition 10 and

$$L_p(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\mathbb{R}^n), \qquad s < 0, \quad 1 < p < \infty, \quad 0 < q < \infty, \tag{3.12}$$

that D_Γ is also dense in $A^s_{pq}(\mathbb{R}^n)$ with (3.12). For any ball K in $\mathbb{R}^n$ (and $s \in \mathbb{R}$, $0 < q < \infty$) there is a number $c_K > 0$ such that

$$\|f\,|A^s_{p_1,q}(\mathbb{R}^n)\| \le c_K \,\|f\,|A^s_{p_2,q}(\mathbb{R}^n)\|, \qquad 0 < p_1 \le p_2 < \infty,$$

for all $f \in A^s_{p_2,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset K$. This well-known assertion follows from Hölder's inequality and characterisations of the spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of local means as it may be found in [4, Sections 2.4.6, 2.5.3, pp. 122, 138]. However then one gets again from Proposition 10 and the corresponding density assertion for the spaces in (3.12) that D_Γ is dense in all spaces in (1.9).

Step 2 We prove Proposition 19(i). For $s > 0$ and $0 < p \le 1$ it follows from Definition 8 that

$$\begin{aligned}\|f\,|L_p(\mathbb{R}^n)\|^p &= \Big\| \sum_{j=0}^{\infty} (\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n)\Big\|^p \\ &\le \sum_{j=0}^{\infty} \|(\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n)\|^p \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\|^p;\end{aligned}$$

similarly for $1 < p < \infty$ (but this is well known). Together with elementary embeddings one gets

$$\|\varphi\,|L_p(\mathbb{R}^n)\| \le c\,\|\varphi\,|A^s_{pq}(\mathbb{R}^n)\|, \quad s > 0,\ 0 < p < \infty,\ 0 < q < \infty,$$

for all $\varphi \in D(\mathbb{R}^n)$ or $\varphi \in S(\mathbb{R}^n)$ as requested in Definition 12 resulting in (2.23). If f is a non-trivial element

$$f \in A^s_{pq}(\mathbb{R}^n) \ \text{ with } \ \Gamma = \operatorname{supp} f \ \text{ compact } \ \text{and} \ |\Gamma| = 0,$$

hence $0 < p < 1$, $0 < s \le n(\frac{1}{p} - 1)$ then it follows by similar arguments to those in Steps 2 and 3 in Section 3.1 (but much simpler) that f can be approximated by $D(\mathbb{R}^n)$-functions with supports in

$$\Gamma_\varepsilon = \{y \in \mathbb{R}^n : \ \operatorname{dist}\,(y, \Gamma) < \varepsilon\}$$

for any $\varepsilon > 0$. Then one gets $\operatorname{supp}\,\operatorname{tr} f \subset \Gamma$ and hence $\operatorname{tr} f = 0$ in $L_p(\mathbb{R}^n)$. This can be extended to all f covered by (2.24).

Step 3 The proof of Proposition 19(ii) is similar. So far we justified in Remark 20 that $\operatorname{tr}_\mu \mu' \in L_p(\Gamma, \mu)$ makes sense. It remains to prove that $\operatorname{tr}_\mu \mu' = 0$ in $L_p(\Gamma, \mu)$. Let

$$\Gamma' \subset \Gamma'_j = \bigcup_{m=1}^{M_j} B(\gamma'_m, 2^{-j}), \qquad M_j \sim 2^{jd'},$$

where $B(\gamma'_m, 2^{-j})$ are open balls centred at Γ' and of radius 2^{-j}, with $j \in \mathbb{N}$. As in Step 2 one can approximate μ' in $B^{\frac{n-d}{p}}_{pp}(\mathbb{R}^n)$ by $D(\mathbb{R}^n)$-functions with supports in Γ'_j. However

$$\mu(\Gamma'_j) \le c\, 2^{-jd}\, 2^{jd'} \to 0 \quad \text{if} \quad j \to \infty.$$

This proves $\mathrm{tr}_\mu \mu' = 0$ in $L_p(\Gamma, \mu)$.

Appendix

A.1 Atoms

Atomic decompositions in the spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and their numerous special cases have a long and substantial history. This is not the subject of the present paper. The interested reader may consult [4, Section 1.9] and [5, Section 1.5] both for (historical) references and additional explanations. We follow here mainly [5, Section 1.5.1].

First we complement the notation introduced in Section 2.1. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}^n_0$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \alpha_j \in \mathbb{N}_0, \quad |\alpha| = \sum_{j=1}^n \alpha_j.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n_0$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \qquad \text{(monomials)}.$$

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 21 (i) *Let $0 < p < \infty$, $0 < q < \infty$. Then b_{pq} is the collection of all sequences*

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\} \tag{A.1}$$

such that

$$\|\lambda \,|b_{pq}\| = \left(\sum_{j=0}^\infty \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \right)^{q/p} \right)^{1/q} < \infty,$$

and f_{pq} is the collection of all sequences λ according to (A.1) *such that*

$$\|\lambda \,| f_{pq}\| = \left\| \left(\sum_{j,m} 2^{jnq/p} \, |\lambda_{jm} \, \chi_{jm}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty.$$

(ii) *Let $s \in \mathbb{R}$, $0 < p < \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d \geq 1$. Then L_∞-functions $a_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called (s,p)-atoms if*

$$\operatorname{supp} a_{jm} \subset dQ_{jm}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n; \tag{A.2}$$

there exist all (*classical*) *derivatives $D^\alpha a_{jm}$ with $|\alpha| \leq K$ such that*

$$|D^\alpha a_{jm}(x)| \leq 2^{-j(s-\frac{n}{p})+j|\alpha|}, \qquad |\alpha| \leq K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n,$$

and

$$\int_{\mathbb{R}^n} x^\beta \, a_{jm}(x) \, \mathrm{d}x = 0, \qquad |\beta| < L, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{A.3}$$

Remark 22 If $L = 0$ then (A.3) is empty (no moment conditions). Of course, the atoms depend on K, L, and d. But this will not be indicated. We put, as usual,

$$\sigma_p = n\left(\frac{1}{\min(p,1)} - 1\right) \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(1,p,q)} - 1\right).$$

Proposition 23 *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$.*

(i) *Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, with*

$$K > s \qquad \text{and} \qquad L > \sigma_p - s \tag{A.4}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\, a_{jm}, \qquad \lambda \in b_{pq}, \tag{A.5}$$

where a_{jm} are (s,p)-atoms according to Definition 21 *with* (A.4) *and fixed $d \geq 1$ in* (A.2). *Furthermore,*

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda\,|b_{pq}\| \tag{A.6}$$

are equivalent quasi-norms where the infimum is taken over all representations (A.5) (*for fixed K, L, d*).

(ii) *Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, with*

$$K > s \qquad \text{and} \qquad L > \sigma_{pq} - s \tag{A.7}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\, a_{jm}, \qquad \lambda \in f_{pq}, \tag{A.8}$$

where a_{jm} are (s,p)-atoms according to Definition 21 *with* (A.7) *and fixed $d \geq 1$ in* (A.2). *Furthermore,*

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda\,|f_{pq}\|$$

are equivalent quasi-norms where the infimum is taken over all representations (A.8) (*for fixed K, L, d*).

Remark 24 This formulation coincides essentially with a corresponding assertion in [5, Section 1.5.1]. There one finds also some technical comments how the convergence in (A.5) must be understood. The above proposition can be extended to $p = \infty$ and/or $q = \infty$. But this will not be needed here.

A.2 Porosity

The arguments in this paper rely on atomic decompositions of B-spaces and F-spaces and the miraculous interplay especially of F-spaces with porous sets in $\mathbb{R}^n$. We collect what we need.

Definition 25 *A closed set Γ in $\mathbb{R}^n$ is said to be porous if there is a number η with $0 < \eta < 1$ such that one finds, for any ball $B(x,r)$ centred at $x \in \mathbb{R}^n$ and of radius r with $0 < r < 1$, a ball $B(y, \eta r)$ with*

$$B(y,\eta r) \subset B(x,r) \quad \text{and} \quad B(y,\eta r) \cap \Gamma = \emptyset. \tag{A.9}$$

Remark 26 If one restricts (A.9) to balls $B(x, r)$ centred at Γ then one gets (A.9) for all admitted balls $B(x, r)$ at the expense of a smaller η. Conditions of this type are well known in fractal geometry. One may consult for example [7, Section 11, p. 156] where also the notation *porous* comes from. We used this property of sets in connection with function spaces several times and refer to [9, Definition 18.10, p. 142] and [11, Definition 9.16] denoted previously as *ball condition*. There one finds also a few further references and discussions. In particular according to [11, 9.17, pp. 138/139] any porous set Γ has Lebesgue measure $|\Gamma| = 0$. For us the following observation is of interest.

Proposition 27 *Any compact d-set in $\mathbb{R}^n$ with $0 < d < n$ is porous.*

Remark 28 A proof has been given in [11, Proposition 9.18, Remark 9.19, pp. 139–141]. There one finds also some relevant references, in particular to [25–26]. One may also consult [5, Definition 9.20, Remark 9.21, Proposition 9.22, pp. 393/394] for further properties.

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