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On Generalized Inverse Transversals

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Abstract Let S be a regular semigroup, S° an inverse subsemigroup of S. S° is called a generalized inverse transversal of S, if $V(x) \cap S^\circ \neq \emptyset$. In this paper, some properties of this kind of semigroups are discussed. In particular, a construction theorem is obtained which contains some recent results in the literature as its special cases.

Keywords generalized inverse transversal, inverse transversal, regular semigroup

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1 Introduction

The study of regular semigroups with inverse transversals was initiated by Blyth and McFadden in 1982 (see [1]). This type of semigroups attracted much attention. Several authors have investigated various types of regular semigroups with inverse transversals (see [1–3] and their references). In particular, a construction theorem for the general case was given by Saito in [3]. Recently, the concept of inverse transversal was generalized by many authors (see [4–7]). In particular, Chen gave an interesting generalization for inverse transversal, namely, the orthodox transversal in [4]. In this paper, we consider another generalization for inverse transversals, the so-called generalized inverse transversals.

Let S be a regular semigroup, $E(S)$ the set of idempotents of S and S_1 a subsemigroup of S. We denote the set of all inverses of $x \in S$ by $V(x)$ and let $V_{S_1}(x) = V(x) \cap S_1$. Recall that S_1 is called an *inverse transversal* of S if $|V_{S_1}(x)| = 1$ for any $x \in S$ and a *quasi-ideal* of S if $S_1 S S_1 \subseteq S_1$.

Let S be a regular semigroup and S° an inverse subsemigroup of S. S° is called a *generalized inverse transversal* of S if $V_{S^\circ}(x) \neq \emptyset$ for all $x \in S$. Clearly, an inverse transversal S° of S is a generalized inverse transversal, but the converse is not true, even if S° is a quasi-ideal of S (see [2]). It is worth remarking that generalized inverse transversals of a completely regular semigroup or orthodox semigroup are inverse transversals (see [8]). Let S be a regular semigroup with a generalized inverse transversal S° and X a subset of S. We denote one of the inverses of $x \in S$ in S° by x°, and denote $(x^\circ)^{-1}$, a unique inverse of x° in S° by $x^{\circ\circ}$. We also denote $V_{S^\circ}(X) = \{x^\circ \in S^\circ | x^\circ \in V(x), x \in X\}$.

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Let S be a regular semigroup with a generalized inverse transversal S° and $a \in S$. Denote $I_a = \{aa^\circ | a^\circ \in V_{S^\circ}(a)\}, \Lambda_a = \{a^\circ a | a^\circ \in V_{S^\circ}(a)\}$ and $I = \cup_{a\in S} I_a, \Lambda = \cup_{a\in S}\Lambda_a$. S° is called a *multiplicative* generalized inverse transversal of S if $\Lambda I \subseteq E(S^\circ)$. It can be proved that a multiplicative generalized inverse transversal of S is an inverse transversal (see [9]). Notations and terminologies not given in this paper can be found in [1–11].

2 Elementary Properties

In this section, we consider some elementary properties of regular semigroups with generalized inverse transversals. We begin our paper with the following proposition:

Proposition 2.1 *Let S be a regular semigroup with a generalized inverse transversal S°. Then*:

(1) $I = \{e \in E(S) | (\exists\, e^\circ \in E(S^\circ)) e\mathscr{L}e^\circ\}, \Lambda = \{e \in E(S) | (\exists\, e^\circ \in E(S^\circ)) e\mathscr{R}e^\circ\}$;

(2) $I \cap \Lambda = E(S^\circ)$;

(3) $IE(S^\circ) \subseteq I,\ E(S^\circ)\Lambda \subseteq \Lambda$.

Proof (1) Let $e = xx^\circ \in I,\ x^{\circ\circ} \in V_{S^\circ}(x^\circ)$. Then $e^\circ = x^{\circ\circ}x^\circ \in E(S^\circ)$ and $e^\circ \mathscr{L} x^\circ \mathscr{L} xx^\circ = e$. The converse inclusion is obvious; a similar argument to Λ.

(2) Clearly, $E(S^\circ) \subseteq I \cap \Lambda$. Suppose that $e \in I \cap \Lambda$. By (1), there exist $e^\circ, f^\circ \in E(S^\circ)$ such that $e^\circ \mathscr{L} e \mathscr{R} f^\circ$. So $e = f^\circ e^\circ \in E(S^\circ)$.

(3) Let $e \in I,\ f \in E(S^\circ)$. By (1), there is $e^\circ \in E(S^\circ)$ such that $e\mathscr{L}e^\circ$. Thus $ef \mathscr{L} e^\circ f \in E(S^\circ)$ and $efef = ee^\circ fef = efe^\circ ef = efe^\circ f = ee^\circ f = ef \in E(S)$. Again by (1), $ef \in I$. Similarly, the case for Λ can be proved.

Proposition 2.2 *Let S be a regular semigroup with a generalized inverse transversal S° and $L^E_a = \{e \in E(S) | e\mathscr{L}a\}$, $R^E_a = \{e \in E(S) | e\mathscr{R}a\}$ for $a \in E(S^\circ)$. Then*:

(1) $I = \cup_{a\in E(S^\circ)} L^E_a,\ \Lambda = \cup_{a\in E(S^\circ)} R^E_a$;

(2) *If $a, b \in E(S^\circ),\ b \le a,\ e \in L^E_a,\ g \in L^E_b$ and $f \in R^E_a,\ h \in R^E_b$, then $eg \in L^E_b$ and $hf \in R^E_b$.*

Proof (1) follows from (1) of Proposition 2.1. Let $e\mathscr{L}a$ and $g\mathscr{L}b$. Then $egeg = egbeg = egbaeg = egbag = egbg = eg$ and $egb = eg,\ beg = baeg = bag = bg = b$. This implies that $eg\mathscr{L}b$, that is to say, $eg \in L^E_b$; a similar argument to Λ.

Proposition 2.3 *Let S be a regular semigroup with a generalized inverse transversal S°. Then $\mathscr{H}$ on S saturates S°.*

Proof Let $x \in S,\ a \in S^\circ$ and $x\mathscr{H}a$. Then there exists $\bar{x} \in V(x)$ such that $x\bar{x} = aa^\circ,\ \bar{x}x = a^\circ a \in S^\circ$. So $x = x\bar{x}\bar{x}^\circ\bar{x}x = aa^\circ\bar{x}^\circ a^\circ a \in S^\circ$, where $\bar{x}^\circ \in V_{S^\circ}(\bar{x})$.

Proposition 2.4 *Let S be a regular semigroup with a generalized inverse transversal S°. Then $\mathscr{D}^{S^\circ} = \mathscr{D}^S \cap (S^\circ \times S^\circ)$.*

Proof Let $x, y \in S^\circ$ and $x\mathscr{D}^S y$. Then there exist $z \in S, x^\circ \in V_{S^\circ}(x)$ and $y^\circ \in V_{S^\circ}(y)$ such that $x^\circ x \mathscr{L} x \mathscr{L} z \mathscr{R} y \mathscr{R} yy^\circ$. Thus there is $\bar{z} \in V(z)$ satisfying $z\bar{z} = yy^\circ,\ \bar{z}z = x^\circ x \in S^\circ$. Hence $z = z\bar{z}\bar{z}^\circ\bar{z}z = yy^\circ\bar{z}^\circ x^\circ x \in S^\circ$, where $\bar{z}^\circ \in V_{S^\circ}(\bar{z})$. We have proved that $\mathscr{D}^S \cap (S^\circ \times S^\circ) \subseteq \mathscr{D}^{S^\circ}$. The reverse inclusion is obvious.

Theorem 2.5 *Let S be a regular semigroup with a generalized inverse transversal S°. Then $V_{S^\circ}(a) = V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ)$ for all $a \in S, a^\circ \in V_{S^\circ}(a)$.*

Proof Let $x \in V_{S^\circ}(a^\circ a)$ and $y \in V_{S^\circ}(aa^\circ)$. Then $axa^\circ ya = a(a^\circ axa^\circ a)a^\circ(aa^\circ yaa^\circ)a = aa^\circ aa^\circ aa^\circ a = a$. Similarly, $xa^\circ yaxa^\circ y = xa^\circ y$. This implies that $V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ) \subseteq V_{S^\circ}(a)$.

Conversely, assume that $\bar{a} \in V_{S^\circ}(a), a^{\circ\circ} \in V_{S^\circ}(a^\circ)$. By Propositions 2.3 and 2.4, we have $\bar{a}aa^\circ, a^\circ a\bar{a} \in S^\circ$. It is easy to prove that $\bar{a}aa^\circ a^{\circ\circ} \in V_{S^\circ}(a^\circ a),\ a^{\circ\circ}a^\circ a\bar{a} \in V_{S^\circ}(aa^\circ)$. So

$$\bar{a} = \bar{a}aa^\circ a^{\circ\circ}a^\circ a^{\circ\circ}a^\circ a\bar{a} \in V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ).$$

Let S be a regular semigroup with a generalized inverse transversal S°. In the following statements, for $x \in S^\circ$ and $e \in I, f \in \Lambda$, we denote $T_{(e,x,f)} = V_{S^\circ}(V_{S^\circ}(e))xV_{S^\circ}(V_{S^\circ}(f))$ and let $T = \{T_{(e,x,f)} | e\mathscr{L}xx^{-1}, f\mathscr{R}x^{-1}x \ \&\ x \in S^\circ\}$, where x^{-1} denotes the unique inverse of x in S°.

Corollary 2.6 *Let S be a regular semigroup with a generalized inverse transversal S°. Suppose that $x \in S^\circ$ and $e \in I, f \in \Lambda$ satisfying $e\mathscr{L}xx^{-1}, f\mathscr{R}x^{-1}x$. Then*

$$T_{(e,x,f)} = V_{S^\circ}(V_{S^\circ}(exf)).$$

Proof Firstly, we assert that $x^{-1} \in V_{S^\circ}(exf)$. In fact,

$$x^{-1}exfx^{-1} = x^{-1}(xx^{-1}e)x(fx^{-1}x)x^{-1} = x^{-1}xx^{-1}xx^{-1}xx^{-1} = x^{-1}$$

and

$$exfx^{-1}exf = ex(fx^{-1}x)x^{-1}(xx^{-1}e)xf = exx^{-1}xx^{-1}xx^{-1}xf = exf.$$

By Theorem 2.5, we have $V_{S^\circ}(exf) = V_{S^\circ}(x^{-1}exf)x^{-1}V_{S^\circ}(exfx^{-1}) = V_{S^\circ}(f)x^{-1}V_{S^\circ}(e)$. Thus $V_{S^\circ}(V_{S^\circ}(exf)) = V_{S^\circ}(V_{S^\circ}(e))xV_{S^\circ}(V_{S^\circ}(f)) = T_{(e,x,f)}$, since S° is an inverse subsemigroup of S.

Remark 2.7 Let S be a regular semigroup with an inverse transversal S°. Then, for every $x \in S^\circ$ and $e \in I, f \in \Lambda$ satisfying $e\mathscr{L}xx^{-1}$ and $f\mathscr{R}x^{-1}x$, we have $T_{(e,x,f)} = x$, since in this case $V_{S^\circ}(e) = xx^{-1}$ and $V_{S^\circ}(f) = x^{-1}x$.

Proposition 2.8 *Let S be a regular semigroup with a generalized inverse transversal S°. Then the following statements are equivalent*:

(1) $V_{S^\circ}(a) \cap V_{S^\circ}(b) \neq \emptyset \Longrightarrow V_{S^\circ}(a) = V_{S^\circ}(b)$ *for all* $a, b \in S$;

(2) $(\forall\, e \in I \cup \Lambda)V_{S^\circ}(e)$ *is a subsemigroup of* S;

(3) $(\forall\, e \in I \cup \Lambda)|V_{S^\circ}(e)| = 1$;

(4) S° *is an inverse transversal of* S.

Proof (1) implies (2). Let $e \in I$. Then, by Proposition 2.1, there exists $e^\circ \in E(S^\circ)$ such that $e\mathscr{L}e^\circ$. Clearly, $e^\circ \in V_{S^\circ}(e) \cap V_{S^\circ}(e^\circ)$. By (1), $V_{S^\circ}(e) = V_{S^\circ}(e^\circ) = \{e^\circ\}$, since S° is an inverse subsemigroup of S. Obviously, $\{e^\circ\}$ is a subsemigroup of S. Dually, we can prove $V_{S^\circ}(e)$ is also a subsemigroup of S for $e \in \Lambda$.

(2) implies (3). Let $e \in I$. Then, by Proposition 2.1, there exists $e^\circ \in E(S^\circ)$ such that $e\mathscr{L}e^\circ$. Evidently, $e^\circ \in V_{S^\circ}(e)$. Let $e^* \in V_{S^\circ}(e)$. Then $e^\circ\mathscr{L}e^*e\mathscr{R}e^*$. By Propositions 2.3 and 2.4, $e^*e \in E(S^\circ)$. This leads to $e^*e = e^\circ$, since S° is an inverse subsemigroup of S. Thus $e^\circ\mathscr{R}e^*$. By (2), $e^*e^\circ \in V_{S^\circ}(e)$, and so $ee^*e^\circ e = ee^*e^\circ = e$. This implies the $\mathscr{H}$-class containing e^* contains an idempotent. Hence $e^\circ\mathscr{H}e^*$, since S° is an inverse subsemigroup of S. Noticing that $e^*, e^\circ \in V_{S^\circ}(e)$, we have $e^\circ = e^*$. Similarly, we can show that $|V_{S^\circ}(e)| = 1$ for $e \in \Lambda$.

(3) implies (4). Let $a \in S$ and $a^\circ \in V_{S^\circ}(a)$. Then $V_{S^\circ}(a) = V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ)$. It is easy to see that $a^\circ a^{\circ\circ} \in V_{S^\circ}(a^\circ a)$ and $a^{\circ\circ}a^\circ \in V_{S^\circ}(aa^\circ)$. By (3), $V_{S^\circ}(a^\circ a) = \{a^\circ a^{\circ\circ}\}$ and $V_{S^\circ}(aa^\circ) = \{a^{\circ\circ}a^\circ\}$. So $V_{S^\circ}(a) = V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ) = a^\circ a^{\circ\circ}a^\circ a^{\circ\circ}a^\circ = a^\circ$. That is, S° is an inverse transversal of S.

(4) implies (1). This is trivial.

Theorem 2.9 *Let S be a regular semigroup with a generalized inverse transversal S° and $T_{(e,x,f)}, T_{(g,y,h)} \in T$. Then*

$$e\mathscr{R}g, T_{(e,x,f)} = T_{(g,y,h)}, \quad f\mathscr{L}h \Longleftrightarrow exf = gyh.$$

Proof For our purpose, the following graph is useful:

x	xx^{-1}			
$x^{-1}x$	x^{-1}	f		
ex	e	exf, gyh	g	gy
		h	y^{-1}	$y^{-1}y$
			yy^{-1}	y

Necessity. If $e\mathscr{R}g, T_{(e,x,f)} = T_{(g,y,h)}, f\mathscr{L}h$, then the above graph is valid, and by Corollary 2.6, $V_{S^\circ}(V_{S^\circ}(exf)) = V_{S^\circ}(V_{S^\circ}(gyh))$. Thus $V_{S^\circ}(exf) = V_{S^\circ}(gyh)$, since S° is an inverse subsemigroup of S. In view of the proof of Corollary 2.6, $x^{-1} \in V_{S^\circ}(exf) = V_{S^\circ}(gyh)$. But $exf\mathscr{H}gyh$, so $exf = gyh$.

Sufficiency. Let $exf = gyh$. By the above graph, $e\mathscr{R}exf = gyh\mathscr{R}g, f\mathscr{L}exf = gyh\mathscr{L}h$. Again, by Corollary 2.6, we have

$$T_{(e,x,f)} = V_{S^\circ}(V_{S^\circ}(exf)) = V_{S^\circ}(V_{S^\circ}(gyh)) = T_{(g,y,h)}.$$

Define

$$M = \{(R_e^I, T_{(e,x,f)}, L_f^\Lambda) \in I/\mathscr{R} \times T \times \Lambda/\mathscr{L} | e \in I, x \in S^\circ, f \in \Lambda\}.$$

For convenience, we denote $(R_e^I, T_{(e,x,f)}, L_f^\Lambda) \in M$ by $(R_e, T_x, L_f) \in M$. The following corollary is obvious:

Corollary 2.10 *Let S be a regular semigroup with a generalized inverse transversal S° and $(R_e^I, T_x, L_f^\Lambda), (R_g^I, T_y, L_h^\Lambda) \in M$. Then*

$$(R_e^I, T_x, L_f^\Lambda) = (R_g^I, T_y, L_h^\Lambda) \Longleftrightarrow exf = gyh.$$

Proposition 2.11 *Let $x, y \in S^\circ, f \in R_{x^{-1}x}^E, g \in L_{yy^{-1}}^E$ and $(fg)^* \in V_{S^\circ}(fg)$. Then $yy^{-1}(fg)^*x^{-1}x \in V_{S^\circ}(fg)$.*

Proof Let $(fg)^* \in V_{S^\circ}(fg)$. Then

$$\begin{aligned} yy^{-1}(fg)^*x^{-1}xfgyy^{-1}(fg)^*x^{-1}x &= yy^{-1}(fg)^*(x^{-1}xf)(gyy^{-1})(fg)^*x^{-1}x \\ &\qquad (\text{since } f\mathscr{R}x^{-1}x, g\mathscr{L}yy^{-1}) \\ &= yy^{-1}(fg)^*fg(fg)^*x^{-1}x \\ &= yy^{-1}(fg)^*x^{-1}x \end{aligned}$$

and

$$\begin{aligned} fgyy^{-1}(fg)^*x^{-1}xfg &= f(gyy^{-1})(fg)^*(x^{-1}xf)g \ (\text{since } f\mathscr{R}x^{-1}x, g\mathscr{L}yy^{-1}) \\ &= fg(fg)^*fg = fg. \end{aligned}$$

We have shown that $yy^{-1}(fg)^*x^{-1}x \in V(fg)$. But $yy^{-1}(fg)^*x^{-1}x \in S^\circ$ is clear. Thus,

$$yy^{-1}(fg)^*x^{-1}x \in V_{S^\circ}(fg).$$

Proposition 2.12 *Let* $(R^I_e, T_x, L^\Lambda_f), (R^I_g, T_y, L^\Lambda_h) \in M$ *and* $(fg)^*, (fg)^\circ \in V_{S^\circ}(fg)$. *Then*

$$(R^I_{exfg(fg)^\circ x^{-1}}, T_{x(fg)^{\circ\circ}y}, L^\Lambda_{y^{-1}(fg)^\circ fgyh}) = (R^I_{exfg(fg)^*x^{-1}}, T_{x(fg)^{*-1}y}, L^\Lambda_{y^{-1}(fg)^*fgyh}) \in M.$$

Proof Let $(fg)^\# = yy^{-1}(fg)^\circ x^{-1}x$. Then, by Proposition 2.11 and the fact that S° is an inverse subsemigroup of S, we have $(fg)^\# \in V_{S^\circ}(fg)$, $(fg)^{\#-1} = x^{-1}x(fg)^{\circ\circ}yy^{-1}$ and

$$x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1} = x(fg)^{\#-1}yy^{-1}(fg)^\# x^{-1}, \quad exfg(fg)^\circ x^{-1} = exfg(fg)^\# x^{-1}.$$

Therefore,

$$\begin{aligned} exfg(fg)^\circ x^{-1}exfg(fg)^\circ x^{-1} &= exfg(fg)^\# x^{-1}exfg(fg)^\# x^{-1} \\ &= exfg(fg)^\# x^{-1}xfg(fg)^\# x^{-1} \quad (\text{since } \mathrm{x}^{-1}\mathscr{L}\mathrm{xx}^{-1}\mathscr{L}\mathrm{e}) \\ &= exfg(fg)^\# fg(fg)^\# x^{-1} = exfg(fg)^\# x^{-1} \\ &= exfg(fg)^\circ x^{-1} \in E(S), \end{aligned}$$

$$\begin{aligned} (exfg(fg)^\circ x^{-1})(x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1}) &= (exfg(fg)^\# x^{-1})(x(fg)^{\#-1}yy^{-1}(fg)^\# x^{-1}) \\ &= exfg(fg)^\# x^{-1} = exfg(fg)^\circ x^{-1}. \end{aligned} \tag{$*$}$$

Similarly, $(x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1})(exfg(fg)^\circ x^{-1}) = (x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1})$. We have shown that

$$exfg(fg)^\circ x^{-1}\mathscr{L}x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1}). \tag{$**$}$$

Dually,

$$y^{-1}(fg)^\circ fgyh \in \mathrm{E(S)} \text{ and } y^{-1}(fg)^\circ fgyh\mathscr{R}y^{-1}(fg)^\circ x^{-1}x(fg)^{\circ\circ}y.$$

This implies that

$$(R^I_{exfg(fg)^\circ x^{-1}}, T_{x(fg)^{\circ\circ}y}, L^\Lambda_{y^{-1}(fg)^\circ fgyh}) \in M.$$

Similarly,

$$(R^I_{exfg(fg)^*x^{-1}}, T_{x(fg)^{*-1}y}, L^\Lambda_{y^{-1}(fg)^*fgyh}) \in M.$$

Moreover,

$$\begin{aligned} (exfg(fg)^\circ x^{-1})(x(fg)^{\circ\circ}y)(y^{-1}(fg)^\circ fgyh) &= (exfg(fg)^\circ x^{-1})(x(fg)^{\circ\circ}yy^{-1}(fg)^\circ \\ &\quad \cdot (x^{-1}xf)gyh) \quad (\text{since } x^{-1}x\mathscr{R}f) \\ &= (exfg(fg)^\circ x^{-1})(x(fg)^{\circ\circ}yy^{-1}(fg)^\circ x^{-1}) \\ &\quad \cdot (xfgyh) \\ &= (exfg(fg)^\circ x^{-1})(xfgyh) \quad (\text{by } (*)) \\ &= exfg(fg)^\circ fgyh \quad (\text{since } x^{-1}x\mathscr{R}f) \\ &= exfgyh. \end{aligned}$$

Similarly, we can show that

$$(exfg(fg)^*x^{-1})(x(fg)^{*^{-1}}y)(y^{-1}(fg)^*fgyh) = exfgyh.$$

The result follows from Corollary 2.10.

By the condition $(**)$ and its dual, for $x, y \in S^\circ$, the following mappings are well-defined:

$$\alpha_{(x,y)} : R^E_{x^{-1}x} \times L^E_{yy^{-1}} \to I, \quad (f,g) \mapsto xfg(fg)^\# x^{-1} \in L^E_{x(fg)^{\#-1}y[x(fg)^{\#-1}y]^{-1}},$$

$$\beta_{(x,y)} : R^E_{x^{-1}x} \times L^E_{yy^{-1}} \to \Lambda, \quad (f,g) \mapsto y^{-1}(fg)^{\#} fgy \in R^E_{[x(fg)^{\#-1}y]^{-1}x(fg)^{\#-1}y},$$

where $(fg)^{\#}$ is a given inverse of fg in S°. In fact, $R^E_{x^{-1}x} = R^{\Lambda}_{x^{-1}x}$ and $L^E_{yy^{-1}} = L^I_{yy^{-1}}$.

Lemma 2.13 *Let S be a regular semigroup with a generalized inverse transversal S° and $x, y, z, w \in S^{\circ}$. Then, for $e \in L^E_{xx^{-1}}, g \in L^E_{yy^{-1}}, k \in L^E_{zz^{-1}}, p \in L^E_{ww^{-1}}$ and $f \in R^E_{x^{-1}x}, h \in R^E_{y^{-1}y}, l \in R^E_{z^{-1}z}, q \in R^E_{w^{-1}w}$, the following conditions are satisfied:*

(a) *If*

$$e\mathscr{R}k, T_{(e,x,f)} = T_{(k,z,l)}, f\mathscr{L}l \text{ and } g\mathscr{R}p, T_{(g,y,h)} = T_{(p,w,q)}, h\mathscr{L}q,$$

then

$$e\alpha_{(x,y)}(f,g) = \theta_1 \mathscr{R} \theta_2 = k\alpha_{(z,w)}(l,p), \ \beta_{(x,y)}(f,g)h = \delta_1 \mathscr{L} \delta_2 = \beta_{(z,w)}(l,p)q$$

and

$$T_{(\theta_1, x(fg)^{\#-1}y, \delta_1)} = T_{(\theta_2, z(lp)^{\#-1}w, \delta_2)};$$

(b)

$$\alpha_{(x,y)}(f,g)\alpha_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k) = m\mathscr{R}n = \alpha_{(x,y(hk)^{\#-1}z)}(f, g\alpha_{(y,z)}(h,k)),$$
$$\beta_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k) = i\mathscr{L}j = \beta_{(x,y(hk)^{\#-1}z)}(f, g\alpha_{(y,z)}(h,k))\beta_{(y,z)}(h,k)$$

and

$$T_{(em,s,il)} = T_{(en,t,jl)},$$

where $s = x(fg)^{\#-1}y(\beta_{(x,y)}(f,g)hk)^{\#-1}z$ and $t = x(fg\alpha_{(y,z)}(h,k))^{\#-1}y(hk)^{\#-1}z$.

Proof (a) By hypothesis, $(R^I_e, T_x, L^{\Lambda}_f), (R^I_g, T_y, L^{\Lambda}_h), (R^I_k, T_z, L^{\Lambda}_l), (R^I_p, T_w, L^{\Lambda}_q) \in M$. In view of Proposition 2.12, we have

$$(R^I_{\theta_1}, T_{x(fg)^{\#-1}y}, L^{\Lambda}_{\delta_1}), (R^I_{\theta_2}, T_{z(lp)^{\#-1}w}, L^{\Lambda}_{\delta_2}) \in M$$

and

$$\begin{aligned}\theta_1 x(fg)^{\#-1}y\delta_1 &= e\alpha_{(x,y)}(f,g)x(fg)^{\#-1}y\beta_{(x,y)}(f,g)h \\ &= exfg(fg)^{\#}x^{-1}x(fg)^{\#-1}yy^{-1}(fg)^{\#}x^{-1}xfgyh \quad (\text{by } (*) \text{ and } x^{-1}x\mathscr{R}f) \\ &= exfgyh.\end{aligned}$$

Similarly, $\theta_2 z(lp)^{\#-1}w\delta_2 = kzlpwq$. If $e\mathscr{R}k, T_{(e,x,f)} = T_{(k,z,l)}, f\mathscr{L}l$ and $g\mathscr{R}p, T_{(g,y,h)} = T_{(p,w,q)}$, $h\mathscr{L}q$, by Theorem 2.9, we have $exf = kzl$ and $gyh = pwq$. So $exfgyh = kzlpwq$. By Corollary 2.10,

$$(R^I_{\theta_1}, T_{x(fg)^{\#-1}y}, L^{\Lambda}_{\delta_1}) = (R^I_{\theta_2}, T_{z(lp)^{\#-1}w}, L^{\Lambda}_{\delta_2}).$$

That is to say, (a) holds.

(b) By hypothesis, $(R^I_e, T_x, L^{\Lambda}_f), (R^I_g, T_y, L^{\Lambda}_h), (R^I_k, T_z, L^{\Lambda}_l) \in M$. In view of Proposition 2.12, $(R^I_{e\alpha_{(x,y)}(f,g)}, T_{x(fg)^{\#-1}y}, L^{\Lambda}_{\beta_{(x,y)}(f,g)h}) \in M$ and

$$\begin{aligned}\alpha_{(x,y)}(f,g)x(fg)^{\#-1}y\beta_{(x,y)}(f,g) &= xfg(fg)^{\#}x^{-1} \cdot x(fg)^{\#-1}y \cdot y^{-1}(fg)^{\#}fgy \\ &= [xfg(fg)^{\#}x^{-1}x(fg)^{\#-1}yy^{-1}(fg)^{\#}x^{-1}]xfgy \\ &\quad (\text{since } x^{-1}x\mathscr{R}f) \\ &= xfg(fg)^{\#}x^{-1}xfgy \quad (\text{by } (*)) \\ &= xfgy. \qquad (***)\end{aligned}$$

Now, we have $(R^I_{e\alpha_{(x,y)}(f,g)}, T_{x(fg)^{\#-1}y}, L^\Lambda_{\beta_{(x,y)}(f,g)h}), (R^I_k, T_z, L^\Lambda_l) \in M$. Again by Proposition 2.12, $(R^I_{em}, T_s, L^\Lambda_{il}) \in M$ and

$$\alpha_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k)x(fg)^{\#-1}y(\beta_{(x,y)}(f,g)hk)^{\#-1}z\beta_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k)$$
$$= x(fg)^{\#-1}y\beta_{(x,y)}(f,g)hkz \qquad \text{(by } (***)).$$

So

$$\begin{aligned} emsil &= e\alpha_{(x,y)}(f,g)[\alpha_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k) \\ &\quad \cdot x(fg)^{\#-1}y(\beta_{(x,y)}(f,g)hk)^{\#-1}z \cdot \beta_{(x(fg)^{\#-1}y,z)}(\beta_{(x,y)}(f,g)h,k)]l \\ &= e[\alpha_{(x,y)}(f,g)x(fg)^{\#-1}y\beta_{(x,y)}(f,g)]hkzl \\ &= exfgyhkzl \qquad \text{(by } (***) \text{ again)}. \end{aligned}$$

Similarly, we can prove $(R^I_{en}, T_t, L^\Lambda_{jl}) \in M$ and $entjl = exfgyhkzl$. By Corollary 2.10, $(R^I_{em}, T_s, L^\Lambda_{il}) = (R^I_{en}, T_t, L^\Lambda_{jl})$. So $em\mathscr{R}en, il\mathscr{L}jl$ and $T_{(em,s,il)} = T_{(en,t,jl)}$. Noticing that $\mathscr{R}$ and $\mathscr{L}$ are a left congruence and a right congruence, respectively, we have $m = xx^{-1}em\mathscr{R}xx^{-1}en = n$ and $i = ilz^{-1}z\mathscr{L}jlz^{-1}z = j$. That is, (b) is true.

3 Main Result

In this section, we construct a regular semigroup with a generalized inverse transversal. Recall that a non-empty set S is called a *partial groupiod* if S is equipped with a partial binary operation. Let E be a partial groupiod. For any $a, b \in E$, the notation $ab \in E$ means that the product of a and b is defined in E. A partial groupiod is called a partial semigroup, if the following condition is valid: For e, f and $g \in E$, if $ef, fg \in E$, then $e(fg) \in E$ if and only if $(ef)g \in E$; in this case, $(ef)g = e(fg)$, and the common product is denoted by efg. In particular, a partial semigroup E is called a partial band if $ee \in E$ and $ee = e$ for any $e \in E$.

Let E be a partial band. We define the relations $\tilde{\mathscr{L}}^E$ and $\tilde{\mathscr{R}}^E$ on E as follows:

$$e\tilde{\mathscr{L}}^E f \Longleftrightarrow ef,\ \ fe \in E,\ \ ef = e,\ \ fe = f;$$
$$e\tilde{\mathscr{R}}^E f \Longleftrightarrow ef,\ \ fe \in E,\ \ ef = f,\ \ fe = e.$$

Remark 3.1 The relations $\tilde{\mathscr{L}}^E$ and $\tilde{\mathscr{R}}^E$ on a partial band E defined above are equivalences. Clearly, they are reflexive and symmetrical. Let $x, y, a \in E$ and $x\tilde{\mathscr{L}}^E a\tilde{\mathscr{L}}^E y$. Then we have $xa = x,\ ax = a,\ ya = y,\ ay = a$. So $x(ay) = xa = x$. By the definition of partial band, it follows that $xy = (xa)y = x(ay) = x$. Similarly, $yx = y$. That is to say, $x\tilde{\mathscr{L}}^E y$. A similar argument for $\tilde{\mathscr{R}}^E$. After that, we let $\tilde{L}^E_a = \{x \in E | x\tilde{\mathscr{L}}^E a\}$, $\tilde{R}^E_a = \{x \in E | x\tilde{\mathscr{R}}^E a\}$ for $a \in E$.

Let E be a partial band and E° a subsemilattice of E. E is called a partial lower (resp. upper) chain of $\{\tilde{L}^E_a | a \in E^\circ\}$ (resp. $\{\tilde{R}^E_a | a \in E^\circ\}$) if:

(1) $E = \cup_{a \in E^\circ} \tilde{L}^E_a$ (resp. $E = \cup_{a \in E^\circ} \tilde{R}^E_a$);

(2) Let $a, b \in E^\circ,\ a \leq b$ and $e \in \tilde{L}^E_b,\ g \in \tilde{L}^E_a$ (resp. $e \in \tilde{R}^E_b, g \in \tilde{R}^E_a$).

Then $eg \in \tilde{L}^E_a$ (resp. $ge \in \tilde{R}^E_a$).

Definition 3.2 *Let $P = I \cup \Lambda \cup S^\circ$ be a partial semigroup, I, Λ partial subbands and S° an inverse subsemigroup of P. Then S° is called a generalized inverse transversal of P if:*

(1) $I \cap \Lambda = E(S^\circ)$;

(2) *I and* Λ *are partial lower chain of* $\{\tilde{L}_a^E | a \in E(S^\circ)\}$ *and partial upper chain of* $\{\tilde{R}_a^E | a \in E(S^\circ)\}$, *respectively*;

(3) *For any* $e \in P, V_{S^\circ}(e) = \{x \in S^\circ | ex \in I, xe \in \Lambda; exe = e, xex = x\} \neq \emptyset$.

In this case, we say that $P = I \cup \Lambda \cup S^\circ$ *is a partial semigroup with a generalized inverse transversal* S°. *Moreover, we denote* $V_{S^\circ}(V_{S^\circ}(e))xV_{S^\circ}(V_{S^\circ}(f))$ *by* $T_{(e,x,f)}$ *for* $e \in I, f \in \Lambda, x \in S^\circ$, *and let* $T = \{T_{(e,x,f)} | e \in I, f \in \Lambda, x \in S^\circ \ \& \ e \in \tilde{L}^I_{xx^{-1}}, f \in \tilde{R}^\Lambda_{x^{-1}x}\}$, *where* x^{-1} *denotes the unique inverse of* x *in* S°.

Now, we can give our main result of this paper.

Theorem 3.3 *Let* $P = I \cup \Lambda \cup S^\circ$ *be a partial semigroup with a generalized inverse transversal* S°. *Let* $* : \Lambda \times I \to S^\circ, (f, g) \mapsto f * g$ *such that*:

(1) $f * x^{-1}x = x^{-1}x, \ yy^{-1} * g = yy^{-1}$ *for* $x, y \in S^\circ$ *and* $f \in \tilde{R}^\Lambda_{x^{-1}x}, g \in \tilde{L}^I_{yy^{-1}}$;

(2) $f * g = fg$ *for* $f, \ g \in E(S^\circ)$.

Moreover, for arbitrary $x, y, z, w \in S^\circ$, *there exist mappings*

$$\alpha_{(x,y)} : \tilde{R}^\Lambda_{x^{-1}x} \times \tilde{L}^I_{yy^{-1}} \to I, \quad \alpha_{(x,y)}(f,g) \in \tilde{L}^I_{x(f*g)y[x(f*g)y]^{-1}},$$
$$\beta_{(x,y)} : \tilde{R}^\Lambda_{x^{-1}x} \times \tilde{L}^I_{yy^{-1}} \to \Lambda, \quad \beta_{(x,y)}(f,g) \in \tilde{R}^\Lambda_{[x(f*g)y]^{-1}x(f*g)y},$$

and for $e \in \tilde{L}^I_{xx^{-1}}, \ g \in \tilde{L}^I_{yy^{-1}}, \ k \in \tilde{L}^I_{zz^{-1}}, p \in \tilde{L}^I_{ww^{-1}}$ *and* $f \in \tilde{R}^\Lambda_{x^{-1}x}, \ h \in \tilde{R}^\Lambda_{y^{-1}y}, \ l \in \tilde{R}^\Lambda_{z^{-1}z}, q \in \tilde{R}^\Lambda_{w^{-1}w}$, *the following conditions are satisfied*:

(3) *If*

$$e\tilde{\mathscr{R}}k, T_{(e,x,f)} = T_{(k,z,l)}, f\tilde{\mathscr{L}}l \text{ and } g\tilde{\mathscr{R}}p, T_{(g,y,h)} = T_{(p,w,q)}, h\tilde{\mathscr{L}}q,$$

then

$$e\alpha_{(x,y)}(f,g) = \theta_1 \tilde{\mathscr{R}} \theta_2 = k\alpha_{(z,w)}(l,p), \ \beta_{(x,y)}(f,g)h = \delta_1 \tilde{\mathscr{L}} \delta_2 = \beta_{(z,w)}(l,p)q$$

and

$$T_{(\theta_1, x(f*g)y, \delta_1)} = T_{(\theta_2, z(l*p)w, \delta_2)};$$

(4)

$$\alpha_{(x,y)}(f,g)\alpha_{(x(f*g)y,z)}(\beta_{(x,y)}(f,g)h,k) = m\tilde{\mathscr{R}}n = \alpha_{(x,y(h*k)z)}(f, g\alpha_{(y,z)}(h,k)),$$
$$\beta_{(x(f*g)y,z)}(\beta_{(x,y)}(f,g)h,k) = i\tilde{\mathscr{L}}j = \beta_{(x,y(h*k)z)}(f, g\alpha_{(y,z)}(h,k))\beta_{(y,z)}(h,k),$$

and

$$T_{(em,s,il)} = T_{(en,t,jl)},$$

where $s = x(f*g)y((\beta_{(x,y)}(f,g)h) * k)z$ *and* $t = x(f * g\alpha_{(y,z)}(h,k))y(h*k)z$;

(5) $\alpha_{(x,y)}(x^{-1}x, yy^{-1}) = xy(xy)^{-1}, \ \beta_{(x,y)}(x^{-1}x, yy^{-1}) = (xy)^{-1}xy$.

Define a multiplication on the set

$$W = \{(\tilde{R}^I_e, T_{(e,x,f)}, \tilde{L}^\Lambda_f) \in I/\tilde{\mathscr{R}}^I \times T \times \Lambda/\tilde{\mathscr{L}}^\Lambda | e \in I, x \in S^\circ, f \in \Lambda\}$$

by

$$(\tilde{R}^I_e, T_{(e,x,f)}, \tilde{L}^\Lambda_f)(\tilde{R}^I_g, T_{(g,y,h)}, \tilde{L}^\Lambda_h) = (\tilde{R}^I_c, T_{(c,x(f*g)y,d)}, \tilde{L}^\Lambda_d),$$

where $c = e\alpha_{(x,y)}(f,g)$ *and* $d = \beta_{(x,y)}(f,g)h$. *Then* W *is a regular semigroup with a generalized inverse transversal isomorphic to* S^0. *Conversely, every regular semigroup with a generalized inverse transversal can be constructed in this way.*

Proof For brevity, we denote $(\tilde{R}_e^I, T_{(e,x,f)}, \tilde{L}_f^\Lambda) \in W$ by (R_e, T_x, L_f). Firstly, we prove that W defined above is a semigroup. In fact, suppose that $(R_e, T_x, L_f), (R_g, T_y, L_h), (R_k, T_z, L_l), (R_p, T_w, L_q) \in W$, then $e \in \tilde{L}^I_{xx^{-1}}$ and $\alpha_{(x,y)}(f,g) \in \tilde{L}^I_{x(f*g)y[x(f*g)y]^{-1}}$. By the fact that I is a partial lower chain of $\{\tilde{L}_a^E | a \in E(S^\circ)\}$ and $x(f*g)y[x(f*g)y]^{-1} \leq xx^{-1}$, we have $e\alpha_{(x,y)}(f,g) \in \tilde{L}^I_{x(f*g)y[x(f*g)y]^{-1}}$. Similarly, using the fact that Λ is a partial upper chain of $\{\tilde{R}_a^E | a \in E(S^\circ)\}$, we can see $\beta_{x,y}(f,g)h \in \tilde{R}^\Lambda_{[x(f*g)y]^{-1}x(f*g)y}$. So $T_{(c,x(f*g)y,d)} \in T$. If $(R_e, T_x, L_f) = (R_k, T_z, L_l)$ and $(R_g, T_y, L_h) = (R_p, T_w, L_q)$, by condition (3), we have

$$(R_e, T_x, L_f)(R_g, T_y, L_h) = (R_k, T_z, L_l)(R_p, T_w, L_q).$$

That is to say, the operation is well defined and closed. Let

$$(R_e, T_x, L_f), (R_g, T_y, L_h), (R_k, T_z, L_l) \in W.$$

Then

$$\begin{aligned}[(R_e, T_x, L_f)(R_g, T_y, L_h)](R_k, T_z, L_l) &= (R_{e\alpha_{(x,y)}(f,g)}, T_{x(f*g)y}, L_{\beta_{(x,y)}(f,g)h})(R_k, T_z, L_l)\\ &= (R_{em}, T_s, L_{il}).\end{aligned}$$

Similarly, we have

$$(R_e, T_x, L_f)[(R_g, T_y, L_h)(R_k, T_z, L_l)] = (R_{en}, T_t, L_{jl}),$$

where m, i, s, n, j, t are defined as in Theorem 3.3. We assert that $(R_{en}, T_t, L_{jl}) = (R_{em}, T_s, L_{il})$. In fact, by condition (4), $T_s = T_t$, so we need to prove only $em\tilde{\mathscr{R}}^I en$ and $il\tilde{\mathscr{L}}^\Lambda jl$. Since I is a partial band and a partial lower chain of $\{\tilde{L}_a^E | a \in E(S^\circ)\}$ and $e \in \tilde{L}^I_{xx^{-1}}, \alpha_{(x,y)}(f,g) \in \tilde{L}^I_{x(f*g)y[x(f*g)y]^{-1}}$ and $\alpha_{(x(f*g)y,z)}(\beta_{(x,y)}(f,g)h, k) \in \tilde{L}^I_{ss^{-1}}$, and $ss^{-1} \leq x(f*g)y[x(f*g)y]^{-1} \leq xx^{-1}$, we have $em, m \in \tilde{L}^I_{ss^{-1}}, mss^{-1} = m$ and $e(mss^{-1}) = em \in I$. Similarly, $n, en \in I$. Noticing that $m\tilde{\mathscr{L}}^I n$, we obtain that $mn = n$ and so $emn = en \in I$. By similar discussions, we can show that $emss^{-1}xx^{-1} = em \in I$. Noticing that $xx^{-1}e = xx^{-1} \in I$, we have $emss^{-1}(xx^{-1}e) = em \in I$. So $(em)(en) = (emss^{-1}xx^{-1})(en) = (emss^{-1}xx^{-1}e)n = emn = en$. Similarly, we have $enem = em$. That is, $em\tilde{\mathscr{R}}^I en$. Dually, $il\tilde{\mathscr{L}}^\Lambda jl$. We have shown that W is a semigroup. Let $W^\circ = \{(R_e, T_x, L_f) \in W | e, f \in S^\circ\}$. If $(R_e, T_x, L_f) \in W^\circ$, then $e = xx^{-1}, f = x^{-1}x, T_x = x$. So $\varphi: \ W^\circ \to S^\circ, (R_{xx^{-1}}, x, L_{x^{-1}x}) \mapsto x$ is well defined. By condition (2), $x(x^{-1}x * yy^{-1})y = xy$, according to condition (5), we have $\sigma = xx^{-1}\alpha_{(x,y)}(x^{-1}x, yy^{-1}) = xy(xy)^{-1}$ and $\tau = \beta_{(x,y)}(x^{-1}x, yy^{-1})y^{-1}y = (xy)^{-1}xy$. So

$$\begin{aligned}\varphi[(R_{xx^{-1}}, x, L_{x^{-1}x})(R_{yy^{-1}}, y, L_{y^{-1}y})] &= \varphi(R_\sigma, T_{x(x^{-1}x*yy^{-1})y}, L_\tau)\\ &= \varphi(R_{xy(xy)^{-1}}, xy, L_{(xy)^{-1}xy}) = xy\\ &= \varphi(R_{xx^{-1}}, x, L_{x^{-1}x})\varphi(R_{yy^{-1}}, y, L_{y^{-1}y}).\end{aligned}$$

That is, φ is a homomorphism. Let $x \in S^\circ$. Because $x = (xx^{-1})x(x^{-1}x)$ and $(R_{xx^{-1}}, x, L_{x^{-1}x}) \in W^\circ$, φ is surjective. The injectivity of φ is clear. We have proved that φ is an isomorphism from W° onto S°. Consequently, W° is an inverse subsemigroup of W.

Next, we show that W° is a generalized inverse transversal of W. Let $(R_e, T_x, L_f) \in W$. Then $(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}}) \in W^\circ$. By condition (1), $x(f * x^{-1}x)x^{-1} = xx^{-1}$. Since $\alpha_{(x,x^{-1})}(f, x^{-1}x) \in \tilde{L}^I_{xx^{-1}}, \beta_{(x,x^{-1})}(f, x^{-1}x) \in \tilde{R}^\Lambda_{xx^{-1}}, e\alpha_{(x,x^{-1})}(f, x^{-1}x) = e$ and $\beta_{(x,x^{-1})}(f, x^{-1}x)xx^{-1} = xx^{-1}$, we have

$$(R_e, T_x, L_f)(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}}) = (R_{e\alpha_{(x,x^{-1})}(f,x^{-1}x)}, T_{x(f*x^{-1}x)x^{-1}}, L_{\beta_{(x,x^{-1})}(f,x^{-1}x)xx^{-1}})$$

$$= (R_e, T_{xx^{-1}}, L_{xx^{-1}}).$$

Again, noticing that $xx^{-1}(xx^{-1} * e)x = x, \alpha_{(xx^{-1},x)}(xx^{-1}, e) \in \tilde{L}^I_{xx^{-1}}$ and $\beta_{(xx^{-1},x)}(xx^{-1}, e) \in \tilde{R}^\Lambda_{x^{-1}x}$, we have $e\alpha_{(xx^{-1},x)}(xx^{-1}, e) = e$ and $\beta_{(xx^{-1},x)}(xx^{-1}, e)f = f$, since $e\tilde{\mathscr{L}}^I xx^{-1}$ and $f\tilde{\mathscr{R}}^\Lambda x^{-1}x$. Hence

$$\begin{aligned}(R_e, T_x, L_f)(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}})(R_e, T_x, L_f) &= (R_e, T_{xx^{-1}}, L_{xx^{-1}})(R_e, T_x, L_f)\\ &= (R_{e\alpha_{(xx^{-1},x)}(xx^{-1},e)}, T_{xx^{-1}(xx^{-1}*e)x},\\ &\quad L_{\beta_{(xx^{-1},x)}(xx^{-1},e)f})\\ &= (R_e, T_x, L_f).\end{aligned}$$

Dually, we can prove

$$(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}})(R_e, T_x, L_f)(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}}) = (R_{x^{-1}x}, x^{-1}, L_{xx^{-1}}).$$

That is to say, $(R_{x^{-1}x}, x^{-1}, L_{xx^{-1}}) \in V_{W^\circ}((R_e, T_x, L_f))$. Up to now, we have proved the direct part of our theorem.

Conversely, let S be a regular semigroup with a generalized inverse transversal S° and I, Λ be defined as in Section 1. In view of Propositions 2.1 and 2.2, $I \cap \Lambda = E(S^\circ)$, I and Λ are partial lower chain of $\{L^E_a | a \in E(S^\circ)\}$ and partial upper chain of $\{R^E_a | a \in E(S^\circ)\}$, respectively. Condition (3) of Definition 3.2 is obvious. So $P = I \cup \Lambda \cup S^\circ$ is a partial semigroup with a generalized inverse transversal S°. For $(f, g) \in \Lambda \times I$, put $f * g = (fg)^{\#-1}$, where $(fg)^\#$ is a given element in $V_{S^\circ}(fg)$. Clearly, $*$ is a mapping from $\Lambda \times I$ into S° satisfying (1) and (2). For each $(x, y) \in S^\circ \times S^\circ$ and for every $(f, g) \in R^E_{x^{-1}x} \times L^E_{yy^{-1}}$, let $\alpha_{(x,y)}(f, g) = xfg(f * g)^{-1}x^{-1}$ and $\beta_{(x,y)}(f, g) = y^{-1}(f * g)^{-1}fgy$. Then, by Lemma 2.13, they satisfy the conditions (3) and (4). Condition (5) follows from condition (2) and the fact that S° is an inverse subsemigroup of S. By the direct part of the theorem, we have constructed a semigroup

$$W = \{(R^I_e, T_{(e,x,f)}, L^\Lambda_f) \in I/\mathscr{R} \times T \times \Lambda/\mathscr{L} | e \in I, x \in S^\circ, f \in \Lambda\}$$

under the multiplication

$$(R^I_e, T_{(e,x,f)}, L^\Lambda_f)(R^I_g, T_{(g,y,h)}, L^\Lambda_h) = (R^I_{exfg(f*g)^{-1}x^{-1}}, T_{(u,x(f*g)y,v)}, L^\Lambda_{y^{-1}(f*g)^{-1}fgyh}),$$

where $u = exfg(f * g)^{-1}x^{-1}$ and $v = y^{-1}(f * g)^{-1}fgyh$. For convenience, denote $(R^I_e, T_{(e,x,f)}, L^\Lambda_f) \in W$ by (R_e, T_x, L_f). By Proposition 2.12, the element

$$(R_{exfg(f*g)^{-1}x^{-1}}, T_{x(f*g)y}, L_{y^{-1}(f*g)^{-1}fgyh})$$

is independent of the choice of $(fg)^\# \in V_{S^\circ}(fg)$.

Define $\psi : W \to S, (R_e, T_x, L_f) \mapsto exf$. By Theorem 2.9, ψ is well defined and injective. Let $x \in S$. Then $\psi(R_{xx^\circ}, T_{x^{\circ\circ}}, L_{x^\circ x}) = x$, since $x = xx^\circ x^{\circ\circ} x^\circ x$, where $x^\circ \in V_{S^\circ}(x)$. So ψ is surjective. Let $(R_e, T_x, L_f), (R_g, T_y, L_h) \in W$. By the proof of Proposition 2.12, we have

$$\begin{aligned}\psi[(R_e, T_x, L_f)(R_g, T_y, L_h)] &= \psi(R_{exfg(f*g)^{-1}x^{-1}}, T_{x(f*g)y}, L_{y^{-1}(f*g)^{-1}fgyh})\\ &= (exfg(f * g)^{-1}x^{-1})(x(f * g)y)(y^{-1}(f * g)^{-1}fgyh)\\ &= exfgyh \qquad (\text{by } (***))\\ &= \psi(R_e, T_x, L_f)\psi(R_g, T_y, L_h).\end{aligned}$$

Thus ψ is an isomorphism from W onto S.

Corollary 3.4 *In Theorem* 3.3, *if we enforce the condition*

$$\alpha_{(x,y)} : \tilde{R}^\Lambda_{x^{-1}x} \times \tilde{L}^I_{yy^{-1}} \to I, \quad \alpha_{(x,y)}(f, g) \in \tilde{L}^I_{x(f*g)y[x(f*g)y]^{-1}},$$

$$\beta_{(x,y)}:\tilde{R}^{\Lambda}_{x^{-1}x}\times\tilde{L}^{I}_{yy^{-1}}\to\Lambda,\quad \beta_{(x,y)}(f,g)\in\tilde{R}^{\Lambda}_{[x(f*g)y]^{-1}x(f*g)y}$$

to

$$\alpha_{(x,y)}:\tilde{R}^{\Lambda}_{x^{-1}x}\times\tilde{L}^{I}_{yy^{-1}}\to I,\quad (f,g)\mapsto x(f*g)y[x(f*g)y]^{-1}$$

and

$$\beta_{(x,y)}:\tilde{R}^{\Lambda}_{x^{-1}x}\times\tilde{L}^{I}_{yy^{-1}}\to\Lambda,\quad (f,g)\mapsto[x(f*g)y]^{-1}x(f*g)y,$$

then the semigroup W defined as in Theorem 3.3 *is a regular semigroup with a quasi-ideal generalized inverse transversal isomorphic to S°; conversely, every regular semigroup with a quasi-ideal generalized inverse transversal can be constructed in this way.*

Proof By Theorem 3.3, it is sufficient to prove W° defined as in the proof of Theorem 3.3 is a quasi-ideal of W. Let $(R_{xx^{-1}},T_x,L_{x^{-1}x}),(R_{zz^{-1}},T_z,L_{z^{-1}z})\in W^\circ$ and $(R_g,T_y,L_h)\in W$. By the enforced condition,

$$\begin{aligned}(R_{xx^{-1}},T_x,L_{x^{-1}x})(R_g,T_y,L_h)(R_{zz^{-1}},T_z,L_{z^{-1}z})&=(R_{aa^{-1}},T_a,L_{a^{-1}ah})(R_{zz^{-1}},T_z,L_{z^{-1}z})\\&=(R_{bb^{-1}},T_b,L_{b^{-1}b})\in W^\circ,\end{aligned}$$

where $a=x(x^{-1}x*g)y$ and $b=a[(a^{-1}ah)*zz^{-1}]z$. This implies that W° is a quasi-ideal of W.

Conversely, let S be a regular semigroup with a quasi-ideal generalized inverse transversal S°. In view of the second part of the proof of Theorem 3.3, for each $(x,y)\in S^\circ\times S^\circ$ and for every $(f,g)\in R^E_{x^{-1}x}\times L^E_{yy^{-1}}$, we have $f*g=fg\in S^\circ$. So

$$\alpha_{(x,y)}(f,g)=xfg(fg)^{-1}x^{-1}=xfgyy^{-1}(fg)^{-1}x^{-1}=x(f*g)y[x(f*g)y]^{-1}$$

and

$$\beta_{(x,y)}(f,g)=y^{-1}(fg)^{-1}fgy=y^{-1}(fg)^{-1}x^{-1}xfgy=[x(f*g)y]^{-1}x(f*g)y.$$

This is the enforced condition exactly. The result follows from Theorem 3.3.

Let S be a regular semigroup with an inverse transversal S°. Then I and Λ defined in Section 1 is a left regular band and a right regular band, respectively (see [8]). As an application of Theorem 3.3, we give a construction for regular semigroups with inverse transversals, which is essentially Saito's Theorem 3.2 of [3].

Theorem 3.5 *Let S° be an inverse semigroup with the semilattice E° of idempotents, and let I be a semilattice of left zero semigroups $\{L_a|a\in E^\circ\}$ and Λ a semilattice of right zero semigroups $\{R_a|a\in E^\circ\}$. Suppose that I and Λ have a common semilattice transversal E°. Let $\Lambda\times I\to S^\circ,(f,g)\mapsto f*g$ be a mapping satisfying*:

(A) $f^\circ(f*g)g^\circ=f*g$;

(B) $f^\circ*g=f*g^\circ=f^\circ g^\circ$.

Suppose that, for each $(x,y)\in S^\circ\times S^\circ$, there exist mappings

$$\begin{aligned}\alpha_{(x,y)}&:R_{x^{-1}x}\times L_{yy^{-1}}\to I,\quad \alpha_{(x,y)}(f,g)\in L_{x(f*g)y[x(f*g)y]^{-1}},\\ \beta_{(x,y)}&:R_{x^{-1}x}\times L_{yy^{-1}}\to\Lambda,\quad \beta_{(x,y)}(f,g)\in R_{[x(f*g)y]^{-1}x(f*g)y}\end{aligned}$$

satisfying the following conditions:

(C) *If $g\in L_{yy^{-1}},k\in L_{zz^{-1}},f\in R_{x^{-1}x},h\in R_{y^{-1}y}$, then*

$$\begin{aligned}\alpha_{(x,y)}(f,g)\alpha_{(x(f*g)y,z)}(\beta_{(x,y)}(f,g)h,k)&=\alpha_{(x,y(h*k)z)}(f,g\alpha_{y,z}(h,k)),\\ \beta_{(x(f*g)y,z)}(\beta_{(x,y)}(f,g)h,k)&=\beta_{(x,y(h*k)z)}(f,g\alpha_{(y,z)}(h,k))\beta_{(y,z)}(h,k),\end{aligned}$$

and

$$(f*g)y((\beta_{(x,y)}(f,g)h)*k)=(f*g\alpha_{(y,z)}(h,k))y(h*k);$$

(D) $\alpha_{(x,y)}(x^{-1}x,yy^{-1})=xy(xy)^{-1},\beta_{(x,y)}(x^{-1}x,yy^{-1})=(xy)^{-1}xy;$

(E) $\alpha_{(f^\circ,g^\circ)}(f^\circ,g)=f^\circ g$ *and* $\beta_{(f^\circ,g^\circ)}(f,g^\circ)=fg^\circ.$

Define a multiplication on the set

$$W=\{(e,x,f)\in I\times S^\circ\times\Lambda|e\in L_{xx^{-1}},x\in S^\circ,f\in R_{x^{-1}x}\}$$

by

$$(e,x,f)(g,y,h)=(e\alpha_{(x,y)}(f,g),x(f*g)y,\beta_{(x,y)}(f,g)h).$$

Then W *is a semigroup with an inverse transversal isomorphic to* S°*. Conversely, every regular semigroup with an inverse transversal can be constructed in this way.*

Proof Let $P=I\cup\Lambda\cup S^\circ$. By hypothesis, we can define an operation "o" on P as follows. $xoy=xy$, if $x,y\in I$ (resp. Λ,S°), where xy is the product of x and y in I(resp. Λ,S°). Clearly, P is a partial semigroup and S° is a generalized inverse transversal in the sense of Definition 3.2. In this case, for $e\in I,f\in\Lambda,x\in S^\circ$ satisfying $e\in L_{xx^{-1}}$ and $f\in R_{x^{-1}x}$, $T_{(e,x,f)}=V_{S^\circ}(V_{S^\circ}(e))xV_{S^\circ}(V_{S^\circ}(f))=x$ and $V_{S^\circ}(V_{S^\circ}(e))=e^\circ,V_{S^\circ}(V_{S^\circ}(f))=f^\circ$, and $R_e=\{e\},L_f=\{f\}$, where e° and f° are the unique inverses of e and f in $E(S^\circ)$, respectively. So (R_e,T_x,L_f) can be denoted by (e,x,f). It is easy to see that (B) implies (1) and (2). (3) is trivial, (C) is equivalent to (4), and (D) is equal to (5). By the proof of Theorem 3.3,

$$W=\{(e,x,f)\in I\times S^\circ\times\Lambda|e\in L_{xx^{-1}},x\in S^\circ,f\in R_{x^{-1}x}\}$$

is a regular semigroup with the multiplication

$$(e,x,f)(g,y,h)=(e\alpha_{(x,y)}(f,g),x(f*g)y,\beta_{(x,y)}(f,g)h)$$

and $W^\circ=\{(e,x,f)\in W|e,f\in S^\circ\}$ is a generalized inverse transversal of W. Using the condition (E), we can prove that W° is also an inverse transversal of W (see [3]). The converse part follows from Theorem 3.2 in [3].

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