Acta Mathematica Sinica, English Series
Dec., 2007, Vol. 23, No. 12, pp. 2189–2204
Published online: Oct. 7, 2007
DOI: 10.1007/s10114-007-0989-6
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

The Maximum Principle for One Kind of Stochastic Optimization Problem and Application in Dynamic Measure of Risk

Shao Lin JI Zhen WU
School of Mathematics and Systems Science, Shandong University, Jinan 250100, *P. R. China*
E-mail: jsl@sdu.edu.cn wuzhen@sdu.edu.cn

Abstract The authors get a maximum principle for one kind of stochastic optimization problem motivated by dynamic measure of risk. The dynamic measure of risk to an investor in a financial market can be studied in our framework where the wealth equation may have nonlinear coefficients.

Keywords backward stochastic differential equation, perturbation method, Ekeland's variational principle, dynamic measure of risk

MR(2000) Subject Classification 60H10, 60H20, 93E20

1 Introduction

It is well known that Bismut [1–3] first introduced a backward stochastic differential equation (BSDE for short) when he was studying adjoint equations of the stochastic optimal control problem. The first well-posedness result for nonlinear BSDEs was introduced by Pardoux and Peng [4] and independently by Duffie and Epstein [5]. Since then, BSDEs have been proved to be a useful tool in stochastic control, mathematical finance and other fields. For example, El Karoui, Peng and Quenez proved the comparison theorem of BSDE and gave some applications in optimal control and financial mathematics in [6]. Using the comparison theorem of BSDE, Hamadene and Lepeltier [7] studied the existence of optimal control and saddle point for the stochastic control and differential game problem. Kohlmann and Zhou [8] combined BSDE with the linear-quadratic stochastic control problem. Tang and Li [9] studied the BSDE with Poisson process as the noise source and gave the maximum principle for stochastic control system with random jump. Xu [10] studied one kind of forward-backward stochastic control system and got the stochastic maximum principle with state constraints. Jiang [11] studied a property of g-expectation which is introduced by BSDE.

In this paper, we deal with one kind of stochastic optimal control problem where the control variable is the terminal condition of a BSDE. This BSDE is regarded as the state equation and needs to satisfy some constraints. Different from classical stochastic control problems (see [12–13]), our problem has the economic background which can be seen in El Karoui, Peng and Quenez [14]. In their paper, they deal with a recursive utility optimization problem where the terminal wealth and consumption are control variables. Introducing Lagrange multiplier

Received October 20, 2004, Accepted January 26, 2007
This work is supported by the National Basic Research Program of China (973 Program, No. 2007CB814900) the Natural Science Foundation of China (10671112), Shandong Province (Z2006A01), and the New Century Excellent Young Teachers Program of Education Ministry of China

and perturbing the terminal condition, they get a maximum principle when the coefficients of the control system satisfy convexity assumption. In this paper, we use the terminal perturbation method and Ekeland's variational principle to deal with the corresponding optimal control problem with state constraints. Using Ekeland's variational principle to deal with state constraints is systematically studied in Ji [15]. Without convexity assumption, we obtain a necessary condition which the optimal objective satisfies, i.e. a maximum principle.

Then we apply the obtained results to study dynamic measure of risk. In a financial market, if an investor has not enough initial wealth x, he cannot perfectly hedge the liability C at future time T. An interesting problem is how to quantify such a risk. There are many excellent works concerning this subject (see Föllmer and Leukert [16–17] , Cvitanic and Karatas [18]). In [18], Cvitanic and Karatas quantified the risk by

$$\inf_{\pi(\cdot)\in\mathscr{A}(x)} E_0\left(\frac{C-X^{x,\pi}(T)}{S_0(T)}\right)^+,$$

where $\pi(\cdot)$ is the portfolio strategy, $S_0(\cdot)$ is the price of the risk-free instrument and $X(\cdot)$ represents the investor's wealth. Föllmer and Leukert [16–17], Edirisinghe, Naik and Uppal [19] proposed the related criteria. This measure associates with a stochastic optimal control problem in which the control (portfolio strategy) and state (wealth) constraints are imposed . Cvitanic and Karatas [18] employed the tools of convex duality and proved the existence of optimal strategy. In this paper, we apply the obtained results to study the problem introduced by Cvitanic and Karatas [18]. In more details, we first rewrite the stochastic optimal control problem in [18] as a backward formulation, i.e. instead of the portfolio strategy we use the terminal wealth as control variable. Then we obtain the same results as those in [18]. Furthermore, we can deal with nonlinear wealth equation even without convex assumption. In nonlinear case the convex duality method doesn't work and we obtain a necessary condition that the terminal wealth of an investor must satisfy. The corresponding results can be applied more widely in the financial market.

This paper is organized as follows: in Section 2, we deal with one kind of stochastic control problem motivated by dynamic measure of risk. Using terminal perturbation method and Ekeland's variational principle, a stochastic maximum principle is obtained. In Section 3, we apply the obtained maximum principle to study dynamic measure of risk in which the wealth equation is linear and obtain the same results as those in [18]. We study the dynamic measure of risk problem when the wealth equation is nonlinear without convexity assumption in Section 4.

2 One Kind of Stochastic Optimization Problem

2.1 Introduction

In this section, we study one kind of stochastic optimization problem with state constraints. Applying Ekeland's variational principle, we introduce a new approach to get the maximum principle for this problem without convexity assumption.

Let

$$W_0(\cdot) = (W_0^1(\cdot), \ldots, W_0^d(\cdot))'$$

be a standard d-dimensional Brownian Motion defined on a complete probability space $(\Omega,\mathscr{F},P_0)$. The information structure is given by a filtration $F = \{\mathscr{F}_t\}_{0\le t\le T}$; which is the σ-algebra generated by the Brownian Motion $W_0(\cdot)$ and augmented. For any given Euclidean space H, we denote by $(\cdot,\cdot)$ (resp.$|\cdot|$) the scalar product (resp. norm) of the space. We denote by $M^2(0,T;H)$, the space of all $\mathscr{F}_t$-adapted processes with values in H, such that

$$E_0\int_0^T \mid x(t)\mid^2 dt < \infty,\ \ \forall\, x(\cdot)\in M^2(0,T;H),$$

where E_0 denotes expectation with respect to the probability measure P_0. Obviously, $M^2(0,T;H)$ is a Hilbert space. Let us denote by $L^2(\Omega,\mathscr{F}_T,P_0)$, the space of all $\mathscr{F}_T$-measurable random variables ξ with values in R, such that $E_0\mid\xi\mid^2<\infty$.

Let $A\in L^2(\Omega,\mathscr{F}_T,P_0;R)$, $C\in L^2(\Omega,\mathscr{F}_T,P_0;R)$ and $A\le C$ a.s.

Definition 2.1 *We set*

$$U=\{\xi\mid\xi\in L^2(\Omega,\mathscr{F}_T,P_0), A\le\xi\le C\ \ a.s.\}$$

and call U the objective domain.

We want to maximize the following cost function

$$J(\xi)\triangleq E_0[u(\xi)]$$

subject to

$$\begin{cases} -dX(t)=f(X(t),Z(t),t)dt-Z(t)dW_0(t),\\ X(T)=\xi, \end{cases} \tag{2.1}$$

where $\xi\in U$, $X(t)$ is the state variable which satisfies an initial constraint $X(0)=x$ and f,u are given maps. This kind of optimization problem has the economic background which can be seen in [14].

We choose the metric in U to be

$$d(v,u)\triangleq(E_0\mid v-u\mid^2)^{\frac{1}{2}},\ \forall\, v,u\in U,$$

and let f,u be such that

$$f(X,Z,t,\omega): R\times R^{1\times d}\times[0,T]\times\Omega\to R,$$
$$u(X,\omega): R\times\Omega\to R.$$

We assume

(H1) f is continuous in $R\times R^{1\times d}\times[0,T]$ for a.a. ω and continuously differentiable with respect to (X,Z).

(H2) u is continuously differentiable with respect to X.

(H3) The derivatives of f and u are bounded.

(H4) For each $(X,Z)\in R\times R^{1\times d}$, $f(X,Z,\cdot)\in M^2(0,T;R)$.

Thanks to assumptions (H1)–(H4), one can find a unique pair $(X(\cdot),Z(\cdot))\in M^2(0,T;R)\times M^2(0,T;R^d)$ which satisfies (2.1) according to the existence and uniqueness theorem of BSDE in [4].

Definition 2.2 *ξ is called admissible objective for given $x\in R$, if the solution of* (2.1) *satisfies $X(0)=x$. We shall denote by $\mathscr{N}(x)$ the set of admissible objectives for x, and denote $X(0)$ by X_0^ξ.*

An admissible objective ξ^* is called optimal objective if it attains the maximum of $J(\xi)$ over $\mathscr{N}(x)$. Our aim is to obtain a characterization of the optimal objective.

2.2 Variational Equation

The purpose of this section is to obtain the first order variational equation.

Let ξ^* be an optimal objective and $(X^*(\cdot), Z^*(\cdot))$ be the corresponding optimal trajectory. Let $\hat{\xi} \in L^2(\Omega, \mathscr{F}, P_0)$ such that $(\xi^* + \hat{\xi}) \in U$. Since U is convex, then for any $0 \le p \le 1$,

$$\xi^p \triangleq \xi^* + p\hat{\xi}$$

is also in U.

Let $(\delta X(\cdot), \delta Z(\cdot))$ be the solution of

$$\begin{cases} -d\delta X(t) = [f_X(X^*(t), Z^*(t), t)\delta X(t) + f_Z(X^*(t), Z^*(t), t)\delta Z(t)]dt - \delta Z(t)dW_0(t), \\ \delta X(T) = \hat{\xi}. \end{cases} \tag{2.2}$$

One can find a unique pair $(\delta X(\cdot), \delta Z(\cdot)) \in M^2(0, T; R) \times M^2(0, T; R^d)$ which satisfies (2.2). (2.2) is called variational equation.

We denote by $(X^p(\cdot), Z^p(\cdot))$ the solution of (2.1) corresponding to $X(T) = \xi^p$.

Set

$$\tilde{X}^p(t) = p^{-1}(X^p(t) - X^*(t)) - \delta X(t),$$
$$\tilde{Z}^p(t) = p^{-1}(Z^p(t) - Z^*(t)) - \delta Z(t).$$

Using the techniques in [20], we have the following convergence results.

Lemma 2.3 *Assume* (H1), (H3) *and* (H4). *Then*

$$\lim_{p\to 0} \sup_{0\le t\le T} E_0 \mid \tilde{X}^p(t) \mid^2 = 0,$$
$$\lim_{p\to 0} E_0 \int_0^T \mid \tilde{Z}^p(t) \mid^2 dt = 0.$$

Proof From (2.1) and (2.2) we have

$$\begin{cases} -d\tilde{X}^p(t) = p^{-1}[f(X^p(t), Z^p(t), t) - f(X^*(t), Z^*(t), t) - pf_X(X^*(t), Z^*(t), t)\delta X(t) \\ \qquad\qquad -f_Z^{'}(X^*(t), Z^*(t), t)\delta Z(t)]dt - \tilde{Z}^p(t)dW_0(t), \\ \tilde{X}^p(T) = 0. \end{cases}$$

Let

$$A^p(t) = \int_0^1 f_X(X^*(t) + \lambda p(\delta X(t) + \tilde{X}^p(t)), Z^*(t) + \lambda p(\delta Z(t) + \tilde{Z}^p(t)), t)d\lambda,$$
$$B^p(t) = \int_0^1 f_Z(X^*(t) + \lambda p(\delta X(t) + \tilde{X}^p(t)), Z^*(t) + \lambda p(\delta Z(t) + \tilde{Z}^p(t)), t)d\lambda,$$
$$C^p(t) = [A^p(t) - f_X(X^*(t), Z^*(t), t)]\delta X(t) + [B^p(t) - f_Z(X^*(t), Z^*(t), t)]\delta Z(t).$$

Thus

$$\begin{cases} -d\tilde{X}^p(t) = (A^p(t)\cdot \tilde{X}^p(t) + B^p(t)\cdot \tilde{Z}^p(t) + C^p(t))dt - \tilde{Z}^p(t)dW_0(t), \\ \tilde{X}^p(T) = 0. \end{cases}$$

Using Itô's formula to $\mid \tilde{X}^p(t) \mid^2$, we get

$$E_0 \mid \tilde{X}^p(t) \mid^2 + E_0 \int_t^T \mid \tilde{Z}^p(s) \mid^2 ds$$
$$= 2E_0 \int_t^T \tilde{X}^p(s)(A^p(s)\cdot \tilde{X}^p(s) + B^p(s)\cdot \tilde{Z}^p(s) + C^p(s))ds$$

$$\leq KE_0 \int_t^T | \tilde{X}^p(s) |^2 \, ds + \frac{1}{2} E_0 \int_t^T | \tilde{Z}^p(s) |^2 \, ds + E_0 \int_t^T | C^p(s) |^2 \, ds,$$

where K is a constant.

So

$$E_0 | \tilde{X}^p (t) |^2 + \frac{1}{2} E_0 \int_t^T | \tilde{Z}^p(s) |^2 \, ds \leq KE_0 \int_t^T | \tilde{X}^p(s) |^2 \, ds + E_0 \int_t^T | C^p(s) |^2 \, ds.$$

Using the Lebesgue dominate convergence theorem, we have

$$\lim_{p \to 0} E_0 \int_0^T | C^p(t) |^2 \, dt = 0.$$

Applying Grownwall's inequality, we obtain the result.

2.3 Variational Inequality

In this section, we will employ the well-known Ekeland's variational principle to get the variational inequality.

Ekeland's Variational Principle *Let $(V, d(\cdot, \cdot))$ be a complete metric space and let $F(\cdot) : V \to R$ be a lower semi-continuous function, bounded from below. If for some $\varepsilon > 0$, there exists $u \in V$ satisfying*

$$F(u) \leq \inf_{v \in V} F(v) + \varepsilon,$$

then there exists $u_\varepsilon \in V$ such that

(i) $F(u_\varepsilon) \leq F(u)$,

(ii) $d(u, u_\varepsilon) \leq \sqrt{\varepsilon}$,

(iii) $F(v) + \sqrt{\varepsilon} d(v, u_\varepsilon) \geq F(u_\varepsilon)$ *for all* $v \in V$.

We now introduce a mapping $F_\varepsilon(\cdot) : U \to R$ by

$$F_\varepsilon(\xi) = \{ | X_0^\xi - x |^2 + [E_0(u(\xi^*) - u(\xi)) + \varepsilon]^2 \}^{\frac{1}{2}},$$

where the complete metric space U is defined in Definition 2.1, x is the given initial state constraint with X_0^ξ, ε is an arbitrary positive constant and ξ^* is the optimal objective.

Lemma 2.4 *$F_\varepsilon(\cdot)$ is a continuous function on U.*

Using the method similar to that for Lemma 2.3, it is easy to prove this lemma.

Lemma 2.5 *Let ξ^* be the optimal objective. Supposing* (H1)–(H4), *then there exist $h^0 \in R$, $h \in R$, for each $\hat{\xi}$ such that $(\xi^* + \hat{\xi}) \in U$ and the following variational inequality holds*

$$h^0 \cdot \delta X_0 + hE[u_X(\xi^*) \cdot \hat{\xi}] \geq 0, \tag{2.3}$$

where δX_0 is the value of (2.2) *at time* 0.

Proof We consider function $F_\varepsilon(\cdot)$ and it is easy to check that

$$F_\varepsilon(\xi^*) = \varepsilon;$$
$$F_\varepsilon(\xi) > 0, \ \forall \, \xi \in U;$$
$$F_\varepsilon(\xi^*) \leq \inf_{\xi \in U} F_\varepsilon(\xi) + \varepsilon.$$

Thus from Ekeland variational principle, there exists $\xi^\varepsilon \in U$, such that

(i) $F_\varepsilon(\xi^\varepsilon) \leq F_\varepsilon(\xi^*)$,

(ii) $d(\xi^\varepsilon, \xi^*) \leq \sqrt{\varepsilon}$,

(iii) $F_\varepsilon(\xi) + \sqrt{\varepsilon} d(\xi, \xi^\varepsilon) \geq F_\varepsilon(\xi^\varepsilon)$.

For each $\bar{\xi} \in U$, we denote $\hat{\xi} = \bar{\xi} - \xi^*$ by $\hat{\xi}$.

Set

$$\hat{\xi}^\varepsilon = \bar{\xi} - \xi^\varepsilon,$$
$$\xi_p^\varepsilon = \xi^\varepsilon + p\hat{\xi}^\varepsilon.$$

Let $(X_p^\varepsilon(\cdot), Z_p^\varepsilon(\cdot))$ (resp. $(X^\varepsilon(\cdot), Z^\varepsilon(\cdot))$) be the solution of (2.1) corresponding to ξ_p^ε (resp. ξ^ε). From Ekeland's variational principle, it follows that

$$F_\varepsilon(\xi_p^\varepsilon) + \sqrt{\varepsilon} d(\xi_p^\varepsilon, \xi^\varepsilon) - F_\varepsilon(\xi^\varepsilon) \geq 0, \tag{2.4}$$

where

$$d(\xi_p^\varepsilon, \xi^\varepsilon) = (E_0 \mid p\hat{\xi}^\varepsilon \mid^2)^{\frac{1}{2}} = p(E_0 \mid \hat{\xi}^\varepsilon \mid^2)^{\frac{1}{2}}.$$

We consider the variational equation

$$\begin{cases} -d\delta X^\varepsilon(t) = [f_X(X^\varepsilon(t), Z^\varepsilon(t), t)\delta X^\varepsilon(t) + f_Z(X^\varepsilon(t), Z^\varepsilon(t), t)\delta Z^\varepsilon(t)]dt - \delta Z^\varepsilon(t) dW_0(t), \\ \delta X^\varepsilon(T) = \hat{\xi}^\varepsilon, \end{cases}$$

and from Lemma 2.3, we get

$$\lim_{p\to 0} \sup_{0\leq t\leq T} E_0 \left| \frac{X_p^\varepsilon(t) - X^\varepsilon(t)}{p} - \delta X^\varepsilon(t) \right| = 0.$$

Thus

$$X_0^{\xi_p^\varepsilon} - X_0^{\xi^\varepsilon} = p\delta X_0^\varepsilon + o(p).$$

This leads to the following expansion

$$\mid X_0^{\xi_p^\varepsilon} - x \mid^2 - \mid X_0^{\xi^\varepsilon} - x \mid^2 = 2p(X_0^{\xi^\varepsilon} - x)\delta X_0^\varepsilon + o(p).$$

Since

$$\begin{aligned} &[E_0(u(\xi^*) - u(\xi_p^\varepsilon)) + \varepsilon]^2 - [E_0(u(\xi^*) - u(\xi^\varepsilon)) + \varepsilon]^2 \\ &\quad = -2pE_0[u_X(\xi^\varepsilon) \cdot \hat{\xi}^\varepsilon][E_0(u(\xi^*) - u(\xi^\varepsilon)) + \varepsilon] + o(p), \end{aligned}$$

we have

$$\begin{aligned} \lim_{p\to 0} \frac{F_\varepsilon(\xi_p^\varepsilon) - F_\varepsilon(\xi^\varepsilon)}{p} &= \lim_{p\to 0} \frac{1}{F_\varepsilon(\xi_p^\varepsilon) + F_\varepsilon(\xi^\varepsilon)} \frac{F_\varepsilon^2(\xi_p^\varepsilon) - F_\varepsilon^2(\xi^\varepsilon)}{p} \\ &= \frac{1}{F_\varepsilon(\xi^\varepsilon)} \{(X_0^{\xi^\varepsilon} - x)\delta X_0^\varepsilon - E_0[u_X(\xi^\varepsilon) \cdot \hat{\xi}^\varepsilon][E_0(u(\xi^*) - u(\xi^\varepsilon)) + \varepsilon]\}. \end{aligned}$$

Set

$$h_\varepsilon^0 = \frac{X_0^{\xi^\varepsilon} - x}{F_\varepsilon(\xi^\varepsilon)},$$
$$h_\varepsilon = -\frac{1}{F_\varepsilon(\xi^\varepsilon)}[E_0(u(\xi^*) - u(\xi^\varepsilon)) + \varepsilon].$$

From (2.4),

$$h_\varepsilon^0 \cdot \delta X_0^\varepsilon + h_\varepsilon E_0[u_X(\xi^\varepsilon) \cdot \hat{\xi}^\varepsilon] \geq -\sqrt{\varepsilon}(E_0 \mid \hat{\xi}^\varepsilon \mid^2)^{\frac{1}{2}}. \tag{2.5}$$

According to the definition of $F_\varepsilon(\cdot)$, it is obvious that $\mid h_\varepsilon^0 \mid^2 + \mid h_\varepsilon \mid^2 = 1$.

Thus there exists a convergence subsequence of (h_ε^0, h^0). We denote the limit by (h^0, h). An analysis similar to that in Lemma 2.3 yields $\delta X_0^\varepsilon \to \delta X_0$. It is also easy to check that

$$E_0[u_X(\xi^\varepsilon) \cdot \hat{\xi}^\varepsilon] \to E_0[u_X(\xi^*) \cdot \hat{\xi}].$$

Let $\varepsilon \to 0$ in (2.5). It follows that (2.3) holds.

2.4 Maximum Principle

Now we introduce the adjoint equation

$$\begin{cases} dq(t) = f_X(X^*(t), Z^*(t), t)q(t)dt + f_Z(X^*(t), Z^*(t), t)q(t)dW_0(t), \\ q(0) = h^0, \end{cases} \tag{2.6}$$

where $(X^*(\cdot), Z^*(\cdot))$ is the corresponding optimal trajectory.

It is easily seen that there is a unique process in $M^2(0, T; R)$ which satisfies (2.6).

Let

$$M \triangleq \{\omega \mid \xi^*(\omega) = A(\omega)\}, \qquad N \triangleq \{\omega \mid \xi^*(\omega) = C(\omega)\}.$$

Theorem 2.6 *We assume* (H1)–(H4). *Let ξ^* be the optimal objective and let $(X^*(\cdot), Z^*(\cdot))$ be the corresponding optimal trajectory. Then there exist constants $h, h^0 \in R$ such that*

$$\begin{aligned} &hu_X(\xi^*(\omega)) + q_T(\omega) \geq 0, \quad \text{if} \;\; \xi^*(\omega) = A(\omega), \quad a.s. \\ &hu_X(\xi^*(\omega)) + q_T(\omega) = 0, \quad \text{if} \;\; A(\omega) < \xi^*(\omega) < C(\omega), \quad a.s. \\ &hu_X(\xi^*(\omega)) + q_T(\omega) \leq 0, \quad \text{if} \;\; \xi^*(\omega) = C(\omega), \quad a.s., \end{aligned}$$

where $q(t)$ is the solution of the adjoint equation (2.6).

Proof Applying Itô's lemma to $\delta X(t)q(t)$ yields

$$\begin{aligned} &E_0(\delta X(T) \cdot q(T) - \delta X_0 \cdot q(0)) \\ &= E_0\Big[- \int_0^T [(f_X(X^*(t), Z^*(t), t)\delta X(t) + f_Z'(X^*(t), Z^*(t), t)\delta Z(t))q(t)\Big]dt \\ &\quad + \int_0^T [(f_X(X^*(t), Z^*(t), t)\delta X(t)q(t) + < \delta Z(t), f_Z(X^*(t)(t), Z^*(t), t)q(t) >)]dt \\ &= 0. \end{aligned}$$

Thus

$$h^0 \cdot \delta X_0 = E_0[\hat{\xi} \cdot q(T)].$$

From (2.3), we have that for each $\bar{\xi} \in U$, the following inequality holds

$$\begin{aligned} E_0[\hat{\xi} \cdot q(T)] + hE_0[u_X(\xi^*) \cdot \hat{\xi}] &= E_0[(q(T) + hu_X(\xi^*)) \cdot \hat{\xi}] \\ &= E_0[(q(T) + hu_X(\xi^*)) \cdot (\bar{\xi} - \xi^*)] \\ &\geq 0. \end{aligned}$$

Let us consider the case where M is a nonempty set. It is easy to check for each $\varepsilon > 0$

$$P\{\omega \mid \omega \in M, hu_X(\xi^*(\omega)) + q_T(\omega) < -\varepsilon\} = 0.$$

Thus, from the continuous property of probability, we have

$$hu_X(\xi^*(\omega)) + q_T(\omega) \geq 0, \quad \forall\, \omega \in M, \text{ a.s.}$$

Using a similar method, we can prove

$$\begin{aligned} &hu_X(\xi^*(\omega)) + q_T(\omega) = 0, \text{ if } A(\omega) < \xi^*(\omega) < C(\omega), \text{ a.s.} \\ &hu_X(\xi^*(\omega)) + q_T(\omega) \leq 0, \text{ if } \xi^*(\omega) = C(\omega), \text{ a.s. .} \end{aligned}$$

The proof is complete.

3 Application in Dynamic Measure of Risk

In this section, we apply the results of Section 2 to study the dynamic measure of risk.

We assume in financial market there are one bank account (risk free instrument) and several stocks(risky instruments). The respective prices $S_0(\cdot)$ and $S_1(\cdot),\ldots,S_d(\cdot)$ of these financial instruments are governed by the equations

$$dS_0(t) = S_0(t)r(t)dt, \qquad S_0(0) = 1,$$
$$dS_i(t) = S_i(t)\Big[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^j(t)\Big], \quad S_i(0) = s_i > 0, i = 1,\ldots,d.$$

We assume:

(H5) The interest rate $r(\cdot)$ is a predictable and bounded process.

(H6) The stock-appreciation rates $b(\cdot) = (b_1(\cdot),\ldots,b_d(\cdot))'$ is a predictable and bounded process.

(H7) The stock-volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le d}$ is a predictable and bounded process. $\sigma(\cdot)$ is assumed to be invertible and $\sigma^{-1}(\cdot)$ is assumed to be bounded uniformly in $(t,\omega)\in[0,T]\times\Omega$.

Then, the risk premium process

$$\theta_0(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\tilde{1}], \quad 0\le t\le T, \tag{3.1}$$

where $\tilde{1} = (1,\ldots,1)' \in R^d$, is bounded and

$$Z_0(t) \triangleq \exp\Big[-\int_0^t \theta_0'(s)dW_0(s) - \frac{1}{2}\int_0^t \|\theta_0(s)\|^2 ds\Big], \quad 0\le t\le T \tag{3.2}$$

is a P_0-martingale. Thus the discounted stock prices $\frac{S_1(\cdot)}{S_0(\cdot)},\ldots,\frac{S_d(\cdot)}{S_0(\cdot)}$ become martingales and

$$W(t) \triangleq W_0(t) + \int_0^t \theta_0(s)ds, \ 0\le t\le T \tag{3.3}$$

becomes Brownian motion under the risk-neutral equivalent martingale measure

$$P(\Lambda) = E_0[Z_0(T)1_\Lambda], \quad \Lambda\in\mathscr{F}. \tag{3.4}$$

Given the above assumptions, we consider a small agent whose actions can't affect the market prices and who can decide at time $t\in[0,T]$ which amount $\pi_i(t)$ to invest in each of the stocks $i = 1,\ldots,d$ with initial capital x .

Let $X(\cdot) = X^{x,\pi}(\cdot)$ denote his wealth process. $X(\cdot)$ satisfies the equation

$$\begin{cases} dX(t) = \Big[X(t) - \sum_{i=1}^{d}\pi_i(t)\Big]r(t)\,dt + \sum_{i=1}^{d}\pi_i(t)\Big[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^j(t)\Big] \\ \qquad\quad = r(t)X(t)dt + \pi'(t)\sigma(t)dW(t), \\ X(0) = x. \end{cases} \tag{3.5}$$

Definition 3.1 (i) *A process $\pi:[0,T]\times\Omega\to R^d$ is called portfolio process if it is a predictable process and satisfies $E_0\int_0^T\|\pi(t)\|^2dt<\infty$.*

(ii) *The process $X(\cdot) = X^{x,\pi}(\cdot)$ defined by* (3.5) *is called the wealth process corresponding to portfolio $\pi(\cdot)$ and initial capital x.*

(iii) *Given a random variable $A\in L^2(\Omega,\mathscr{F}_T,P)$, a portfolio process $\pi(\cdot)$ is said admissible for the initial capital x, and we write $\pi(\cdot)\in\mathscr{A}(x)$, if*

$$X^{x,\pi}(t) \ge S_0(t)\cdot E\Big[\frac{A}{S_0(T)}\Big|\mathscr{F}(t)\Big] \triangleq A(t), \quad 0\le t\le T \tag{3.6}$$

holds almost surely. Here E denotes expectation with respect to the probability measure P of (3.4).

Then we study the problem of dynamic measures of risk in [18].

Suppose that the total liabilities of the investor at time T are described by a contingent claim C : a random variable in $L^2(\Omega, \mathscr{F}_T, P)$, with $P[C \geq A] = 1$ and $P[C > A] > 0$.

Define

$$\frac{C(t)}{S_0(t)} \triangleq E\left[\frac{C}{S_0(T)}\middle|\mathscr{F}(t)\right],\ 0 \leq t \leq T. \tag{3.7}$$

From the standard results on complete financial markets [18], we know that we can't hedge the liability C perfectly if $A(0) \leq x < C(0)$. In this case, we shall adopt the following value function of the stochastic control problem introduced by Cvitanic and Karatzas [18]

$$V_0(x) \triangleq V_0(x; C) \triangleq \inf_{\pi(\cdot)\in\mathscr{A}(x)} E_0\left(\frac{C - X^{x,\pi}(T)}{S_0(T)}\right)^+ \tag{3.8}$$

(least expected discounted net loss, over all admissible portfolios) as a reasonable measure of risk.

Now we give an equivalent form of the problem (3.8) in order to resolve it by results in Section 2. Given the investor's terminal wealth $X(T) = \xi$ ($\xi \in L^2(\Omega, \mathscr{F}_{\mathscr{T}}, P)$), we can describe the wealth process of the investor by BSDE:

$$\begin{cases} dX(t) = \left[X(t) - \sum_{i=1}^{d}\pi_i(t)\right] r(t)\, dt + \sum_{i=1}^{d}\pi_i(t)\left[b_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_0^j(t)\right] \\ \qquad\quad = r(t)X(t)dt + \pi'(t)\sigma(t)dW(t), \\ X(T) = \xi. \end{cases} \tag{3.9}$$

Thanks to assumptions (H5), (H6) and (H7), then equation (3.9) has a unique pair of square-integrable solution $(X(\cdot), \pi(\cdot))$ such that $(X(\cdot), \pi(\cdot)) \in M^2(0, T; R) \times M^2(0, T; R^d)$ (see [6]). We denote by $(X^\xi(\cdot), \pi^\xi(\cdot))$ the solution of (3.9) with respect to the terminal condition $X(T) = \xi$.

Definition 3.2 *Given a random variable $A \in L^2(\Omega, \mathscr{F}_{\mathscr{T}}, P)$ and the initial wealth x, the terminal wealth ξ is called admissible for the initial wealth x, and we write $\xi \in \mathscr{A}'(x)$, if $X^\xi(0) = x$ and $\xi \geq A, a.s.$*

Lemma 3.3 (i) *We assume $\xi \in \mathscr{A}'(x)$. Then we have $\pi^\xi(\cdot) \in \mathscr{A}(x)$, where $(X^\xi(\cdot), \pi^\xi(\cdot))$ is the solution of* (3.9).

(ii) *We assume $\pi(\cdot) \in \mathscr{A}(x)$. Then we have $X^{x,\pi}(T) \in \mathscr{A}'(x)$, where $X^{x,\pi}(\cdot)$ is the solution of* (3.5).

Proof (i) Let $\xi \in \mathscr{A}'(x)$. By Definition 3.2, $X^\xi(0) = x, X^\xi(T) = \xi \geq A$, a.s.

Then $(X^\xi(\cdot), \pi^\xi(\cdot))$ satisfies the following forward stochastic differential equation

$$\begin{cases} dX^\xi(t) = r(t)X^\xi(t)dt + [\pi^\xi(t)]^T\sigma(t)dW(t), \\ X^\xi(0) = x. \end{cases}$$

So $X^\xi(\cdot) = X^{x,\pi^\xi}(\cdot)$.

Let $X(T) = A$ in equation (3.9). We have an explicit solution for $X^A(\cdot)$

$$X^A(t) = S_0(t) \cdot E\left[\frac{A}{S_0(T)}\middle|\mathscr{F}(t)\right] = A(t), \quad 0 \leq t \leq T.$$

Since $\xi \geq A$, it yields $X^\xi(t) = X^{x,\pi^\xi}(t) \geq X^A(t) = A(t)$ according to the comparison theorem of BSDE in [6]. By Definition 3.1, $\pi^\xi(\cdot)$ is admissible.

(ii) Let $\pi(\cdot) \in \mathscr{A}(x)$. According to Definition 3.1, $X^{x,\pi}(0) = x, X^{x,\pi}(T) \geq A$, a.s.

We consider equation (3.5), $(X^{x,\pi}(\cdot), \pi(\cdot))$ satisfies the following BSDE

$$\begin{cases} dX(t) = r(t)X(t)dt + \pi'(t)\sigma(t)dW(t), \\ X(T) = X^{x,\pi}(T). \end{cases}$$

By Definition 3.2, $X^{x,\pi}(T) \in \mathscr{A}'(x)$. The proof is complete.

Remark 3.4 By Lemma 3.3, we have the equivalent value function of problem (3.8):

$$V_0(x) = \inf_{\xi \in \mathscr{A}'(x)} E_0 \left(\frac{C - \xi}{S_0(T)} \right)^+ .$$

Definition 3.5 *ξ^* is called optimal objective if $\xi^* \in \mathscr{A}'(x)$ and $V_0(x) = E_0(\frac{C-\xi^*}{S_0(T)})^+$.*

Lemma 3.6 *We assume* (H5)–(H7). *Then if the optimal objective ξ^* exists, it satisfies $\xi^* \leq C$.*

Proof Set $B \triangleq \{\omega \mid \xi^*(\omega) > C(\omega)\}$.

Suppose $P(B) > 0$. Then we have $0 < P(B) < 1$.

Set

$$\xi'(\omega) \triangleq \begin{cases} \xi^*(\omega), & \omega \in B^C, \\ C(\omega), & \omega \in B. \end{cases}$$

We have

$$E_0 \left(\frac{C - \xi^*}{S_0(T)} \right)^+ = E_0 \left(\frac{C - \xi'}{S_0(T)} \right).$$

Since $\xi^*(\omega) > \xi'(\omega)$, it follows that $X^{\xi^*}(0) = x > X^{\xi'}(0) \triangleq x'$ by the comparison theorem of BSDE.

$X^{\xi'}(\cdot)$ satisfies the following forward stochastic differential equation:

$$\begin{cases} dX^{\xi'}(t) = r(t)X^{\xi'}(t)dt + [\pi^{\xi'}(t)]^T\sigma(t)dW(t), \\ X^{\xi'}(0) = x'. \end{cases} \tag{3.10}$$

We consider the equation

$$\begin{cases} dy(t) = r(t)y(t)dt, \\ y(0) = x - x'. \end{cases} \tag{3.11}$$

Because $x > x'$, (3.11) has a positive solution

$$y(t) = (x - x')\exp\left[\int_0^t r(s)ds\right] = (x - x')S_0(T).$$

From (3.10) and (3.11), we obtain

$$\begin{cases} d(X^{\xi'}(t) + y(t)) = r(t)(X^{\xi'}(t) + y(t))dt + [\pi^{\xi'}(t)]^T\sigma(t)dW(t), \\ X^{\xi'}(0) + y(0) = (x - x') + x' = x. \end{cases}$$

Since $X^{\xi'}(0) + y(0) = x$ and $X^{\xi'}(T) + y(T) = \xi' + (x - x')S_0(T) \triangleq \eta \geq A$, η is an admissible objective.

$$\begin{aligned} E_0\left(\frac{C-\eta}{S_0(T)}\right)^+ &= E_0\left(\frac{C-\xi'}{S_0(T)}-(x-x')\right)^+ \\ &= E_0\left(\left(\frac{C-\xi^*}{S_0(T)}-(x-x')\right)1_{B^C}\right)^+ \\ &< E_0\left(\left(\frac{C-\xi^*}{S_0(T)}\right)1_{B^C}\right)^+ \\ &= E_0\left(\frac{C-\xi^*}{S_0(T)}\right)^+. \end{aligned}$$

This contradicts the fact that ξ^* is the optimal objective. The proof is complete.

Remark 3.7 Lemma 3.6 implies that we can define

$$V_0(x) \triangleq V_0(x;C) \triangleq \inf_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{C-\xi}{S_0(T)}\right),$$

where

$$\mathscr{A}^0(x) \triangleq \{\xi \mid X^\xi(0)=x, \quad A\le\xi\le C, \text{ a.s.}\}$$

and it doesn't change the stochastic optimal control problem (3.8).

Since

$$\begin{aligned} V_0(x) &= \inf_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{C-\xi}{S_0(T)}\right) \\ &= E_0\left(\frac{C}{S_0(T)}\right) - \inf_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{\xi}{S_0(T)}\right) \\ &= E_0\left(\frac{C}{S_0(T)}\right) + \sup_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{\xi}{S_0(T)}\right), \end{aligned}$$

it is sufficient to study

$$V_0'(x) \triangleq V_0'(x;C) \triangleq \sup_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{\xi}{S_0(T)}\right). \tag{3.12}$$

Now we know the stochastic optimal control problem (3.8) is equivalent to finding ξ^* which makes

$$E_0\left(\frac{\xi^*}{S_0(T)}\right) = \sup_{\xi\in\mathscr{A}^0(x)} E_0\left(\frac{\xi}{S_0(T)}\right).$$

This is the typical form of the optimization problem discussed in Section 2, so we can use the results in Section 2 to resolve this problem.

First, we describe (3.9) by BSDE

$$\begin{cases} -dX(t) = -[r(t)X(t)+Z(t)\theta_0(t)]dt - Z(t)dW_0(t), \\ X(T)=\xi, \end{cases} \tag{3.13}$$

where

$$Z(t)\triangleq \pi'(t)\sigma(t), \qquad \theta_0(t)=\sigma^{-1}(t)[b(t)-r(t)\tilde{1}], \quad 0\le t\le T.$$

Theorem 3.8 *We assume* (H5)–(H7). *Then if the optimal objective ξ^* exists, there exist a constant $\lambda\in R$ and a random variable U with $A\le U\le C$, a.s. such that*

$$\xi^* = C1_{[Z_0(T)<\lambda]} + A1_{[Z_0(T)>\lambda]} + U1_{[Z_0(T)=\lambda]}$$

or

$$\xi^* = C1_{[Z_0(T)>\lambda]} + A1_{[Z_0(T)<\lambda]} + U1_{[Z_0(T)=\lambda]}.$$

Proof In this case, $u(X)=\frac{X}{S_0(T)},\ f(X,Z,t)=-[r(t)X+\theta_0(t)Z],\ u_X(\xi^*)=\frac{1}{S_0(T)}$.

The adjoint equation is

$$\begin{cases} -dq(t)=r(t)q(t)dt+q(t)\theta_0'(t)dW_0(t), \\ q(0)=h^0. \end{cases} \tag{3.14}$$

(3.14) has an explicit solution,

$$\begin{aligned} q(t) &= h^0\exp\Big[-\int_0^t r(s)ds-\int_0^t \theta_0'(s)dW_0(s)-\frac{1}{2}\int_0^t \|\theta_0(s)\|^2ds\Big] \\ &= \frac{h^0}{S_0(t)}Z_0(t). \end{aligned}$$

By Theorem 2.6, the necessary condition becomes

$$\begin{aligned} &\frac{h}{S_0(T)}+\frac{h^0}{S_0(T)}Z_0(T)\geq 0, \quad \text{if}\ \ \xi^*(\omega)=A(\omega), \quad \text{a.s.} \\ &\frac{h}{S_0(T)}+\frac{h^0}{S_0(T)}Z_0(T)= 0, \quad \text{if}\ \ A(\omega)<\xi^*(\omega)<C(\omega), \quad \text{a.s.} \\ &\frac{h}{S_0(T)}+\frac{h^0}{S_0(T)}Z_0(T)\leq 0, \quad \text{if}\ \ \xi^*(\omega)=C(\omega), \quad \text{a.s.} \end{aligned}$$

(i) We assume $h^0>0$. Let $\lambda\triangleq\frac{-h}{h^0}$.

The necessary condition becomes $\exists\lambda\in R$,

$$\begin{cases} Z_0(T)\geq\lambda, & \text{if}\ \ \xi^*(\omega)=A(\omega), \quad \text{a.s.} \\ Z_0(T)=\lambda, & \text{if}\ \ A(\omega)<\xi^*(\omega)<C(\omega), \quad \text{a.s.} \\ Z_0(T)\leq\lambda, & \text{if}\ \ \xi^*(\omega)=C(\omega). \quad \text{a.s.} \end{cases}$$

Thus the optimal objective ξ^* has the form:

$$\xi^*=C1_{[Z_0(T)<\lambda]}+A1_{[Z_0(T)>\lambda]}+U1_{[Z_0(T)=\lambda]}.$$

(ii) We assume $h^0<0$. The necessary condition is

$$\begin{cases} Z_0(T)\leq\lambda, & \text{if}\ \ \xi^*(\omega)=A(\omega), \quad \text{a.s.} \\ Z_0(T)=\lambda, & \text{if}\ \ A(\omega)<\xi^*(\omega)<C(\omega), \quad \text{a.s.} \\ Z_0(T)\geq\lambda, & \text{if}\ \ \xi^*(\omega)=C(\omega). \quad \text{a.s.} \end{cases}$$

Then the optimal objective ξ^* has the form:

$$\xi^*=C1_{[Z_0(T)>\lambda]}+A1_{[Z_0(T)<\lambda]}+U1_{[Z_0(T)=\lambda]}.$$

(iii) We assume $h^0=0$. Then we have $h^2=1$. It is easy to check that we can't find $\xi\in U$ which satisfies the necessary condition and the initial constraint. The proof is complete.

Theorem 3.9 *We assume* (H5)–(H7). *Then* $\xi^*=C1_{[Z_0(T)<\hat\lambda]}+A1_{[Z_0(T)>\hat\lambda]}+U1_{[Z_0(T)=\hat\lambda]}$ *is the optimal objective if there exists* $\hat\lambda\in R^+$ *such that*

$$E\Big[\frac{C}{S_0(T)}1_{[Z_0(T)<\hat\lambda]}+\frac{A}{S_0(T)}1_{[Z_0(T)>\hat\lambda]}+\frac{U}{S_0(T)}1_{[Z_0(T)=\hat\lambda]}\Big]=x.$$

Proof For any $\eta\in\mathscr{A}^0(x)$, we have

$$E\Big(\frac{\xi^*}{S_0(T)}-\frac{\eta}{S_0(T)}\Big)=E_0\Big(Z_0(T)\Big[\frac{\xi^*}{S_0(T)}-\frac{\eta}{S_0(T)}\Big]\Big)$$

$$= E_0\left(Z_0(T)\left[\frac{C-\eta}{S_0(T)}1_{[Z_0(T)<\hat\lambda]}+\frac{A-\eta}{S_0(T)}1_{[Z_0(T)>\hat\lambda]}+\frac{U-\eta}{S_0(T)}1_{[Z_0(T)=\hat\lambda]}\right]\right)$$
$$\le \hat\lambda E_0\left[\frac{C-\eta}{S_0(T)}1_{[Z_0(T)<\hat\lambda]}+\frac{A-\eta}{S_0(T)}1_{[Z_0(T)>\hat\lambda]}+\frac{U-\eta}{S_0(T)}1_{[Z_0(T)=\hat\lambda]}\right]$$
$$= \hat\lambda E_0\left(\frac{\xi^*}{S_0(T)}-\frac{\eta}{S_0(T)}\right).$$

Since $\xi^*\in\mathscr{A}^0(x)$ and $E\big(\frac{\eta}{S_0(T)}\big)=x$, it follows that $E\big(\frac{\xi^*}{S_0(T)}-\frac{\eta}{S_0(T)}\big)=0$. Thus

$$E_0\left(\frac{\xi^*}{S_0(T)}\right)\ge E_0\left(\frac{\eta}{S_0(T)}\right),\quad \forall\,\eta\in\mathscr{A}^0(x).$$

The proof is complete.

Remark 3.10 *This result gives a sufficient condition for the optimal objective.*

Theorem 3.11 *We assume that there exists $\tilde\lambda\in R^+$ such that*

$$E\left[\frac{C}{S_0(T)}1_{[Z_0(T)>\tilde\lambda]}+\frac{A}{S_0(T)}1_{[Z_0(T)<\tilde\lambda]}+\frac{U}{S_0(T)}1_{[Z_0(T)=\tilde\lambda]}\right]=x.$$

Then $\xi^*=C1_{[Z_0(T)>\tilde\lambda]}+A1_{[Z_0(T)<\tilde\lambda]}+U1_{[Z_0(T)=\tilde\lambda]}$ makes $E_0\big(\frac{\xi}{S_0(T)}\big)$ attain minimum over $\mathscr{A}^0(x)$.

Proof This follows by the same method as in Theorem 3.9.

Remark 3.12 Set

$$\hat\lambda\triangleq\sup\left\{\lambda\in R^+\,\middle|\,E\left[\frac{C}{S_0(T)}1_{[Z_0(T)<\lambda]}+\frac{A}{S_0(T)}1_{[Z_0(T)\ge\lambda]}\right]\le x\right\}.\tag{3.15}$$

According to proposition 3.1 of [18], we know that for every $x\in(A(0),C(0))$ and $\hat\lambda$ as in (3.15), there exists an $\mathscr{F}_T$ measurable random variable U with $A\le U\le C$ such that

$$E\left[\frac{C}{S_0(T)}1_{[Z_0(T)<\hat\lambda]}+\frac{A}{S_0(T)}1_{[Z_0(T)>\hat\lambda]}+\frac{U}{S_0(T)}1_{[Z_0(T)=\hat\lambda]}\right]=x.\tag{3.16}$$

By Theorem 3.9, we know that the optimal objective exists for this stochastic control problem.

4 Nonlinear Case

In this section, we consider the stochastic control problem (3.8) where the wealth process is described by a nonlinear BSDE. Here we employ all assumptions in Section 2.

Corresponding to (3.9), the wealth process satisfies

$$\begin{cases} -dX(t)=f(X(t),Z(t),t)dt-Z(t)dW_0(t),\\ X(T)=\xi,\end{cases}\tag{4.1}$$

where

$$f:R\times R^d\times[0,T]\times\Omega\to R$$

is a nonlinear function and satisfies assumptions (H1)–(H4).

We denote by $(A(\cdot),Z^A(\cdot))$ (resp. $(C(\cdot),Z^C(\cdot))$) the solution of (4.1) with respect to $X(T)=A$ (resp. $X(T)=C$). Suppose $A(0)<x<C(0)$.

Definition 4.1 *Given a random variable $A\in L^2(\Omega,\mathscr{F}(T),P_0)$ and the initial wealth x, the terminal wealth ξ corresponding to (4.1) is called admissible for the initial wealth x, and we write $\xi\in\mathscr{N}'(x)$, if $X^\xi(0)=x$ and $\xi\ge A$, a.s.*

We still suppose, at time T, the agent's liability is C: a random variable in $L^2(\Omega, \mathscr{F}_T, P_0)$ with $P_0[C \geq A] = 1$ and $P_0[C > A] > 0$.

Then we can give the value function of the extended stochastic control problem

$$V_0(x) \triangleq V_0(x; C) \triangleq \inf_{\xi \in \mathscr{N}'(x)} E_0 \left(\frac{C - \xi}{S_0(T)} \right)^+ \tag{4.2}$$

which means the least expected discounted net loss of the agent.

Definition 4.2 *ξ^* is called optimal objective, if $\xi^* \in \mathscr{N}'(x)$ and $V_0(x) = E_0(\frac{C-\xi^*}{S_0(T)})^+$.*

Lemma 4.3 *We assume* (H1)–(H4)*. Then if ξ^* exists, it satisfies $\xi^* \leq C$.*

Proof Set $B_0 \triangleq \{\omega \mid \xi^*(\omega) > C(\omega)\}$.

Suppose $P_0(B_0) > 0$. Let

$$\hat{\xi}(\omega) \triangleq \begin{cases} \xi^*(\omega), & \omega \in B_0^C, \\ C(\omega), & \omega \in B_0. \end{cases}$$

Thus

$$E_0 \left(\frac{C - \xi^*}{S_0(T)} \right)^+ = E_0 \left(\frac{C - \hat{\xi}}{S_0(T)} \right)^+ = E_0 \left(\frac{C - \hat{\xi}}{S_0(T)} \right).$$

We denote by $(X^{\hat{\xi}}(\cdot), Z^{\hat{\xi}}(\cdot))$ the solution of (4.1) corresponding to $X(T) = \hat{\xi}$.

Because $\xi^* > \hat{\xi}$, we have

$$X^{\xi^*}(0) = x > X^{\hat{\xi}}(0) \triangleq x'$$

by the comparison theorem of BSDE.

$X^{\hat{\xi}}(\cdot)$ satisfies the following forward stochastic differential equation:

$$\begin{cases} dX^{\hat{\xi}}(t) = f(X^{\hat{\xi}}(t), Z^{\hat{\xi}}(t), t)dt + Z^{\hat{\xi}}(t)dW_0(t), \\ X^{\hat{\xi}}(0) = x'. \end{cases}$$

Suppose $X(\cdot)$ satisfies the equation:

$$\begin{cases} dX(t) = f(X(t), Z^{\hat{\xi}}(t), t)dt + Z^{\hat{\xi}}(t)dW_0(t), \\ X(0) = x. \end{cases}$$

Since $x > x'$, it follows that $X(t) > X^{\hat{\xi}}(t), 0 \leq t \leq T$ according to the comparison theorem of stochastic differential equation.

Thus we have

$$\begin{aligned} &X(T) > X^{\hat{\xi}}(T) = \hat{\xi}, \\ &E_0 \left(\frac{C(\omega) - \xi^*(\omega)}{S_0(T)} \right)^+ = E_0 \left(\frac{C(\omega) - \hat{\xi}(\omega)}{S_0(T)} \right)^+ \\ &\qquad > E_0 \left(\frac{C(\omega) - X(T)}{S_0(T)} \right)^+. \end{aligned}$$

This contradicts the fact that ξ^* is the optimal objective. The proof is complete.

Remark 4.4 Applying this theorem , we can define

$$V_0(x) \triangleq V_0(x; C) \triangleq \inf_{\xi \in \mathscr{N}(x)} E_0 \left(\frac{C - \xi}{S_0(T)} \right),$$

where $\mathscr{N}(x) \triangleq \{\xi \mid X^{\xi}(0) = x, \ A \leq \xi \leq C, \ \text{a.s.}\}$ is defined in Section 2.

Thus, it is sufficient to study

$$V_0(x) \triangleq V_0(x;C) \triangleq \max_{\xi \in \mathcal{N}(x)} E_0\left(\frac{\xi}{S_0(T)}\right).$$

Theorem 4.5 *We assume* (H1)–(H4). *Then if the optimal objective* ξ^* *exists , there exist a constant* $\lambda \in R$ *and a random variable* U *with* $A \le U \le C$ *such that*

$$\xi^* = C1_{[q_T(\omega)S_0(T)<\lambda]} + A1_{[q_T(\omega)S_0(T)>\lambda]} + U1_{[q_T(\omega)S_0(T)=\lambda]}.$$

Proof Notice $u(x) = E_0(\frac{\xi}{S_0(T)})$.

According to Theorem 2.6, the optimal objective satisfies the following necessary condition:

$$\begin{cases} \dfrac{h}{S_0(T)} + q_T(\omega) \ge 0, & \text{if } \xi^*(\omega) = A(\omega), \quad \text{a.s.} \\ \dfrac{h}{S_0(T)} + q_T(\omega) = 0, & \text{if } A(\omega) < \xi^*(\omega) < C(\omega), \quad \text{a.s.} \\ \dfrac{h}{S_0(T)} + q_T(\omega) \le 0, & \text{if } \xi^*(\omega) = C(\omega), \quad \text{a.s.,} \end{cases}$$

where q_T is the value of the solution of the adjoint equation at $t = T$:

$$\begin{cases} dq_t = f_x(X_t^*, Z_t^*, t)q_t dt + f_z(X_t^*, Z_t^*, t)q_t dW_0(t), \\ q(0) = h^0. \end{cases}$$

Here (X_t^*, Z_t^*) is the optimal trajectory corresponding to ξ^*.

Set $\lambda = -h \in R$. We have

$$\begin{cases} q_T(\omega)S_0(T) \ge \lambda, & \text{if } \xi^*(\omega) = A(\omega), \quad \text{a.s.} \\ q_T(\omega)S_0(T) = \lambda, & \text{if } A(\omega) < \xi^*(\omega) < C(\omega), \quad \text{a.s.} \\ q_T(\omega)S_0(T) \le \lambda, & \text{if } \xi^*(\omega) = C(\omega), \quad \text{a.s.} \end{cases}$$

Thus the optimal objective ξ^* has the form:

$$\xi^* = C1_{[q_T S_0(T)<\lambda]} + A1_{[q_T S_0(T)>\lambda]} + U1_{[q_T S_0(T)=\lambda]}.$$

The proof is completed.

References

[1] Bismut, J.: Analyse convexe et probabilités. Thèse de Faculté des Sciences de Paris, Paris, 1973

[2] Bismut, J.: Théorie probabiliste du contrôle des diffusions. *Mem. Amer. Math. Soc.*, **4**, 1–130 (1976)

[3] Bismut, J.: An introductory approach to duality in optimal stochastic control. *SIAM Rev.*, **20**, 62–78 (1978)

[4] Pardoux, E., Peng, S. G.: Adapted solution of a Backward stochastic differential equations. *System and Control Letters*, **14**, 55–61 (1990)

[5] Duffie, D., Epstein, L.: Stochastic Differential utility. *Econometrica*, **60**, 353–394 (1992)

[6] El Karoui, N., Peng, S. G., Quenez, M. C.: Backward Stochastic Differential Equation in Finance. *Math. Finance*, **7**, 1–71 (1997)

[7] Hamadene, S., Lepeltier, J. P.: Zero-sum stochastic differential games and backward equations. *System and Control Letters*, **24**, 259–263 (1995)

[8] Kohlmann, M., Zhou, X. Y.: Relationship between and stochastic controls: A linear-quadratic approach. *SIAM J. Control Optim.*, **38**, 1392–1407 (2000)

[9] Tang, S. J., Li, X. J.: Necessary condition for optimal control of stochastic systems with random jumps. *SIAM.J. Control Optim.*, **32**, 1447–1475 (1994)

[10] Xu, W. S.: Stochastic maximum principle for optimal control problem of forward and backward system. *J. Austral. Math. Soc. Ser. B*, **37**, 172–185 (1995)

[11] Jiang, L.: A Property of g-Expectation. *Acta Mathematica Sinica, English Series*, **20**(5), 769–778 (2004)

[12] Liu, D. B.: Mean-variance Hedging for Pricing European-type Contingent Claims with Transaction Costs. *Acta Mathematica Sinica, English Series*, **19**(4), 655–670 (2003)

[13] Yong, J. M., Zhou, X. Y.: Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999

[14] El Karoui, N., Peng, S. G., Quenez, M. C.: Backward Stochastic Differential Equation Finance and Optimization, Ph. D. Thesis of Paris VI University, 1993

[15] Ji, S. L.: A stochastic optimization problem with state constraints and its applications in finance, Ph. D. Thesis of Shandong University, 1999

[16] Föllmer, H., Leukert, P.: Quantile hedging. *Finance and Stochastics*, **3**, 251–273 (1999)

[17] Föllmer, H., Leukert, P.: Efficient hedges: cost versus shortfall risk. *Finance and Stochastics*, **4**, 117–146 (2000)

[18] Cvitanic, J., Karatzas, I.: On dynamic measures of risk. *Finance and Stochastics*, **3**(4), 451–482 (1999)

[19] Edirisinghe, C., Naik, V., Uppal, R.: Optimal replication of options with transaction costs and trading restrictions. *J. Financ* & *Quantit Anal.*, **28**, 117–138 (1993)

[20] Peng, S. G.: Backward Stochastic Differential Equations and Applications to Optimal Control. *Appl. Math. Optim.*, **27**, 125–144 (1993)