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Quasi L_p-Intersection Bodies

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Abstract The purpose of this paper is to generalize the notion of intersection bodies to that of quasi L_p-intersection bodies. The L_p-analogs of the Busemann intersection inequality and the Brunn–Minkowski inequality for the quasi L_p-intersection bodies are obtained. The Aleksandrov–Fenchel inequality for the mixed quasi L_p-intersection bodies is also established.

Keywords quasi L_p-intersection bodies, Busemann intersection inequality, Brunn–Minkowski inequality, Aleksandrov–Fenchel inequality

MR(2000) Subject Classification 52A39, 52A40

1 Introduction

Projection bodies, centroid bodies and intersection bodies have a long and complicated history [1–10]. Projection bodies go back to Minkowski (see [11], Section 30, p. 50) The fact that affinely equivalent convex bodies have affinely equivalent projection bodies makes the Minkowski map especially interesting in Banach space theory. Centroid bodies were defined by Petty [7], but they appear in another guise in the work of Dupin, connected with floating bodies [12], and of Blaschke [13]. Intersection bodies were first explicitly defined and named by Lutwak in the important paper [4]. It was here that the duality between intersection bodies and projection bodies was first made clear. A number of important results regarding these bodies were proved in which there are three fundamental inequalities: the Petty projection inequality [9], the Busemann–Petty centroid inequality [7] and the Busemann intersection inequality [2].

The L_p-analogs of centroid bodies and projection bodies have received considerable attention in recent years [14–19]. Lutwak, Yang and Zhang established the L_p-analog of the Petty projection inequality [16]. It states that *if K is a convex body in $\mathbb{R}^n$, then for $1 < p < \infty$,*

$$V(\Pi_p^* K)V(K)^{(n-p)/p} \leq \omega_n^{n/p}, \tag{1.1}$$

with equality if and only if K is an ellipsoid.

Here $\Pi_p^* K = (\Pi_p K)^*$ is used to denote the polar body of the L_p-projection body, $\Pi_p K$, of K and write ω_n for $V(B_n)$, the n-dimensional volume of the unit ball B_n.

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They also established the L_p-analog of the Busemann–Petty centroid inequality [16]. It states that *if K is a star body (about the origin) in $\mathbb{R}^n$, then for $1 < p < \infty$,*

$$V(\Gamma_p K) \geq V(K), \tag{1.2}$$

with equality if and only if K is an ellipsoid centered at the origin.

Here $\Gamma_p K$, is the L_p-centroid body of K. Note that in the definitions of $\Pi_p K$ and $\Gamma_p K$, the normalization is chosen so that for the unit ball, B_n, there are $\Pi_p B_n = \Gamma_p B_n = B_n$.

A quite different proof of the L_p-analog of the Busemann–Petty centroid inequality is obtained by Campi and Gronchi [14].

The purpose of this paper is to generalize the notion of intersection bodies to that of quasi L_p-intersection bodies. Moreover, we will establish the L_p-analog of the Busemann intersection inequality which states that, *if K is a convex body in $\mathbb{R}^n$, for $1 < p < n$,*

$$V(I_p K) V(K)^{(p-n)/p} \leq \omega_n^{(2p-n)/p}; \tag{1.3}$$

if $p > n$, then the inequality is reversed; for $p \neq n$, the equality holds if and only if K is an origin-centered ball.

Here $I_p K$ is the quasi L_p-intersection body of K. Note that the normalization in the definition of quasi L_p-intersection bodies is also chosen so that for the unit ball, B_n, we have $I_p B_n = B_n$.

Remark From the definition of $I_p K$ in the next section, it's clear that this newly defined notion is the right generalization of the classical intersection body. On the other hand, $I_p K$ isn't affine associated when $p > 1$. So we call $I_p K$ the quasi L_p-intersection body of K.

The Brunn–Minkowski inequality and the monotone property for the quasi L_p-intersection bodies will also be established in this article. In the last section, we show the Aleksandrov-Fenchel inequality for the mixed quasi L_p-intersection bodies.

2 The L_p-analog of the Busemann Intersection Inequality

For quick reference we recall some basic results from the Brunn–Minkowski theory; good references are Gardner [20], Leichtweiß [21], Schneider [22], and Thompson [23].

Let S^{n-1} denote the unit sphere in Euclidean n-space, $\mathbb{R}^n$. Let B_n denote the origin-centered standard unit ball in $\mathbb{R}^n$, and write ω_n for $V(B_n)$, the n-dimensional volume of B_n. If $u \in S^{n-1}$, then $u^\perp$ is the $(n-1)$-dimensional subspace orthogonal to u.

If K is a convex body (i.e., compact convex subset with nonempty interior) in $\mathbb{R}^n$ that contains the origin in its interior, then we will use K^* to denote the *polar body* of K, i.e.,

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}. \tag{2.1}$$

A star body in $\mathbb{R}^n$ is a nonempty compact set K satisfying $[o, x] \subset K$ for all $x \in K$ such that the *radial function* $\rho_K(\cdot) = \rho(K, \cdot)$, defined by

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \tag{2.2}$$

is positive and continuous. Two star bodies K and L are said to be dilations if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a convex body in $\mathbb{R}^n$, then its *support function*, $h_K(\cdot) = h(K,\cdot) : \mathbb{R}^n \to R$, is defined for $x \in \mathbb{R}^n$ by $h(K,x) = \max\{x\cdot y : y \in K\}$. If K is a centered (i.e., symmetric about the origin) convex body, then it follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_K^* = \frac{1}{\rho_K} \quad \text{and} \quad \rho_K^* = \frac{1}{h_K}. \tag{2.3}$$

For $p \geq 1$, convex bodies K, L, and $\varepsilon > 0$ the L_p-Firey combination $K +_p \varepsilon L$ is defined as the convex body whose support function is given by

$$h(K +_p \varepsilon L, \cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p. \tag{2.4}$$

For $p \geq 1$, the *L_p-mixed volume*, $V_p(K,L)$, of the convex bodies K, L was defined in [24] by

$$\frac{n}{p}V_p(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K +_p \varepsilon L) - V(K)}{\varepsilon}. \tag{2.5}$$

It was shown in [24] that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u). \tag{2.6}$$

In order to define the I_pK appropriately, we introduce the notion of *L_p-dual mixed volume* first. For $p \geq 1$, star bodies $K,\ L$, and $\varepsilon > 0$ the radial L_p-Firey combination $K \tilde{+}_p\ \varepsilon \cdot L$ is defined as the star body whose radial function is given by

$$\rho(K \tilde{+}_p\ \varepsilon \cdot L,\ \cdot)^p = \rho(K,\ \cdot)^p + \varepsilon\rho(L,\ \cdot)^p. \tag{2.7}$$

For $p \geq 1$, the L_p-dual mixed volume, $\tilde{V}_p(K,L)$, of the star bodies K and L can be defined by

$$\frac{n}{p}\tilde{V}_p(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K \tilde{+}_p\ \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.8}$$

Lemma 2.1 *For star bodies $K,\ L$, and $p \geq 1$,*

$$\tilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-p}\rho_L(u)^p\ dS(u). \tag{2.9}$$

Proof When $p \geq 1$, from (2.7) and (2.8), it follows that

$$\begin{aligned}
\tilde{V}_p(K,L) &= \frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{V(K \tilde{+}_p\ \varepsilon \cdot L) - V(K)}{\varepsilon} \\
&= \frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{1}{n\varepsilon}\int_{S^{n-1}} \left[\rho(K \tilde{+}_p\ \varepsilon \cdot L, u)^n - \rho(K,u)^n\right] dS(u) \\
&= \frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{1}{n\varepsilon}\int_{S^{n-1}} \left[(\rho(K,u)^p + \varepsilon\rho(L,u)^p)^{\frac{n}{p}} - \rho(K,u)^n\right] dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u),
\end{aligned}$$

where the last equality can be obtained by L'Hospital's rule [25].

Note: When $p = n$, $\tilde{V}_p(K,L) = V(L)$.

Recall the definition of the L_p-projection body, Π_pK, of K in [16]

$$h(\Pi_pK, u) = \left(\frac{1}{n\omega_n c_{n-2,p}}\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v)\right)^{\frac{1}{p}}, \tag{2.10}$$

for $u \in S^{n-1}$, where the normalization is chosen so that, for $K = B_n$, there is $\Pi_pB_n = B_n$.

If we are using the notion of the L_p-mixed volume to rewrite the definition of $\Pi_p K$, then the geometric essence of this definition can be revealed,

$$h(\Pi_p K, u) = \left(\frac{V_p(K, [-u, u])}{V_p(B_n, [-u, u])} \right)^{\frac{1}{p}}, \tag{2.11}$$

for $u \in S^{n-1}$, where $[-u, u]$ denotes the line segment connecting the points $-u$ to u.

By the duality between intersection bodies and projection bodies, we can define the quasi L_p-intersection body, $I_p K$, of K as follows:

Definition 2.1 *Let K be a star body and $p \geq 1$. Then the quasi L_p-intersection body, $I_p K$, of K can be defined by*

$$\rho(I_p K, u) = \left(\frac{\tilde{V}_p(K, B_n \cap u^{\perp})}{\tilde{V}_p(B_n, B_n \cap u^{\perp})} \right)^{\frac{1}{p}}, \tag{2.12}$$

for $u \in S^{n-1}$.

Suppose that f is a Borel function on S^{n-1}. The *spherical Radon transform* Rf [26] of f is defined by

$$(Rf)(u) = \int_{S^{n-1} \cap u^{\perp}} f(v) dS_{n-2}(v), \tag{2.13}$$

for $u \in S^{n-1}$. Two important properties of the spherical Radon transform are:

(1) The spherical Radon transform is a continuous bijection of $C_e^{\infty}(S^{n-1})$ to itself;

(2) The spherical Radon transform is self-adjoint, that is, if f and g are bounded Borel functions on S^{n-1}, then

$$\int_{S^{n-1}} f(u) Rg(u) du = \int_{S^{n-1}} Rf(u) g(u) du. \tag{2.14}$$

Using the spherical Radon transform we can rewrite the definition of $I_p K$ as:

$$\rho(I_p K, u) = \left(\frac{\tilde{V}_p(K, B_n \cap u^{\perp})}{\tilde{V}_p(B_n, B_n \cap u^{\perp})} \right)^{\frac{1}{p}} = \left(\frac{1}{(n-1)\omega_{n-1}} R(\rho_K^{n-p})(u) \right)^{\frac{1}{p}}, \tag{2.15}$$

for $u \in S^{n-1}$.

Remark 1 The intersection body, IK, of K was first explicitly defined and named by Lutwak [4], which can be defined by: $\rho_{IK}(u) = \frac{v(K \cap u^{\perp})}{v(B_n \cap u^{\perp})} = \frac{\tilde{V}_1(K, B_n \cap u^{\perp})}{\tilde{V}_1(B_n, B_n \cap u^{\perp})}$, for $u \in S^{n-1}$, where $v(\cdot)$ denotes the $(n-1)$-dimensional volume and the normalization is chosen so that $IB_n = B_n$. So it's clear that $IK = I_1 K$. Moreover, from a geometric point of view, the quasi-L_p-intersection body is the right generalization of the classical intersection body.

2 For a general star body K, $IK \neq I_p K$ if $p \neq 1$, and it's possible that $IK_1 = I_p K_2$, i.e., the class of intersection bodies and the class of quasi L_p-intersection bodies may be the same one.

3 Note that for the unit ball, B_n, we have $I_p B_n = B_n$. Moreover, $I_p K$ degenerates into B_n when $p = n$.

Figure 1 shows that $I_3 K$ may be different when $K \in \mathbb{R}^2$ in different positions.

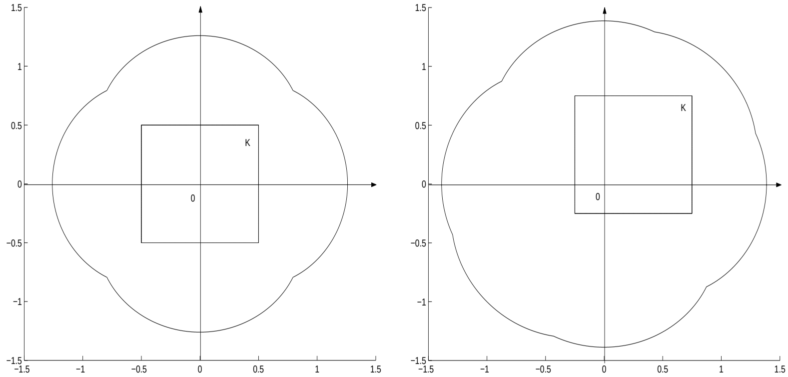

Figure 1 I_3K, where $K \in \mathbb{R}^2$ is a unit square in different positions

One of the classical affine isoperimetric inequalities is the Busemann intersection inequality [2]:

Theorem 2.1* *Let K be a convex body in $\mathbb{R}^n$. Then*

$$V(IK)V(K)^{1-n} \leq \omega_n^{2-n}, \tag{2.16}$$

with equality if and only if K is an ellipsoid.

We will establish the L_p-analog of the Busemann intersection inequality:

Theorem 2.1 *Let K be a convex body in $\mathbb{R}^n$. Then for $1 < p < n$,*

$$V(I_pK)V(K)^{(p-n)/p} \leq \omega_n^{(2p-n)/p}; \tag{2.17}$$

if $p > n$, then the inequality is reversed; for $p \neq n$, the equality holds if and only if K is an origin-centered ball.

In order to prove Theorem 2.1, we introduce a lemma first.

Lemma 2.2 [27] (Hölder inequality) *For nonnegative bounded Borel functions f and g on X, and $p,\ q \in \mathbb{R}^+$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\int_X f(x)g(x)d\mu(x) \leq \left[\int_X f(x)^p d\mu(x)\right]^{\frac{1}{p}} \left[\int_X g(x)^q d\mu(x)\right]^{\frac{1}{q}}; \tag{2.18}$$

if one of p and q is negative, then the inequality is reversed, with equality if and only if one of the functions is zero μ-almost everywhere, or there exist nonnegative constants a and b, not both zero, such that $af^p = bg^q$ μ-almost everywhere.

Proof of Theorem 2.1 If $1 < p < n$, then from (2.13), (2.15) and Lemma 2.2, it follows that

$$\begin{aligned} \rho_{I_pK}(u)^n &= \left[\frac{1}{\omega_{n-1}(n-1)} \int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-p} dS_{n-2}(v)\right]^{\frac{n}{p}} \\ &\leq \left\{\left[\frac{1}{\omega_{n-1}(n-1)} \int_{S^{n-1}\cap u^\perp} \rho_K(v)^{n-1} dS_{n-2}(v)\right]^{\frac{n-p}{n-1}} \right. \\ &\quad \left. \cdot \left[\frac{1}{\omega_{n-1}(n-1)} \int_{S^{n-1}\cap u^\perp} 1^{\frac{n-1}{p-1}} dS_{n-2}(v)\right]^{\frac{p-1}{n-1}}\right\}^{\frac{n}{p}} = \rho_{IK}(u)^{\frac{(n-p)n}{(n-1)p}}. \end{aligned} \tag{2.19}$$

From the polar coordinate formula for volume, we have

$$V(I_pK) = \frac{1}{n}\int_{S^{n-1}} \rho_{I_pK}(u)^n dS(u) \leq \frac{1}{n}\int_{S^{n-1}} \rho_{IK}(u)^{\frac{(n-p)n}{(n-1)p}} dS(u)$$

$$\leq \left[\frac{1}{n}\int_{S^{n-1}} \rho_{IK}(u)^n dS(u)\right]^{\frac{n-p}{(n-1)p}} \left[\frac{1}{n}\int_{S^{n-1}} 1^{\frac{(n-1)p}{(p-1)n}} dS(u)\right]^{\frac{(p-1)n}{(n-1)p}}$$
$$= \omega_n^{\frac{(p-1)n}{(n-1)p}}\ V(IK)^{\frac{n-p}{(n-1)p}}. \tag{2.20}$$

Theorem 2.1*, together with (2.20), shows that

$$\begin{aligned} V(I_pK)V(K)^{1-\frac{n}{p}} &= V(I_pK)V(K)^{(1-n)\frac{n-p}{(n-1)p}} \\ &\leq \omega_n^{\frac{(p-1)n}{(n-1)p}}\ [V(IK)V(K)^{1-n}]^{\frac{n-p}{(n-1)p}} \\ &\leq \omega_n^{\frac{(p-1)n}{(n-1)p}}\ (\omega_n^{2-n})^{\frac{n-p}{(n-1)p}} = \omega_n^{2-\frac{n}{p}}. \end{aligned} \tag{2.21}$$

This establishes the first statement of the theorem for $1 < p < n$, and the corresponding equality condition follows from that of the Hölder inequality.

If $p > n$, then $\frac{n-p}{(n-1)p} < 0$; from Lemma 2.2 we know the inequality in (2.21) should be reversed. So the inequality in (2.21) should be reversed too. This yields the second statement.

Suppose that equality holds in the statements of the theorem for some $p \neq n$. By the Hölder inequality, we see that $\rho_K(\cdot)$ is constant, this means that K is an origin-centered ball in $\mathbb{R}^n$. This completes the proof.

The well-known Blaschke–Santaló inequality [28] states that, *if K is an origin-centered convex body, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.22}$$

with equality if and only if K is an ellipsoid.

In light of the Blaschke–Santaló inequality, the following corollary is an immediate consequence of Theorem 2.1:

Corollary 2.1 *Let K be an origin-centered convex body and $1 \leq p < n$. Then*

$$V(I_pK^*)V(K)^{(n-p)/p} \leq \omega_n^{n/p}; \tag{2.23}$$

if $p > n$, then the inequality is reversed; for $p \neq n$, the equality holds if and only if K is an origin-centered ball.

Remark When $p = n$, Theorem 2.1 and Corollary 2.1 always hold because $V(I_nK) = V(I_nK^*) = \omega_n$.

3 The Brunn–Minkowski Inequality for the Quasi L_p-intersection Bodies

In this section we will establish the Brunn–Minkowski inequality for the quasi L_p-intersection bodies.

For star bodies $K,\ L$, with $\varepsilon > 0$ the radial Minkowski linear combination $K\tilde{+}\ \varepsilon \cdot L$ is the star body defined by $\rho(K\tilde{+}\ \varepsilon \cdot L,\ \cdot) = \rho(K,\ \cdot) + \varepsilon\rho(L,\ \cdot)$.

Lemma 3.1 [27] (Minkowski integral inequality) *For nonnegative bounded Borel functions f and g on X, we have, for $p \geq 1$,*

$$\left[\int_X (f(x)+g(x))^p d\mu(x)\right]^{\frac{1}{p}} \leq \left[\int_X f(x)^p d\mu(x)\right]^{\frac{1}{p}} + \left[\int_X g(x)^p d\mu(x)\right]^{\frac{1}{p}}; \tag{3.1}$$

if $p < 0$ or $0 < p < 1$, then the inequality is reversed; for $p \neq 1$, the equality holds if and only if f and g are proportional.

Theorem 3.1 *Let $K,\ L$ be star bodies. Then for $1 \le p \le n-1$,*

$$V(I_p(K\tilde{+}L))^{\frac{p}{n(n-p)}} \le V(I_pK)^{\frac{p}{n(n-p)}} + V(I_pL)^{\frac{p}{n(n-p)}}; \tag{3.2}$$

if $p > n$ or $\frac{n^2}{n+1} < p < n$, then the inequality is reversed, with equality if and only if K and L are dilations.

Proof If $1 \le p \le n-1$, then from (2.15) and Lemma 3.1, it follows that

$$\begin{aligned}
\rho_{I_p(K\tilde{+}L)}(u)^{\frac{p}{n-p}} &= \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^\perp}\rho_{K\tilde{+}L}(v)^{n-p}dS_{n-2}(v)\right]^{\frac{1}{n-p}}\\
&= \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^\perp}(\rho_K(v)+\rho_L(v))^{n-p}dS_{n-2}(v)\right]^{\frac{1}{n-p}}\\
&\le \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^\perp}\rho_K(v)^{n-p}dS_{n-2}(v)\right]^{\frac{1}{n-p}}\\
&\quad + \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^\perp}\rho_L(v)^{n-p}dS_{n-2}(v)\right]^{\frac{1}{n-p}}\\
&= \rho_{I_pK}(u)^{\frac{p}{n-p}} + \rho_{I_pL}(u)^{\frac{p}{n-p}}.
\end{aligned} \tag{3.3}$$

Since $\frac{(n-p)n}{p} \ge 1$, from (3.3) and again Lemma 3.1 we have

$$\begin{aligned}
V(I_p(K\tilde{+}L))^{\frac{p}{(n-p)n}} &= \left[\frac{1}{n}\int_{S^{n-1}}\rho_{I_p(K\tilde{+}L)}(u)^n dS(u)\right]^{\frac{p}{(n-p)n}}\\
&\le \left[\frac{1}{n}\int_{S^{n-1}}(\rho_{I_pK}(u)^{\frac{p}{n-p}}+\rho_{I_pL}(u)^{\frac{p}{n-p}})^{\frac{(n-p)n}{p}}dS(u)\right]^{\frac{p}{(n-p)n}}\\
&\le \left[\frac{1}{n}\int_{S^{n-1}}\rho_{I_pK}(u)^n dS(u)\right]^{\frac{p}{(n-p)n}} + \left[\frac{1}{n}\int_{S^{n-1}}\rho_{I_pL}(u)^n dS(u)\right]^{\frac{p}{(n-p)n}}\\
&= V(I_pK)^{\frac{p}{(n-p)n}} + V(I_pL)^{\frac{p}{(n-p)n}}.
\end{aligned} \tag{3.4}$$

From the equality condition of Lemma 3.1, we know that the equality holds in (3.4) if and only if $\rho_K(\cdot)$ and $\rho_L(\cdot)$ are proportional, i.e., K and L are dilations.

If $p > n$, then we have $n-p < 0$ and $\frac{(n-p)n}{p} < 0$. Similarly, if $\frac{n^2}{n+1} < p < n$, then we have $0 < n-p < 1$ and $0 < \frac{(n-p)n}{p} < 1$. Thus from Lemma 3.1, we know that the inequality in (3.4) is reversed, with equality if and only if K and L are dilations. This complete the proof.

The following corollary is an immediate consequence of Theorem 3.1:

Corollary 3.1 *Let $K,\ L$ be star bodies. Then*

$$V(I(K\tilde{+}L))^{\frac{1}{n(n-1)}} \le V(IK)^{\frac{1}{n(n-1)}} + V(IL)^{\frac{1}{n(n-1)}}, \tag{3.5}$$

with equality if and only if K and L are dilations.

Note that this result is a dual form of the Brunn–Minkowski inequality for projection bodies given by Lutwak [6]. *Let $K,\ L$ be convex bodies. Then*

$$V(\Pi(K+L))^{\frac{1}{n(n-1)}} \ge V(\Pi K)^{\frac{1}{n(n-1)}} + V(\Pi L)^{\frac{1}{n(n-1)}}, \tag{3.6}$$

with equality if and only if K and L are homothetic.

For $p,\ q \ge 1$, we will establish the relationship among $I_{p+q}K$, I_pK and I_qK.

Theorem 3.2 *Let K be a star body and $p,\ q \ge 1$. Then for $p > q$,*

$$V(I_{p+q}K)^{p^2-q^2} \ge V(I_pK)^{p^2}V(I_qK)^{-q^2}, \tag{3.7}$$

with equality if and only if K is an origin-centered ball.

Proof From the definition of the quasi L_p-intersection bodies and Lemma 2.1, we have

$$\begin{aligned}\rho_{I_{p+q}K}(u)^{p+q} &= \frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^{\perp}} \rho_K(v)^{n-p-q} dS_{n-2}(v)\\ &= \frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^{\perp}} [\rho_K(v)^{n-p}]^{\frac{p}{p-q}}[\rho_K(v)^{n-q}]^{\frac{-q}{p-q}} dS_{n-2}(v)\\ &\geq \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^{\perp}} \rho_K(v)^{n-p} dS_{n-2}(v)\right]^{\frac{p}{p-q}}\\ &\quad \cdot \left[\frac{1}{\omega_{n-1}(n-1)}\int_{S^{n-1}\cap u^{\perp}} \rho_K(v)^{n-q} dS_{n-2}(v)\right]^{\frac{-q}{p-q}}\\ &= \rho_{I_pK}(u)^{\frac{p^2}{p-q}}\rho_{I_qK}(u)^{\frac{-q^2}{p-q}}. \end{aligned} \tag{3.8}$$

So we get

$$\begin{aligned} V(I_{p+q}K)^{p^2-q^2} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I_{p+q}K}(u)^n dS(u)\right]^{p^2-q^2} = \left\{\frac{1}{n}\int_{S^{n-1}} [\rho_{I_{p+q}K}(u)^{p+q}]^{\frac{n}{p+q}} dS(u)\right\}^{p^2-q^2}\\ &\geq \left\{\frac{1}{n}\int_{S^{n-1}} [\rho_{I_pK}(u)^{\frac{p^2}{p-q}}\rho_{I_qK}(u)^{\frac{-q^2}{p-q}}]^{\frac{n}{p+q}} dS(u)\right\}^{p^2-q^2}\\ &= \left\{\frac{1}{n}\int_{S^{n-1}} [\rho_{I_pK}(u)^n]^{\frac{p^2}{p^2-q^2}}[\rho_{I_qK}(u)^n]^{\frac{-q^2}{p^2-q^2}} dS(u)\right\}^{p^2-q^2}\\ &\geq \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I_pK}(u)^n dS(u)\right]^{p^2}\left[\frac{1}{n}\int_{S^{n-1}} \rho_{I_qK}(u)^n dS(u)\right]^{-q^2}\\ &= V(I_pK)^{p^2}V(I_qK)^{-q^2}. \end{aligned} \tag{3.9}$$

From the equality condition of the Hölder inequality, we know that the equality holds in (3.7) if and only if K is an origin-centered ball.

Corollary 3.2 *Let K be a star body and $p+q=n$. Then for $p>q\geq 1$,*

$$V(I_pK)^{p^2}V(I_qK)^{-q^2} \leq \omega_n^{p^2-q^2}, \tag{3.10}$$

with equality if and only if K is an origin-centered ball.

4 The Monotone Property of the Quasi L_p-intersection Bodies

The work of Lutwak [4] represents the beginning of the Busemann–Petty problem's [29] eventual solution. In fact, Lutwak's result is the monotone property of intersection bodies [4]. Let $\mathbb{I}^n$ denote the set of n-dimensional intersection bodies. We have

Theorem 4.1* *Let $K\in\mathbb{I}^n$ and L be a star body in $\mathbb{R}^n$. If $IK\subseteq IL$, then*

$$V(K)\leq V(L), \tag{4.1}$$

with equality if and only if K=L.

We will establish a similar result for the quasi L_p-intersection bodies. Let $\mathbb{I}_p^n$ denote the set of n-dimensional quasi L_p-intersection bodies.

Theorem 4.1 *Let $K\in\mathbb{I}_p^n$ and L be a star body in $\mathbb{R}^n$. If for $1<p<n$, there is*

$$I_pK\subseteq I_pL,$$

then

$$V(K) \leq V(L), \tag{4.2}$$

with equality if and only if K=L. If $p > n$ and $L \in \mathbb{I}_p^n$, K is a star body in $\mathbb{R}^n$, then the above inequality is reversed.

In order to prove Theorem 4.1, we need the following two lemmas for L_p-dual mixed volume:

Lemma 4.1 *Let K, L be star bodies. Then for $1 \leq p < n$,*

$$\tilde{V}_p(K,L)^n \leq V(K)^{n-p}V(L)^p; \tag{4.3}$$

if $p > n$, then the inequality is reversed; for $p \neq n$, equality holds if and only if K and L are dilations.

Proof If $1 \leq p < n$, then from Lemma 2.1 and Lemma 2.2, we can get

$$\begin{aligned}\tilde{V}_p(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-p}\rho(L,u)^p dS(u) \\ &\leq \left[\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u)\right]^{\frac{n-p}{n}} \left[\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^n dS(u)\right]^{\frac{p}{n}} \\ &= V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}.\end{aligned} \tag{4.4}$$

When $p > n$, from Lemma 2.1, the inequality in (4.4) is reversed and from the equality condition of Lemma 2.2, for $p \neq n$, the equality holds if and only if K and L are dilations.

Lemma 4.2 *For star bodies K, L, and $p \geq 1$,*

$$\tilde{V}_p(K, I_pL) = \tilde{V}_p(L, I_pK). \tag{4.5}$$

Proof From (2.14), (2.15) and Lemma 2.1, we obtain

$$\begin{aligned}\tilde{V}_p(K, I_pL) &= \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-p}\rho_{I_pL}(u)^p dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-p} R(\frac{1}{\omega_{n-1}(n-1)}\rho_L^{n-p})(u) dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} R\left(\frac{1}{\omega_{n-1}(n-1)}\rho_K^{n-p}\right)(u)\rho_L(u)^{n-p}(u) dS(u) \\ &= \tilde{V}_p(L, I_pK).\end{aligned}$$

Proof of Theorem 4.1 Let M be a star body in $\mathbb{R}^n$. From (2.9) and (4.5), it follows that

$$\tilde{V}_p(K, I_pM) = \tilde{V}_p(M, I_pK) = \frac{1}{n}\int_{S^{n-1}} \rho_M(u)^{n-p}\rho_{I_pK}(u)^p \, dS(u).$$

Similarly,

$$\tilde{V}_p(L, I_pM) = \tilde{V}_p(M, I_pL) = \frac{1}{n}\int_{S^{n-1}} \rho_M(u)^{n-p}\rho_{I_pL}(u)^p \, dS(u).$$

From the condition $I_pK \subseteq I_pL$, we obtain that

$$\tilde{V}_p(K, I_pM) \leq \tilde{V}_p(L, I_pM) \text{ holds for arbitrary } M. \tag{4.6}$$

If $1 < p < n$, taking $I_pM = K$ in (4.6) and using (4.3) we can get

$$V(K) = \tilde{V}_p(K,K) \leq \tilde{V}_p(L,K) \leq V(L)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}, \tag{4.7}$$

which implies $V(K) \leq V(L)$.

From the equality condition of (4.3) and the condition $I_pK \subseteq I_pL$, we know that the equality holds if and only if $K = L$.

If $p > n$ and $L \in \mathbb{I}_p^n$, K is a star body in $\mathbb{R}^n$, taking $I_pM = L$ in (4.6) and using (4.3), we obtain

$$V(L) = \tilde{V}_p(L, L) \geq \tilde{V}_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{4.8}$$

which implies $V(K) \geq V(L)$ with equality if and only if $K = L$.

5 The Aleksandrov–Fenchel Inequality for the Mixed Quasi L_p-intersection Bodies

In this section we will generalize the quasi L_p-intersection bodies to the mixed quasi L_p-intersection bodies. We will establish the Aleksandrov–Fenchel inequality for the mixed quasi L_p-intersection bodies.

Definition 5.1 *Let $K_1, \ldots, K_{n-1}$ be star bodies and $p \geq 1$. Then the mixed quasi L_p-intersection body, $I_p(K_1, \ldots, K_{n-1})$, of $K_1, \ldots, K_{n-1}$ can be defined by*

$$\rho_{I_p(K_1,\ldots,K_{n-1})}(u) = \left(\frac{1}{(n-1)\omega_{n-1}} R((\rho_{K_1} \cdots \rho_{K_{n-1}})^{\frac{n-p}{n-1}})(u)\right)^{\frac{1}{p}}, \tag{5.1}$$

for $u \in S^{n-1}$.

We will establish the Aleksandrov–Fenchel inequality for the mixed quasi L_p-intersection bodies using the extension form of the Hölder inequality.

Theorem 5.1 *Let $K_1, \ldots, K_{n-1}$ be star bodies and $p \geq 1$. Then for $m \leq n-1$,*

$$V(I_p(K_1, \ldots, K_{n-1}))^m \leq \prod_{j=1}^{m} V(I_p(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_{n-1})), \tag{5.2}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are dilations.

Proof The extension of the the Hölder inequality states that

$$\int_X \prod_{i=1}^{m} f_i(u) du \leq \prod_{i=1}^{m} \left(\int_X f_i(u)^m du\right)^{\frac{1}{m}}, \tag{5.3}$$

with equality if and only if f_i are proportional.

From (2.14), (5.1) and (5.3), it follows that

$$\begin{aligned}
&\rho_{I_p(K_1,\ldots,K_{n-1})}(u)^p \\
&\quad = \frac{1}{\omega_{n-1}(n-1)} \int_{S^{n-1}\cap u^\perp} (\rho_{K_1}(v) \cdots \rho_{K_{n-1}}(v))^{\frac{n-p}{n-1}} dS_{n-2}(v) \\
&\quad \leq \left[\prod_{j=1}^{m} \frac{1}{\omega_{n-1}(n-1)} \int_{S^{n-1}\cap u^\perp} \rho_{K_j}(v)^{\frac{(n-p)m}{n-1}} (\rho_{K_{m+1}}(v) \cdots \rho_{K_{n-1}}(v))^{\frac{n-p}{n-1}} dS_{n-2}(v)\right]^{\frac{1}{m}} \\
&\quad = \prod_{j=1}^{m} \rho(I_p(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_{n-1}), u)^{\frac{p}{m}}.
\end{aligned}$$

From the above inequality and again (5.3), we get

$$V(I_p(K_1, \ldots, K_{n-1})) = \frac{1}{n} \int_{S^{n-1}} \rho_{I_p(K_1,\ldots,K_{n-1})}(u)^n dS(u)$$

$$\leq \frac{1}{n}\int_{S^{n-1}}\prod_{j=1}^{m}\rho(I_p(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_{n-1}),u)^{\frac{n}{m}}dS(u)$$

$$\leq \prod_{j=1}^{m}\left[\frac{1}{n}\int_{S^{n-1}}\rho(I_p(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_{n-1}),u)^{n}dS(u)\right]^{\frac{1}{m}}$$

$$= \prod_{j=1}^{m}V(I_p(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_{n-1}))^{\frac{1}{m}}. \tag{5.4}$$

From the equality condition of (5.3), we know that the equality holds in (5.2) if and only if $K_1,\ldots,K_{n-1}$ are dilations.

The following corollary is an immediate consequence of Theorem 5.1:

Corollary 5.1 *Let $K_1,\ldots,K_{n-1}$ be star bodies. Then for $m \leq n-1$,*

$$V(I(K_1,\ldots,K_{n-1}))^m \leq \prod_{j=1}^{m}V(I(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_{n-1})), \tag{5.5}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are dilations.

Obviously, this result is a dual form of the following one given by Lutwak [6]: *Let $K_1,\ldots,$ K_{n-1} be convex bodies. Then, for $m \leq n-1$,*

$$V(\Pi(K_1,\ldots,K_{n-1}))^m \geq \prod_{j=1}^{m}V(\Pi(\underbrace{K_j,\ldots,K_j}_{m},K_{m+1},\ldots,K_{n-1})). \tag{5.6}$$

Taking $m = n-1$ in Theorem 5.1, we have

Corollary 5.2 *Let $K_1,\ldots,K_{n-1}$ be star bodies and $p \geq 1$. Then for $m \leq n-1$,*

$$V(I_p(K_1,\ldots,K_{n-1}))^{n-1} \leq V(I_pK_1)\cdots V(I_pK_{n-1}), \tag{5.7}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are dilations.

Note that the case of $p = 1$ in Corollary 5.2 is a dual form of the following one given by Lutwak [6]: *Let $K_1,\ldots,K_{n-1}$ be convex bodies. Then for $m \leq n-1$,*

$$V(\Pi(K_1,\ldots,K_{n-1}))^{n-1} \geq V(\Pi K_1)\cdots V(\Pi K_{n-1}), \tag{5.8}$$

with equality if and only if $K_1,\ldots,K_{n-1}$ are homothetic.

Open Problem Comparing (1.1) and (2.23), for $p > n$ we have $V(I_pK^*) \geq V(\Pi_p^*K)$. For $p = 1$, from the following result given by Lutwak [30]: If ξ is a subspace of $\mathbb{R}^n$, and K is a convex body in $\mathbb{R}^n$, which contains the origin in its interior, then $K^* \cap \xi = (K|\xi)^*$, a direct computation shows that $V(IK^*) \leq V(\Pi^*K)$. Is it true that $V(I_pK^*) \leq V(\Pi_p^*K)$ for $1 < p < n$?

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