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Maximal Moment Inequality for Partial Sums of Strong Mixing Sequences and Application

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Abstract Some maximal moment inequalities for partial sums of the strong mixing random variable sequence are established. These inequalities use moment sums as up-boundary and improve the corresponding ones obtained by Shao (1996). To show the application of the inequalities, we apply them to discuss the asymptotic normality of the weight function estimate for the fixed design regression model.

Keywords strong mixing, maximal moment inequality, fixed design regression model, weight function estimate, asymptotic normality

MR(2000) Subject Classification 60E15, 62G05, 62E20, 62J02

1 Introduction

Suppose $\{X_i : i \in \mathbf{Z}\}$ is a real-valued random variable sequence on a probability space $(\Omega, \mathscr{B}, P)$. Let $\mathscr{F}_m^n$ denote the σ-field generated by $(X_i : m \le i \le n)$. Let

$$\alpha(n) = \sup\left\{|P(AB) - P(A)P(B)| : A \in \mathscr{F}_{-\infty}^m, B \in \mathscr{F}_{m+n}^\infty\right\}.$$

The sequence $\{X_i\}$ is said to be α-mixing or strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

The moment inequalities of partial sums $S_n := \sum_{i=1}^n X_i$ play a very important role in various proofs of limit theorems, for example, the Marcinkiewicz–Zygmund inequality and the Rosenthal inequality for independent random variables, the Burkholder inequality for martingales. For dependent random variables, many scholars have also been trying to develop these inequalities. One can refer to Billingsley (1968, [1]), Peligrad (1982, 1985, 1987, [2–4]), Roussas and Ioannides (1987, [5]), Shao (1988, 1989, 1995, [6–8]), Yang (1997, [9]) and Zhang (1998, 2000, [10, 11]) for ϕ-mixing or ρ-mixing sequences, Birkel (1988, [12]) and Shao and Yu (1996, [13]) for positively associated sequences, Su, Zhao and Wang (1997, [14]), Shao and Su (1999, [15]), Shao (2000, [16]), Zhang and Wen (2001, [17]) and Yang (2001, [18]) for negatively associated sequences.

In this paper, we consider the strong mixing sequences. In this case, Yokoyama (1980, [19]) first got that

$$E|S_n|^r \le Cn^{r/2} \tag{1.1}$$

for $r > 2$ and a strictly stationary sequence. Shao and Yu (1996, [13]) and Yang (2000, [20]) investigated some general inequalities. The follow inequalities are due to Shao and Yu (1996 [13], Theorem 4.1; also see Lemma 12.2.2 in Lin and Lu (1996, [21])).

Theorem A *Let $r > 2, \delta > 0, 2 < v \le r + \delta$ and $\{X_i, i \ge 1\}$ be an α-mixing sequence of random variables with $EX_i = 0$ and $||X_i||_{r+\delta} := (E|X_i|^{r+\delta})^{1/(r+\delta)} < \infty$. Assume that $\alpha(n) \le$*

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$Cn^{-\theta}$ for some $C > 0$ and $\theta > 0$. Then , for any $\varepsilon > 0$, there exists $K = K(\varepsilon, r, \delta, v, \theta, C) < \infty$ such that

$$E|S_n|^r \le K\Big\{(nC_n)^{r/2} \max_{1\le i\le n} ||X_i||_v^r + n^{(r-\delta\theta/(r+\delta))\vee(1+\varepsilon)} \max_{1\le i\le n} ||X_i||_{r+\delta}^r\Big\}, \tag{1.2}$$

where $C_n = \big(\sum_{i=0}^n (i+1)^{2/(v-2)}\alpha(i)\big)^{(v-2)/v}$. In particular, for any $\varepsilon > 0$,

$$E|S_n|^r \le K\Big\{n^{r/2} \max_{1\le i\le n} ||X_i||_v^r + n^{1+\varepsilon} \max_{1\le i\le n} ||X_i||_{r+\delta}^r\Big\} \tag{1.3}$$

if $\theta > v/(v-2)$ and $\theta \ge (r-1)(r+\delta)/\delta$, and

$$E|S_n|^r \le Kn^{r/2} \max_{1\le i\le n} ||X_i||_{r+\delta}^r \tag{1.4}$$

if $\theta \ge r(r+\delta)/(2\delta)$.

These inequalities use the maximal moments $\max_{1\le i\le n} ||X_i||_v^r$ and $\max_{1\le i\le n} ||X_i||_{r+\delta}^r$ as up-boundary. In some cases, it will lose the information of the moment sums $\sum_{i=1}^n ||X_i||_v^r$ and $\sum_{i=1}^n ||X_i||_{r+\delta}^r$. Indeed, we have the well-known Rosenthal inequality

$$E \max_{1\le j\le n} |S_j|^r \le C\Big\{\sum_{i=1}^n E|X_i|^r + \Big(\sum_{i=1}^n E|X_i|^2\Big)^{r/2}\Big\} \tag{1.5}$$

for independent random variable sequences.

The main purpose of this paper is to develop some maximal moment inequalities which use moment sums as up-boundary for partial sums of strong mixing sequences. Our inequalities are very near to (1.5) and improve Theorem A. To show the application of the inequalities, we apply them to discuss the asymptotic normality of the weight function estimate for the fixed design regression model.

Throughout this paper, it is supposed that C denotes constant which only depends on some given numbers, $[x]$ denotes the integral part of x, $||X||_r := (E|X|^r)^{1/r}$, $a \wedge b := \min\{a, b\}$. The paper is organized as follows. Section 2 contains the maximal moment inequalities and their proofs, Section 3 gives the application of the inequalities.

2 Moment Inequality

We will prove the following results.

Theorem 2.1 *Let $r > 2, \delta > 0, 2 < v \le r + \delta$ and $\{X_i, i \ge 1\}$ be an α-mixing sequence of random variables with $EX_i = 0$ and $E|X_i|^{r+\delta} < \infty$. Suppose that*

$$\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\} \tag{2.1}$$

and $\alpha(n) \le Cn^{-\theta}$ for some $C > 0$. Then, for any $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon, r, \delta, v, \theta, C) < \infty$ such that

$$E \max_{1\le j\le n} |S_j|^r \le K\Big\{n^\varepsilon \sum_{i=1}^n E|X_i|^r + \sum_{i=1}^n ||X_i||_{r+\delta}^r + \Big(\sum_{i=1}^n ||X_i||_v^2\Big)^{r/2}\Big\}. \tag{2.2}$$

Theorem 2.2 *Let $r > 2, \delta > 0$ and $\{X_i, i \ge 1\}$ be an α-mixing sequence of random variables with $EX_i = 0$ and $E|X_i|^{r+\delta} < \infty$. Suppose that*

$$\theta > r(r+\delta)/(2\delta) \tag{2.3}$$

and $\alpha(n) \le Cn^{-\theta}$ for some $C > 0$. Then, for any $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon, r, \delta, \theta, C) < \infty$ such that

$$E \max_{1\le j\le n} |S_j|^r \le K\Big\{n^\varepsilon \sum_{i=1}^n E|X_i|^r + \Big(\sum_{i=1}^n ||X_i||_{r+\delta}^2\Big)^{r/2}\Big\}. \tag{2.4}$$

Remark 2.1 Since $E|X_i|^r \le ||X_i||_{r+\delta}^r$, we have (1.3) and (1.4) in Theorem A from (2.2) and (2.4). Obviously, $r(r+\delta)/(2\delta) < (r-1)(r+\delta)/\delta$ for $r > 2$. Hence the mixing rate of the condition (2.3) is weaker than that of the condition (2.1). And the condition (2.1) is almost the same as $\theta > v/(v-2)$ and $\theta \ge (r-1)(r+\delta)/\delta$ in Theorem A, the condition (2.3) is almost the same as $\theta \ge r(r+\delta)/(2\delta)$ in Theorem A.

Remark 2.2 Note that (2.2) and (2.4) contain the information of moment sums. They are more efficient in researching weight sums than (1.3) and (1.4). Indeed, there are many weight estimates in statistics, such as least squares regression estimate, non-parameter regression estimate and non-parameter density estimate. So Theorem 2.1 and Theorem 2.2 are very useful results.

Remark 2.3 In application of the inequalities (2.2) and (2.4), we may choose a sufficiently small $\varepsilon > 0$. If the random sequence is independent, then (2.2) is near to (1.5) by taking $v = 2+\delta$ for a sufficiently small $\delta > 0$.

To prove our theorems, we first give the following lemmas.

Lemma 2.1 (i) *Suppose that ξ and η are $\mathscr{F}_1^k$ - measurable and $\mathscr{F}_{k+n}^\infty$ - measurable random variables, respectively. If $E|\xi|^p < \infty, E|\eta|^p < \infty$ for some $p, q, s > 1$ with $1/p + 1/q + 1/s = 1$, then*

$$|E(\xi\eta) - (E\xi)(E\eta)| \le 10\alpha^{1/s}(n)||\xi||_p \cdot ||\eta||_q.$$

(ii) *If $\sum_{i=1}^\infty \alpha^{(q-2)/q}(i) < \infty$ for some $q > 2$, then $E\left(\sum_{i=1}^n X_i\right)^2 \le C\sum_{i=1}^n E||X_i||_q^2$.*

One can see the result (i) in Roussas and Ioannides (1987, [5]). It is easy to get the result (ii) from (i). Also see Lemma 1.2.4 in Lin and Lu (1996, [21]).

Lemma 2.2 *For any $x, y \in R^1$, we have*

$$|x+y|^r \le |y|^r + d_1|x|^r + rx|y|^{r-1}\mathrm{sgn}(y) + d_2x^2|y|^{r-2} \quad \textit{for} \;\; r > 2 \tag{2.5}$$

where $d_1 = 2^r, d_2 = 2^r \cdot r^2$.

Proof For $r > 2, t \in R^1$, it is easy to show that $|1+t|^r \le 1 + d_1|t|^r + rt + d_2t^2$. From this, we have the result by taking $t = y/x$ as $x \ne 0$. It is clear as $x = 0$.

Let $k = [(n/2)^\lambda]$ and $m = [(n/2)^{1-\lambda}]$, where $0 < \lambda < 1$ which will be given later on. Clearly,

$$n < 2(m+1)k, \quad \frac{1}{4}n^\lambda < k < n^\lambda, \quad m < n^{1-\lambda}. \tag{2.6}$$

Fix n and redefine X_i as $X_i = X_i$ for $1 \le i \le n$ and $X_i = 0$ for $i > n$. For $j = 1, 2, \ldots, m+1$, set

$$Y_j = \sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} X_i, \quad Z_j = \sum_{i=(2j-1)k+1}^{n\wedge 2jk} X_i$$

and $S_{1,j} = \sum_{i=1}^j Y_i,\ S_{2,j} = \sum_{i=1}^j Z_i$.

Lemma 2.3

$$\max_{1\le j\le n} |S_j|^r \le C\left\{\max_{1\le j\le m+1}|S_{1,j}|^r + \max_{1\le j\le m+1}|S_{2,j}|^r + \sum_{j=1}^{2(m+1)}\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i\right|^r\right\}.$$

Proof Noting that $S_j = \sum_{i=1}^{[j/k]k} X_i + \sum_{i=[j/k]k+1}^j X_i$, we have

$$\max_{1\le j\le n}|S_j|^r \le 2^{r-1}\max_{1\le j\le n}\left|\sum_{i=1}^{[j/k]k} X_i\right|^r + 2^{r-1}\max_{1\le j\le n}\left|\sum_{i=[j/k]k+1}^{j} X_i\right|^r := I_1 + I_2$$

and

$$I_1 \le 2^{2(r-1)} \max_{1\le j\le m+1} |S_{1,j}|^r + 2^{2(r-1)} \max_{1\le j\le m+1} |S_{2,j}|^r,$$

$$I_2 \le 2^{r-1} \max_{1\le j\le 2(m+1)} \max_{1\le l<k} \left| \sum_{i=(j-1)k+1}^{(j-1)k+l} X_i \right|^r \le 2^{r-1} \sum_{j=1}^{2(m+1)} \max_{1\le l<k} \left| \sum_{i=(j-1)k+1}^{(j-1)k+l} X_i \right|^r.$$

These equations above imply the desired result.

Clearly

$$\max_{1\le j\le m+1} |S_{1,j}|^r \le \left| \max_{1\le j\le m+1} S_{1,j} \right|^r + \left| \max_{1\le j\le m+1} (-S_{1,j}) \right|^r. \tag{2.7}$$

Denote

$$\begin{aligned} M_j &= \max\{0, Y_{j+1}, Y_{j+1}+Y_{j+2}, \ldots, Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\}, \\ N_j &= \max\{Y_{j+1}, Y_{j+1}+Y_{j+2}, \ldots, Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\}, \\ \widetilde{M}_j &= \max\{0, -Y_{j+1}, -Y_{j+1}-Y_{j+2}, \ldots, -Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\}, \\ \widetilde{N}_j &= \max\{-Y_{j+1}, -Y_{j+1}-Y_{j+2}, \ldots, -Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\}. \end{aligned}$$

Then

$$\max_{1\le j\le m+1} S_{1,j} = N_0,\ N_j = Y_{j+1} + M_{j+1},\ 0 \le M_j \le |N_j|, \tag{2.8}$$

$$\max_{1\le j\le m+1} (-S_{1,j}) = \widetilde{N}_0,\ \widetilde{N}_j = -Y_{j+1} + \widetilde{M}_{j+1},\ 0 \le \widetilde{M}_j \le |\widetilde{N}_j|, \tag{2.9}$$

and

$$M_j = \max\{S_{1,j}, S_{1,j+1}, \ldots, S_{1,m+1}\} - S_{1,j} \le \max_{j\le i\le m+1} |S_{1,i}| + |S_{1,j}| \le 2 \max_{1\le j\le m+1} |S_{1,j}|, \tag{2.10}$$

$$\begin{aligned} \widetilde{M}_j &= \max\{-S_{1,j}, -S_{1,j+1}, \ldots, -S_{1,m+1}\} + S_{1,j} \\ &\le \max_{j\le i\le m+1} |S_{1,i}| + |S_{1,j}| \le 2 \max_{1\le j\le m+1} |S_{1,j}|. \end{aligned} \tag{2.11}$$

Lemma 2.4 *If $\theta > (r-1)(r+\delta)/\delta$, then for any $\rho > 0$, there exist positive constants $C_\rho = C(\rho, r, \delta, \theta) < \infty$ and $C_r = C(r) < \infty$ such that*

$$\sum_{j=1}^{m} E\left(Y_j M_j^{r-1}\right) \le C_\rho \sum_{i=1}^{n} \| X_i \|_{r+\delta}^r + \rho C_r E \max_{1\le j\le m+1} |S_{1,j}|^r, \tag{2.12}$$

$$\sum_{j=1}^{m} E(Y_j \widetilde{M}_j^{r-1}) \le C_\rho \sum_{i=1}^{n} \| X_i \|_{r+\delta}^r + \rho C_r E \max_{1\le j\le m+1} |S_{1,j}|^r. \tag{2.13}$$

If $\theta > r(r+\delta)/(2\delta)$, then for any $\rho > 0$, there exist positive constants $C_\rho = C(\rho, r, \delta, \theta) < \infty$ and $C_r = C(r) < \infty$ such that

$$\sum_{j=1}^{m} E(Y_j M_j^{r-1}) \le C_\rho \left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^2 \right)^{r/2} + \rho C_r E \max_{1\le j\le m+1} |S_{1,j}|^r, \tag{2.14}$$

$$\sum_{j=1}^{m} E(Y_j \widetilde{M}_j^{r-1}) \le C_\rho \left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^2 \right)^{r/2} + \rho C_r E \max_{1\le j\le m+1} |S_{1,j}|^r. \tag{2.15}$$

Proof Let $\beta = \delta/[r(r+\delta)]$. By Lemma 1 with $p = r/(r-1), q = r+\delta$ and $s = r(r+\delta)/\delta$, and (2.10), we obtain that

$$\sum_{j=1}^{m} E\left(Y_j M_j^{r-1}\right) \le 10\alpha^\beta(k) \sum_{j=1}^{m} \| Y_j \|_{r+\delta} \cdot \| M_j \|_r^{r-1}$$

$$\leq 10 \cdot 2^{r-1}\alpha^{\beta}(k)\sum_{j=1}^{m} \| Y_j \|_{r+\delta} \cdot \left(E \max_{1\leq j\leq m+1} |S_{1,j}|^r\right)^{(r-1)/r}$$

$$\leq 5 \cdot 2^{r}\rho^{-(r-1)/r}\alpha^{\beta}(k)\sum_{i=1}^{n} \| X_i \|_{r+\delta} \cdot \left(\rho E \max_{1\leq j\leq m+1} |S_{1,j}|^r\right)^{(r-1)/r}$$

$$\leq \frac{5^r \cdot 2^{r^2}\alpha^{\beta r}(k)}{r\rho^{(r-1)}}\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}\right)^{r} + \frac{\rho(r-1)}{r}E \max_{1\leq j\leq m+1} |S_{1,j}|^r, \quad (2.16)$$

using the Hölder inequality $a^{1/r}b^{(r-1)/r} \leq \frac{1}{r}a + \frac{r-1}{r}b$ in the last inequality above. Put $B = \alpha^{\beta r}(k)\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}\right)^{r}$. If $\theta > (r-1)(r+\delta)/\delta$, then

$$B \leq n^{r-1}\alpha^{\beta r}(k)\sum_{i=1}^{n} \| X_i \|_{r+\delta}^{r} . \quad (2.17)$$

Taking $\lambda = (r-1)(r+\delta)/(\theta\delta)$, we have $0 < \lambda < 1$ and

$$n^{r-1}\alpha^{\beta r}(k) \leq Cn^{r-1}k^{-\theta\beta r} \leq Cn^{r-1-\lambda\theta\beta r} \leq Cn^{r-1-\lambda\theta\delta/(r+\delta)} = C. \quad (2.18)$$

A combination of (2.16) with (2.17) and (2.18) yields (2.12). If $\theta > r(r+\delta)/(2\delta)$, then

$$B \leq n^{r/2}\alpha^{\beta r}(k)\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^{2}\right)^{r/2} . \quad (2.19)$$

Taking $\lambda = r(r+\delta)/(2\theta\delta)$, we have $0 < \lambda < 1$ and

$$n^{r/2}\alpha^{\beta r}(k) \leq Cn^{r/2}k^{-\theta\beta r} \leq Cn^{r/2-\lambda\theta\beta r} \leq Cn^{r/2-\lambda\theta\delta/(r+\delta)} = C. \quad (2.20)$$

From (2.16), (2.19) and (2.20), then (2.14) follows. Similarly, we get (2.13) and (2.15).

Lemma 2.5 *If $\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\}$, then for any $\rho > 0$, there exist positive constants $C_\rho = C(\rho, r, v, \delta, \theta) < \infty$ and $C_r = C(r) < \infty$ such that*

$$\sum_{j=1}^{m} E\left(Y_j^2 M_j^{r-2}\right) \leq C_\rho \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{r/2} + C_\rho \sum_{i=1}^{n} \| X_i \|_{r+\delta}^{r} + \rho C_r E \max_{1\leq j\leq m+1} |S_{1,j}|^r, \quad (2.21)$$

$$\sum_{j=1}^{m} E\left(Y_j^2 \widetilde{M}_j^{r-2}\right) \leq C_\rho \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{r/2} + C_\rho \sum_{i=1}^{n} \| X_i \|_{r+\delta}^{r} + \rho C_r E \max_{1\leq j\leq m+1} |S_{1,j}|^r. \quad (2.22)$$

If $\theta > r(r+\delta)/(2\delta)$, then for any $\rho > 0$, there exist positive constants $C_\rho = C(\rho, r, \delta, \theta) < \infty$ and $C_r = C(r) < \infty$ such that

$$\sum_{j=1}^{m} E\left(Y_j^2 M_j^{r-2}\right) \leq C_\rho \left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^{2}\right)^{r/2} + \rho C_r E \max_{1\leq j\leq m+1} |S_{1,j}|^r, \quad (2.23)$$

$$\sum_{j=1}^{m} E(Y_j^2 \widetilde{M}_j^{r-2}) \leq C_\rho \left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^{2}\right)^{r/2} + \rho C_r E \max_{1\leq j\leq m+1} |S_{1,j}|^r. \quad (2.24)$$

Proof Let $\beta = \delta/[r(r+\delta)]$. By Lemma 2.1 (i) with $p = r/(r-2), q = (r+\delta)/2$ and $s = r(r+\delta)/(2\delta)$, and (2.10), we obtain that

$$\sum_{j=1}^{m} E(Y_j^2 M_j^{r-2}) = \sum_{j=1}^{m} E(Y_j^2)E(M_j^{r-2}) + \sum_{j=1}^{m} Cov\left(Y_j^2, M_j^{r-2}\right)$$

$$
\begin{aligned}
&\le \sum_{j=1}^{m} E(Y_j^2)E(M_j^{r-2}) + 10\alpha^{2\beta}(k)\sum_{j=1}^{m} \| Y_j \|_{r+\delta}^2 \| M_j \|_r^{r-2} \\
&\le 2^{r-2}\sum_{j=1}^{m} E(Y_j^2)E\max_{1\le j\le m+1}|S_{1,j}|^{r-2} + C\alpha^{2\beta}(k)\sum_{j=1}^{m} \| Y_j \|_{r+\delta}^2 \left(E\max_{1\le j\le m+1}|S_{1,j}|^r\right)^{(r-2)/r} \\
&\le 2^{r-2}\left(\sum_{j=1}^{m} EY_j^2\right)\left(E\max_{1\le j\le m+1}|S_{1,j}|^r\right)^{(r-2)/r} \\
&\quad + Ck\alpha^{2\beta}(k)\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^2\right)\left(E\max_{1\le j\le m+1}|S_{1,j}|^r\right)^{(r-2)/r} \\
&\le \frac{C_1}{\rho^{r(r-2)/4}}\left(\sum_{j=1}^{m} EY_j^2\right)^{r/2} + \frac{C_2}{\rho^{r(r-2)/4}}k^{r/2}\alpha^{\beta r}(k)\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^2\right)^{r/2} \\
&\quad + \frac{2(r-2)\rho}{r}E\max_{1\le j\le m+1}|S_{1,j}|^r, \qquad (2.25)
\end{aligned}
$$

using the Hölder inequality $a^{2/r}b^{(r-2)/r} \le \frac{2}{r}a + \frac{r-2}{r}b$ in the last inequality above.

If $\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\}$, then $\sum_{i=1}^{\infty}\alpha^{(v-2)/v}(i) \le C\sum_{i=1}^{\infty} i^{-\theta(v-2)/v} < \infty$. By Lemma 2.1 (ii),

$$
\left(\sum_{j=1}^{m} EY_j^2\right)^{r/2} \le C\left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{r/2}. \qquad (2.26)
$$

Taking $\lambda = (r-1)(r+\delta)/(\theta\delta)$, we have $0 < \lambda < 1$ and

$$
n^{r/2-1}k^{r/2}\alpha^{\beta r}(k) \le Cn^{r/2-1}k^{r/2-\theta\beta r} \le Cn^{r/2-1+\lambda(r/2-\theta\beta r)} = Cn^{r(\lambda-1)/2} \le C.
$$

Hence

$$
k^{r/2}\alpha^{\beta r}(k)\left(\sum_{i=1}^{n} \| X_i \|_{r+\delta}^2\right)^{r/2} \le n^{r/2-1}k^{r/2}\alpha^{\beta r}(k)\sum_{i=1}^{n} \| X_i \|_{r+\delta}^r \le C\sum_{i=1}^{n} \| X_i \|_{r+\delta}^r. \qquad (2.27)
$$

A combination of (2.25) with (2.26) and (2.27) yields (2.21).

If $\theta > r(r+\delta)/(2\delta)$, then $\sum_{i=1}^{\infty}\alpha^{\delta/(r+\delta)}(i) \le C\sum_{i=1}^{\infty} i^{-\theta\delta/(r+\delta)} \le C\sum_{i=1}^{\infty} i^{-r/2} < \infty$. By Lemma 2.1 (ii),

$$
\left(\sum_{j=1}^{m} EY_j^2\right)^{r/2} \le C\left(\sum_{i=1}^{n} ||X_i||_{r+\delta}^2\right)^{r/2}. \qquad (2.28)
$$

Taking $\lambda = r(r+\delta)/(2\theta\delta)$, then we have $0 < \lambda < 1$ and

$$
k^{r/2}\alpha^{\beta r}(k) \le Ck^{r/2-\theta\beta r} \le Cn^{\lambda(r/2-\theta\beta r)} = Cn^{r(\lambda-1)/2} \le C. \qquad (2.29)
$$

From (2.25), (2.28) and (2.29), then (2.23) follows. Similarly, we get (2.22) and (2.24).

Lemma 2.6 *If* $\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\}$, *then*

$$
E\max_{1\le j\le m+1}|S_{1,j}|^r \le C\left\{\sum_{j=1}^{m+1} E|Y_j|^r + \sum_{i=1}^{n} \| X_i \|_{r+\delta}^r + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{r/2}\right\}, \qquad (2.30)
$$

$$
E\max_{1\le j\le m+1}|S_{2,j}|^r \le C\left\{\sum_{j=1}^{m+1} E|Z_j|^r + \sum_{i=1}^{n} \| X_i \|_{r+\delta}^r + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{r/2}\right\}. \qquad (2.31)
$$

If $\theta > r(r+\delta)/(2\delta)$, then

$$E \max_{1\le j\le m+1} |S_{1,j}|^r \le C\bigg\{ \sum_{j=1}^{m+1} E|Y_j|^r + \bigg(\sum_{i=1}^{n} \parallel X_i \parallel_{r+\delta}^2 \bigg)^{r/2} \bigg\}, \tag{2.32}$$

$$E \max_{1\le j\le m+1} |S_{2,j}|^r \le C\bigg\{ \sum_{j=1}^{m+1} E|Z_j|^r + \bigg(\sum_{i=1}^{n} \parallel X_i \parallel_{r+\delta}^2 \bigg)^{r/2} \bigg\}. \tag{2.33}$$

Proof By (2.8) and Lemma 2.2

$$\begin{aligned}\Big|\max_{1\le j\le m+1} S_{1,j}\Big|^r &= |N_0|^r = |Y_1 + M_1|^r \le d_1|Y_1|^r + rY_1M_1^{r-1} + d_2Y_1^2M_1^{r-2} + M_1^r \\ &\le d_1|Y_1|^r + rY_1M_1^{r-1} + d_2Y_1^2M_1^{r-2} + |N_1|^r \le \cdots \\ &\le d_1\sum_{j=1}^{m+1}|Y_j|^r + r\sum_{j=1}^{m}Y_jM_j^{r-1} + d_2\sum_{j=1}^{m}Y_j^2M_j^{r-2}. \end{aligned}\tag{2.34}$$

In the same way,

$$\Big|\max_{1\le j\le m+1}(-S_{1,j})\Big|^r \le d_1\sum_{j=1}^{m+1}|Y_j|^r + r\sum_{j=1}^{m}Y_j\widetilde{M}_j^{r-1} + d_2\sum_{j=1}^{m}Y_j^2\widetilde{M}_j^{r-2}. \tag{2.35}$$

As $\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\}$, we have

$$\begin{aligned}&E|\max_{1\le j\le m+1} S_{1,j}|^r \\ &\le d_1\sum_{j=1}^{m+1}E|Y_j|^r + C_\rho\sum_{i=1}^{n}\parallel X_i\parallel_{r+\delta}^r + C_\rho\bigg(\sum_{i=1}^{n}||X_i||_v^2\bigg)^{r/2} + \rho C_r E\max_{1\le j\le m+1}|S_{1,j}|^r\end{aligned}$$

by combining (2.34) with (2.12) and (2.21), and

$$\begin{aligned}&E|\max_{1\le j\le m+1}(-S_{1,j})|^r \\ &\le d_1\sum_{j=1}^{m+1}E|Y_j|^r + C_\rho\sum_{i=1}^{n}\parallel X_i\parallel_{r+\delta}^r + C_\rho\bigg(\sum_{i=1}^{n}||X_i||_v^2\bigg)^{r/2} + \rho C_r E\max_{1\le j\le m+1}|S_{1,j}|^r\end{aligned}$$

by combining (2.35) with (2.13) and (2.22). Hence, from (2.7) and the two equations above,

$$\begin{aligned}&E\max_{1\le j\le m+1}|S_{1,j}|^r \\ &\le C_1\sum_{j=1}^{m+1}E|Y_j|^r + C_\rho\sum_{i=1}^{n}\parallel X_i\parallel_{r+\delta}^r + C_\rho\bigg(\sum_{i=1}^{n}||X_i||_v^2\bigg)^{r/2} + \rho C_r E\max_{1\le j\le m+1}|S_{1,j}|^r.\end{aligned}$$

Thus

$$(1-\rho C_r)E\max_{1\le j\le m+1}|S_{1,j}|^r \le C_1\sum_{j=1}^{m+1}E|Y_j|^r + C_\rho\sum_{i=1}^{n}\parallel X_i\parallel_{r+\delta}^r + C_\rho\bigg(\sum_{i=1}^{n}||X_i||_v^2\bigg)^{r/2}$$

yields (2.30) by taking a sufficiently small ρ. Similarly, we obtain (2.31)–(2.33).

Proof of Theorem 2.1 By Lemmas 2.3 and 2.6 (the case of $\theta > \max\{v/(v-2), (r-1)(r+\delta)/\delta\}$)

$$E\max_{1\le j\le n}|S_j|^r \le C\bigg\{\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + \sum_{j=1}^{2(m+1)}E\max_{1\le l\le k}\bigg|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\bigg|^r$$

$$+\sum_{i=1}^{n} \| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n} ||X_i||_v^2\Big)^{r/2}\Big\}. \tag{2.36}$$

Using the Minkowski inequality to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\le l\le k}|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i|^r$ in the above, and noting (2.6), we have

$$\begin{aligned} E\max_{1\le j\le n}|S_j|^r &\le C\Big\{k^{r-1}\sum_{i=1}^{n}E|X_i|^r + \sum_{i=1}^{n}\| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n}||X_i||_v^2\Big)^{r/2}\Big\} \\ &\le C\Big\{n^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r + \sum_{i=1}^{n}\| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n}||X_i||_v^2\Big)^{r/2}\Big\}. \end{aligned}$$

Applying the inequality above to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\le l\le k}|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i|^r$ in (2.36),

$$\begin{aligned} E\max_{1\le j\le n}|S_j|^r &\le C\Big\{k^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r + \sum_{i=1}^{n}\| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n}||X_i||_v^2\Big)^{r/2}\Big\} \\ &\le C\Big\{n^{\lambda^2(r-1)}\sum_{i=1}^{n}E|X_i|^r + \sum_{i=1}^{n}\| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n}||X_i||_v^2\Big)^{r/2}\Big\}. \end{aligned}$$

Again, applying the inequality above to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i\right|^r$ in (2.36), and repeating t times in this way, we have

$$E\max_{1\le j\le n}|S_j|^r \le C\Big\{n^{\lambda^t(r-1)}\sum_{i=1}^{n}E|X_i|^r + \sum_{i=1}^{n}\| X_i \|_{r+\delta}^{r} + \Big(\sum_{i=1}^{n}||X_i||_v^2\Big)^{r/2}\Big\}$$

for integer $t \ge 1$. Since $0 < \lambda < 1$, $\lambda^t(r-1) < \varepsilon$ for some $t > 1$. Thus (2.2) holds.

Proof of Theorem 2.2 By Lemma 2.3 and Lemma 2.6 (the case of $\theta > r(r+\delta)/(2\delta)$)

$$\begin{aligned} E\max_{1\le j\le n}|S_j|^r &\le C\Big\{\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + \sum_{j=1}^{2(m+1)} E\max_{1\le l\le k}\Big|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i\Big|^r \\ &\quad + \Big(\sum_{i=1}^{n}\| X_i \|_{r+\delta}^{2}\Big)^{r/2}\Big\}, \end{aligned}$$

Which implies (2.5) in the same way as in the proof of Theorem 2.1.

3 Application

To show the application of the inequalities in Section 2, here we discuss the asymptotic normality of the general linear estimator for the fixed design regression. Consider observations

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni},\ 1 \le i \le n, \tag{3.1}$$

where the design points $x_{n1}, \dots, x_{nn} \in A$, which is a compact set of R^d, g is a bounded real valued function on A, and $\varepsilon_{n1}, \dots, \varepsilon_{nn}$ are regression errors with zero mean and finite variance σ^2. A common estimate of g is

$$g_n(x) = \sum_{i=1}^{n} w_{ni}(x)Y_{ni}, \tag{3.2}$$

where weight function $w_{ni}(x), i = 1, 2, \dots, n$, depend on the fixed design points $x_{n1}, \dots, x_{nn}$ and on the number of observations n.

In the independent case, the estimate (3.2) has been considered in many literatures, such as, Priestly and Chao (1972, [22]), Clark (1997, [23]), Georgiev (1984a, 1984b, 1988, [24–26]), Georgiev and Greblicki (1986, [27]), and the references therein. In various dependence cases, $g_n(x)$ has been also researched very much. Fox example, Fan (1990, [28]), Roussas (1989, [29]), Roussas et al. (1992, [30]), Tran, et al (1986, [31]) and the references therein.

Under the strong mixing condition, asymptotic normality of (3.2) has been established by Roussas et al. (1992, [30]). Here our purpose is to use the moment inequalities in Section 2 to give some more weaker conditions for asymptotic normality of the estimate (3.2). Adopting the basic assumptions of Roussas et al. (1992, [30]), we assume the following.

Assumption (A1). (i) g:$A \to R$ is a bounded function defined on the compact subset A of R^d; (ii) $\{\xi_t : t = 0, \pm 1, \dots\}$ is a strictly stationary and α-mixing time series with $E\xi_1 = 0$, $\text{var}(\xi_1) = \sigma^2 \in (0, \infty)$; (iii) For each n, the joint distribution of $\{\varepsilon_{ni} : 1 \le i \le n\}$ is the same as that of $\{\xi_1, \dots, \xi_n\}$.

Denote

$$w_n(x) := \max\{|w_{ni}(x)| : 1 \le i \le n\}, \quad \sigma_n^2(x) := \text{Var}(g_n(x)). \tag{3.3}$$

Assumption (A2). (i) $\Sigma_{i=1}^n |w_{ni}(x)| \le C$ for all $n \ge 1$; (ii) $w_n(x) = O(\Sigma_{i=1}^n w_{ni}^2(x))$; (iii) $\Sigma_{i=1}^n w_{ni}^2(x) = O(\sigma_n^2(x))$.

Assumption (A3). There exist positive integers $p := p(n)$ and $q := q(n)$ such that $p + q \le n$ for sufficiently large n and as $n \to \infty$,

$$qp^{-1} \to o, \quad np^{-1}\alpha(q) \to 0, \quad nqp^{-1} \sum_{i=1}^n w_{ni}^2(x) \to 0, \tag{3.4}$$

$$p \sum_{i=1}^n w_{ni}^2(x) \to 0. \tag{3.5}$$

Here we will prove the following result.

Theorem 3.1 *Let Assumptions* (A1) $\sim$ (A3) *be satisfied. If for some* $s > 0$, $E|\xi_1|^{2+s} < \infty$, *and*

$$\alpha(n) = O(n^{-\theta}) \quad \textit{for some} \quad \theta > (2+s)/s, \tag{3.6}$$

then

$$\sigma_n(x)^{-1}\{g_n(x) - Eg_n(x)\} \xrightarrow{d} N(0,\ 1). \tag{3.7}$$

Remark 3.1 Compare Theorem 3.1 here with Theorem 3.1 in Roussas et al. (1992, [30]), who use the conditions

$$p^2 \sum_{i=1}^n w_{ni}^2(x) \to \infty \quad (\text{as} \quad n \to \infty), \tag{3.8}$$

$$\sum_{i=1}^\infty \alpha^{s/(2+s)}(i) < \infty \quad \text{for some } s > 0. \tag{3.9}$$

Clearly, (3.5) is weaker than (3.8). Furthermore, (3.6) is almost as weak as (3.9). In addition, our proof is much more simple than that of Roussas et al. (1992, [30]).

Proof of Theorem 3.1 We first give some denotations. For convenience of writing, omit everywhere the argument x and set $S_n = \sigma_n^{-1}(g_n - Eg_n)$, $Z_{ni} = \sigma_n^{-1} w_{ni}\varepsilon_{ni}$ for $i = 1, 2, \dots, n$, so that $S_n = \sum_{i=1}^n Z_{ni}$.

Let $k = [n/(p+q)]$. Then S_n may be split as $S_n = S_n' + S_n'' + S_n'''$, where

$$S_n' = \sum_{m=1}^k y_{nm}, \quad S_n'' = \sum_{m=1}^k y_{nm}', \quad S_n''' = y_{nk+1}',$$

$$y_{nm} = \sum_{i=k_m}^{k_m+p-1} Z_{ni}, \quad y_{nm}' = \sum_{j=l_m}^{l_m+q-1} Z_{nj}, \quad y_{nk+1}' = \sum_{i=k(p+q)+1}^{n} Z_{ni},$$

$k_m = (m-1)(p+q)+1,\ l_m = (m-1)(p+q)+p+1, m = 1, \ldots, k$. Thus, to prove (3.7), it suffices to show that

$$E(S''_n)^2 \to 0,\ E(S'''_n)^2 \to 0 \tag{3.10}$$

and

$$S'_n \xrightarrow{d} N(0,\ 1). \tag{3.11}$$

By Lemma 2.1(ii), Assumption (A2) (ii) and (iii), and (3.4), we have

$$\begin{aligned} E(S''_n)^2 &\leq C\sum_{m=1}^{k}\sum_{i=k_m}^{k_m+q-1}\sigma_n^{-2}w_{ni}^2 \leq Ckq\sigma_n^{-2}w_n^2 \leq C\frac{n}{p+q}qw_n \\ &\leq C(1+qp^{-1})^{-1}nqp^{-1}\sum_{i=1}^{n}w_{ni}^2 \to 0 \end{aligned}$$

and

$$\begin{aligned} E(S'''_n)^2 &= E(y'_{nk+1})^2 \leq C\sum_{i=k(p+q)+1}^{n}\sigma_n^{-2}w_{ni}^2 \leq C(n-k(p+q))\sigma_n^{-2}w_n^2 \\ &\leq C\left(\frac{n}{p+q}-k\right)(p+q)w_n \leq C(1+qp^{-1})p\sum_{i=1}^{n}w_{ni}^2 \to 0. \end{aligned}$$

So (3.10) holds.

Now to prove (3.11). Put $s_n^2 = \Sigma_{m=1}^k \mathrm{var}(y_{nm})$. From Lemma 2.2 of Roussas et al. (1992, [30])

$$E(S'_n)^2 \to 1 \ \text{ and } \ s_n^2 \to 1. \tag{3.12}$$

Let Φ_x stand for the characteristic function of the *r. v.* X. Then, by Theorem 7.2 in Roussas and Ioannides (1987, [5]) and (3.4),

$$\left|\Phi_{s'_n}(t) - \prod_{m=1}^{k}\Phi_{y_{nm}}(t)\right| \leq C(k-1)\alpha(q) \leq Cnp^{-1}\alpha(q) \to 0. \tag{3.13}$$

Hence, $\{y_{nm} : m = 1, \ldots, k\}$ may be assumed to be independent random variables. From (3.12) and according to the Berry–Esseen central limit theorem, for (3.11) it suffices to show that

$$\sum_{m=1}^{k}E|y_{nm}|^r \to 0 \ \text{ for some } r > 2. \tag{3.14}$$

Since $\theta > (2+s)/s$ in (3.6), we may choose positive t such that $0 < t < s/2$ and $(2+s)/s < (1+t)(2+s)/(s-2t) < \theta$. Let $r = 2(1+t)$ and $\delta = s-2t$. Then $r+\delta = 2+s$ and

$$\frac{r(r+\delta)}{2\delta} = \frac{(1+t)(2+s)}{s-2t} < \theta.$$

Given positive $\varepsilon < (r-2)/2$, using Theorem 2.2, Assumption (A2) and (3.5), we have

$$\begin{aligned} \sum_{m=1}^{k}E|y_{nm}|^r &\leq C\sum_{m=1}^{k}\left\{p^{\varepsilon}\sum_{i=k_m}^{k_m+p-1}E|Z_{ni}|^r + \left(\sum_{i=k_m}^{k_m+p-1}\sigma_n^{-2}w_{ni}^2||\xi_l||_{r+\delta}^2\right)^{r/2}\right\} \\ &\leq C\left\{p^{\varepsilon}\sum_{i=1}^{n}|w_{ni}|^{r/2} + p^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}|^{r/2}\right\} \end{aligned}$$

$$\begin{aligned} &\leq \quad Cp^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}|^{r/2} \leq Cp^{(r-2)/2}w_{n}^{(r-2)/2}\sum_{i=1}^{n}|w_{ni}| \\ &\leq \quad C\bigg(p\sum_{i=1}^{n}w_{ni}^{2}\bigg)^{(r-2)/2} \to 0, \end{aligned}$$

so (3.14) holds, thus completing the proof.

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