Acta Mathematica Sinica, English Series
Nov., 2006, Vol. 22, No. 6, pp. 1725–1734
Published online: Apr. 12, 2006
DOI: 10.1007/s10114-005-0745-8
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Precise Large Deviations for Sums of Negatively Associated Random Variables with Common Dominatedly Varying Tails

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Abstract In this paper, we obtain results on precise large deviations for non-random and random sums of negatively associated nonnegative random variables with common dominatedly varying tail distribution function. We discover that, under certain conditions, three precise large-deviation probabilities with different centering numbers are equivalent to each other. Furthermore, we investigate precise large deviations for sums of negatively associated nonnegative random variables with certain negatively dependent occurrences. The obtained results extend and improve the corresponding results of Ng, Tang, Yan and Yang (*J. Appl. Prob.*, **41**, 93–107, 2004).

Keywords negatively associated, dominatedly varying tail, precise large deviation
MR(2000) Subject Classification 60F10, 60F05, 60G50

1 Introduction

In this section, we first introduce some classes of heavy-tailed distribution functions (d.f.s), for details, see Embrechts, Klüppelberg and Mikosch [1], etc. Let X be an r.v. with d.f. F on $[0,\infty)$ and $\overline{F}(x) = 1 - F(x)$. Denote by

$$\mathscr{K} = \left\{F : \int_0^\infty e^{\lambda x}\, F(dx) = \infty \text{ for any } \lambda > 0\right\}$$

the heavy-tailed distributions class. Denote by

$$\mathscr{L} = \left\{F : \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \text{ for any } y \in (-\infty,\infty)\right\}$$

the long-tailed distributions class. Denote by

$$\mathscr{D} = \left\{F : \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty \text{ for any } 0 < y < 1\right\}$$

the dominated-tailed distributions class, which is the main object of the present paper. Clearly, $\mathscr{L}$ and $\mathscr{D}$ are subclasses of $\mathscr{K}$ with better properties.

Received December 7, 2004, Revised April 18, 2005, Accepted November 3, 2005
Research supported by National Science Foundation of China (No. 10271087)

Furthermore, for any $y > 1$, set $\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$, $\overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$ and then define $L_F = \lim_{y\downarrow 1} \overline{F}_*(y)$, $\gamma_F = \inf\big\{ -\frac{\log \overline{F}_*(y)}{\log y}$ for all $y > 1\big\}$. The following proposition is well known.

Proposition 1.1 *$F \in \mathscr{D} \Longleftrightarrow \overline{F}_*(y) > 0$ for all $y > 1 \Longleftrightarrow \overline{F}_*(y) > 0$ for some $y > 1 \Longleftrightarrow L_F > 0 \Longleftrightarrow \gamma_F < \infty$.*

Some subclasses of $\mathscr{D}$ are as follows. Denote by $\mathscr{C} = \{F : L_F = 1\}$ the class of d.f. with consistently varying tail, which was first introduced by Cline and Samorodnitsky [2]. For some $0 < \alpha \le \beta < \infty$, denote by $\mathrm{ERV}(\alpha, \beta) = \{F : y^{-\beta} \le \overline{F}_*(y) \le \overline{F}^*(y) \le y^{-\alpha}$ for all $y > 1\}$ the extended regularly varying class. Particularly, if $\alpha = \beta$, it reduces to the regularly varying class, denoted by $\mathscr{R}_{-\alpha}$. Another well-known result is the following proposition.

Proposition 1.2 *The following relations are proper*:

$$\mathscr{R}_{-\alpha} \subset ERV(\alpha, \beta) \subset \mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{D} \subset \mathscr{K}.$$

Throughout this paper, $\{X_k : k \ge 1\}$ denotes a sequence of identically distributed r.v.s with d.f. F on $[0, \infty)$. For $n \ge 1$, we denote by S_n the nth partial sum or non-random sum of the sequence $\{X_k : k \ge 1\}$. Some earlier work on precise large deviations of non-random sums can be found in Nagaev [3, 4, 5] and Heyde [6, 7, 8]. Nagaev [9], [10] studied the precise large deviations for $\mathscr{R}_{-\alpha}$. The precise deviations for $ERV(\alpha, \beta)$ have been investigated by many researchers such as Cline and Hsing [11], Klüppelberg and Mikosch [12], Mikosch and Nagaev [13], Tang, Su, Jiang and Zhang [14], Ng, Tang, Yan and Yang [15], Hu [16], among others. The following result is due to Ng, Tang, Yan and Yang [15]:

Theorem 1.A *Let $\{X_k : k \ge 1\}$ be a sequence of independent, identically distributed (i.i.d.) nonnegative r.v.s with d.f. $F \in \mathscr{C}$ and finite expectation μ. Then, for any fixed $\gamma > 0$,*

$$\lim_{n\to\infty} \sup_{x \ge \gamma n} \left| \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} - 1 \right| = 0. \tag{1.1}$$

Just as it was pointed out in Embrechts, Klüppelberg and Mikosch [1] that, random sums are the bread and butter of insurance mathematics, many researchers have investigated precise large deviations for random sums on the basis of research on non-random sums. We refer the reader to Cline and Hsing [11], Klüppelberg and Mikosch [12], Mikosch and Nagaev [13], Tang, Su, Jiang and Zhang [14], Ng, Tang, Yan and Yang [15], among others. In this paper we always suppose that $\{N(t) : t \ge 0\}$ is a nonnegative integer-valued process and $\lambda(t) = EN(t) \to \infty$ as $t \to \infty$. $S_{N(t)} = \sum_{k=1}^{N(t)} X_k, t \ge 0$ is the so-called random sum. Under the following condition:

$$N(t)(\lambda(t))^{-1} \xrightarrow{P} 1 \quad \text{as} \quad t \to \infty \tag{1.2}$$

and the following assumption:

Assumption 1.1 *For any $\delta > 0$ and some $\varepsilon (= \varepsilon(\delta)) > 0$,*

$$E(1+\varepsilon)^{N(t)} I(N(t) > (1+\delta)\lambda(t)) = o(1) \quad \textit{as} \quad t \to \infty. \tag{1.3}$$

Klüppelberg and Mikosch [12] studied the precise large deviations for random sums of r.v.s belonging to $\mathrm{ERV}(\alpha, \beta)$, where $1 < \alpha \le \beta < \infty$.

Tang, Su, Jiang and Zhang [14] obtained the same result under a weaker condition:

Assumption 1.2 *For some $\varepsilon > 0$ and any $\delta > 0$,*

$$E(N(t))^{\beta+\varepsilon} I(N(t) > (1+\delta)\lambda(t)) = O(\lambda(t)) \quad \textit{as} \quad t \to \infty. \tag{1.4}$$

Recently, Ng, Tang, Yan and Yang [15] obtained a more general result (Theorem 1.B below) under a reasonable condition:

Assumption 1.3 *For some $p > \gamma_F$ and any $\delta > 0$*

$$E(N(t))^p I(N(t) > (1+\delta)\lambda(t)) = O(\lambda(t)) \quad \textit{as} \quad t \to \infty. \tag{1.5}$$

Theorem 1.B *Let $\{X_k : k \ge 1\}$ be a sequence of i.i.d., nonnegative r.v.s with d.f. $F \in \mathscr{C}$ and finite expectation μ, independent of a nonnegative and integer-valued process $\{N(t) : t \ge 0\}$.*

Assume that $N(\cdot)$ *satisfies Assumption* 1.3. *Then for any* $\gamma > 0$,

$$\lim_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \left| \frac{P(S_{N(t)} - \mu\lambda(t) > x)}{\lambda(t)\overline{F}(x)} - 1 \right| = 0. \tag{1.6}$$

From the development of the precise large deviations, we find that researchers weaken the restrictions on $N(\cdot)$ and extend the scope of F step by step. We will go further in this direction and do some work in the following three aspects:

1) In former research, $\{X_k : k \geq 1\}$ was always assumed to be a sequence of i.i.d., nonnegative r.v.s with common d.f. $F \in \mathscr{C}$, where X_k in most cases was the kth claim size or the stocks held by the kth investor. In this paper, the precise large deviations of negatively associated r.v.s with common d.f. $F \in \mathscr{D}$ will be investigated.

2) On the basis of the above research, we can conveniently investigate the precise large deviations for sums of negatively associated r.v.s with certain negatively dependent occurrences. Compared with Proposition 5.1 of Ng, Tang, Yan and Yang [15], we extend the scope of occurrence r.v.s.

3) We will discuss and compare the precise large deviations with centering numbers 0, $\lambda(t)\mu$ and $N(t)\mu$ respectively. We discover that, under certain conditions, the above-mentioned three large-deviation probabilities are equivalent to each other. Furthermore, in the last case, the condition on $N(\cdot)$ can be weakened to:

Assumption 1.4 *For any* $\delta > 0$

$$EN(t)I(N(t) > (1+\delta)\lambda(t)) = o(\lambda(t)) \quad as \quad t \to \infty. \tag{1.7}$$

The outline of the present paper is as follows: Section 2 investigates the precise large deviations for non-random and random sums of a sequence of negatively associated r.v.s $\{X_k : k \geq 1\}$. Section 3 investigates the precise large deviations for sums of negatively associated r.v.s with certain negatively dependent occurrences. Without special statements, the notation in this section are still valid and all limit relationships are as $n \to \infty$ or $t \to \infty$.

2 Precise Large Deviations of Sums

The concept of negatively associated was first introduced by Joag-Dev and Proschan [17]. By definition, a sequence $\{X_k : k \geq 1\}$ is said to be negatively associated (NA) if, for any disjoint nonempty subsets A and B of $\{1, \ldots, m\}, m \geq 2$ and any coordinatewise nondecreasing functions f and g, the inequality

$$\mathrm{Cov}(f(X_i : i \in A), g(X_j : j \in B)) \leq 0$$

holds whenever the moment involved exists. For details, one can also refer to Matula [18], Su, Zhao and Wang [19], Shao [20], among others.

Let $A \subset \{1, 2, \ldots\}$, $\sigma(X_i : i \in A)$ be a σ-field generated by $X_i, i \in A$ and define

$$\varphi(1) = \sup_{n\geq 2} \sup_{k\geq 1} \sup_{C\in\sigma(X_i:1\leq i\leq n, i\neq k)\, D\in\sigma(X_k), P(D)>0} |P(C|D) - P(C)|.$$

Obviously, $0 \leq \varphi(1) \leq 1$ and when $\{X_k : k \geq 1\}$ is independent, $\varphi(1) = 0$.

Theorem 2.1 *Let* $\{X_k : k \geq 1\}$ *be a sequence of NA, identically distributed nonnegative r.v.s with d.f.* $F \in \mathscr{D}$ *and finite expectation* μ. *Then, for any fixed* $\gamma > 0$,

$$1 \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n > x)}{n\overline{F}(x)} \leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n > x)}{n\overline{F}(x)} \leq L_F^{-1} \tag{2.1}$$

and

$$\max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\} \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)}$$
$$\leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \leq L_F^{-1}. \tag{2.2}$$

Theorem 2.2 *Let* $\{X_k : k \geq 1\}$ *be a sequence of NA, identically distributed nonnegative r.v.s with d.f.* $F \in \mathscr{D}$ *and finite expectation* μ, *independent of a nonnegative and integer-valued* *process* $\{N(t) : t \geq 0\}$. *Assume that* $N(\cdot)$ *satisfies Assumption* 1.3. *Then, for any fixed* $\gamma > 0$,

$$1 \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} > x)}{\lambda(t)\overline{F}(x)} \leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} > x)}{\lambda(t)\overline{F}(x)} \leq L_F^{-1} \tag{2.3}$$

and

$$\max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} - \lambda(t)\mu > x)}{\lambda(t)\overline{F}(x)}$$

$$\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} - \lambda(t)\mu > x)}{\lambda(t)\overline{F}(x)} \leq L_F^{-1}. \tag{2.4}$$

If $N(\cdot)$ *only satisfies Assumption* 1.4, *then for any fixed* $\gamma > 0$,

$$\max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} - N(t)\mu > x)}{\lambda(t)\overline{F}(x)}$$

$$\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_{N(t)} - N(t)\mu > x)}{\lambda(t)\overline{F}(x)} \leq L_F^{-1}. \tag{2.5}$$

Remark 2.1 In particular, if $F \in \mathscr{C}$ and $\{X_k : k \geq 1\}$ is a sequence of i.i.d. r.v.s, then $L_F = 1$ and $\varphi(1) = 0$. Then, by the above two theorems, we can immediately obtain Theorem 1.A and 1.B of this paper and

$$P(S_{N(t)} > x) \sim \lambda(t)\overline{F}(x) \quad \text{uniformly for } x \geq \gamma\lambda(t).$$

And if $N(\cdot)$ only satisfies Assumption 1.4, we have

$$P\left(\sum_{k=1}^{N(t)} (X_k - \mu) > x\right) \sim \lambda(t)\overline{F}(x) \quad \text{uniformly for } x \geq \gamma\lambda(t).$$

Hence, under a proper condition, $P(\sum_{k=1}^{N(t)} X_k > x)$, $P(\sum_{k=1}^{N(t)} X_k - \lambda(t)\mu > x)$ and $P(\sum_{k=1}^{N(t)} (X_k - \mu) > x)$ are equivalent to each other uniformly for $x \geq \gamma\lambda(t)$.

Proof of Theorem 2.1 We first prove (2.1). By Property P_3 of Joag-Dev and Proschan [17], $\mu = EX_1 < \infty$ and the standard argument, we have the left-hand inequality of (2.1) immediately. Now we prove the right-hand inequality of (2.1) along the line of the proof of Theorem 3.1 of Ng, Tang, Yan and Yang [15]. For any $\theta \in (0, 1)$, define

$$\widetilde{X}_k = X_k I(X_k \leq \theta x) + \theta x I(X_k > \theta x),\ k \geq 1 \ \text{ and } \ \widetilde{S}_n = \sum_{k=1}^{n} \widetilde{X}_k,\ n \geq 1.$$

By Property P_6 of Joag-Dev and Proschan [17], it follows that $\{\widetilde{X}_k : k \geq 1\}$ is also NA. It is easy to show that

$$P(S_n > x) \leq n\overline{F}(\theta x) + P(\widetilde{S}_n > x). \tag{2.6}$$

We only estimate the second term in (2.6). Let $a = \max\{-\log(n\overline{F}(\theta x)), 1\}$, which tends to ∞ uniformly for $x \geq \gamma n$. For any fixed $h > 0$, by Property P_2 of Joag-Dev and Proschan [17],

$$\frac{P(\widetilde{S}_n > x)}{n\overline{F}(\theta x)} \leq e^{-hx+a}\left(\int_0^{\theta x} e^{ht}\, F(dt) + e^{h\theta x}\overline{F}(\theta x)\right)^n$$

$$\leq \exp\left\{n\int_0^{\theta x} (e^{ht} - 1)\, F(dt) + n(e^{h\theta x} - 1)\overline{F}(\theta x) - hx + a\right\}. \tag{2.7}$$

For arbitrarily fixed $\tau > 1$, let $\rho > \max\{\gamma_F, \tau^{-1}\}$ and $h = \frac{a-\rho\tau\log a}{\theta x}$. Applying an elementary inequality, $e^u - 1 \leq ue^u$, we obtain that

$$n(e^{h\theta x} - 1)\overline{F}(\theta x) \leq nh\theta x e^{h\theta x}\overline{F}(\theta x) = (a - \rho\tau\log a)\, a^{-\rho\tau}$$

$$\leq a^{1-\rho\tau} \to 0 \qquad \text{as} \quad n \to \infty. \tag{2.8}$$

By the proof of Theorem 3.1 of Ng, Tang, Yan and Yang [15], we have that

$$n\int_0^{\theta x} (e^{ht} - 1)\, \overline{F}(dt) - hx + a \to -\infty \qquad \text{as} \quad n \to \infty. \tag{2.9}$$

Hence, by (2.6)–(2.9) and $F \in \mathscr{D}$, we have

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_n > x)}{n\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(\theta x)}{\overline{F}(x)} = (\overline{F}_*(\theta^{-1}))^{-1}.$$

Letting $\theta \uparrow 1$, yields the right-hand inequality of (2.1).

Now we prove (2.2). The right-hand inequality of (2.2) follows immediately from the right-hand inequality of (2.1). We start to prove the left-hand inequality of (2.2). By the left-hand inequality of (2.1) and $x \geq \gamma n$,

$$\begin{aligned}\liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} &\geq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n > x(1+\mu\gamma^{-1}))}{n\overline{F}(x(1+\mu\gamma^{-1}))} \frac{\overline{F}(x(1+\mu\gamma^{-1}))}{\overline{F}(x)} \\ &\geq \liminf_{x\to\infty} \frac{\overline{F}(x(1+\mu\gamma^{-1}))}{\overline{F}(x)} = \overline{F}_*(1+\mu\gamma^{-1}). \end{aligned} \tag{2.10}$$

On the other hand, if $\varphi(1) = 1$, then the left-hand inequality of (2.2) obviously holds. If $\varphi(1) < 1$, let $S_n^{(k)} = S_n - X_k, k = 1, \ldots, n,\ n \geq 1$. Then for any $\lambda > 1$,

$$\begin{aligned} P(S_n - n\mu > x) &\geq P(S_n - n\mu > x,\ \max_{1\leq k\leq n} X_k > \lambda x) \\ &\geq \sum_{k=1}^{n} P(S_n^{(k)} - n\mu > (1-\lambda)x,\ X_k > \lambda x) - (n\overline{F}(\lambda x))^2 \\ &\geq \sum_{k=1}^{n} (P(S_n^{(k)} - n\mu > (1-\lambda)x)\overline{F}(\lambda x) - \varphi(1)\overline{F}(\lambda x)) - (n\overline{F}(\lambda x))^2 \\ &= n\overline{F}(\lambda x)\left(n^{-1}\sum_{k=1}^{n} P(S_n^{(k)} - n\mu > (1-\lambda)x) - \varphi(1) - n\overline{F}(\lambda x)\right). \end{aligned} \tag{2.11}$$

Since $\mu = EX_1 < \infty$, $n\overline{F}(\lambda x) \to 0$ as $n \to \infty$ uniformly for $x \geq \gamma n$. By Theorem 1 of Matula [18], we have $P(S_n^{(k)} - n\mu > (1-\lambda)x) \to 1$ as $n \to \infty$ uniformly for $x \geq \gamma n$, which, together with the Toeplitz lemma, implies that

$$n^{-1}\sum_{k=1}^{n} P(S_n^{(k)} - n\mu > (1-\lambda)x) \to 1 \quad \text{as } n \to \infty \text{ uniformly for } x \geq \gamma n. \tag{2.12}$$

By (2.11) and (2.12), we obtain that

$$\liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} = \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(\lambda x)} \frac{\overline{F}(\lambda x)}{\overline{F}(x)} \geq (1-\varphi(1))\overline{F}_*(\lambda).$$

Let $\lambda \downarrow 1$ in the above inequality. Then we have

$$\liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_n - n\mu > x)}{n\overline{F}(x)} \geq (1-\varphi(1))L_F. \tag{2.13}$$

Combining (2.10) and (2.13), we obtain the left-hand inequality of (2.2). This completes the proof of Theorem 2.1.

Proof of Theorem 2.2 By Lemma 2.4 of Ng, Tang, Yan and Yang [15] and Assumption 1.3, we obtain (1.2), i.e. $N(t)(\lambda(t))^{-1} \xrightarrow{P} 1$ as $t \to \infty$. We first prove (2.3). For any $0 < \delta < 1$, we use the idea of the proof of Theorem 4.1 of Ng, Tang, Yan and Yang [15] and divide $P(S_{N(t)} > x)$ into three parts as

$$\begin{aligned} P(S_{N(t)} > x) &= \left(\sum_{n<(1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t)\leq n\leq(1+\delta)\lambda(t)} + \sum_{n>(1+\delta)\lambda(t)}\right) P(S_n > x)P(N(t) = n) \\ &=: K_{11} + K_{12} + K_{13}. \end{aligned} \tag{2.14}$$

We first estimate K_{11}. By (1.2) and the right-hand inequality of (2.1) of Theorem 2.1, for all large t,

$$\begin{aligned} K_{11} &\leq P(S_{[(1-\delta)\lambda(t)]} > x)P(N(t) < (1-\delta)\lambda(t)) \\ &\leq 2[(1-\delta)\lambda(t)]\overline{F}(x)L_F^{-1}P(N(t) < (1-\delta)\lambda(t)) = o(\lambda(t)\overline{F}(x)). \end{aligned} \tag{2.15}$$

Now we deal with K_{12}. For any $\varepsilon > 0$, by the right-hand inequality of (2.1) of Theorem 2.1, for all large t,

$$K_{12} \le P(S_{[(1+\delta)\lambda(t)]} > x) P\left((1-\delta)\lambda(t) \le N(t) \le (1+\delta)\lambda(t)\right) \le (1+\varepsilon)(1+\delta)\lambda(t)\overline{F}(x)L_F^{-1}. \quad (2.16)$$

Besides, for any $0 < \varepsilon < 1$, by (1.2) and the left-hand inequality of (2.1) of Theorem 2.1, for all large t,

$$K_{12} \ge P(S_{[(1-\delta)\lambda(t)]} > x) P\left((1-\delta)\lambda(t) \le N(t) \le (1+\delta)\lambda(t)\right) \ge (1-\varepsilon)(1-\delta)\lambda(t)\overline{F}(x). \quad (2.17)$$

In order to estimate K_{13}, we need the following lemma:

Lemma 2.1 *Let $\{X_k : k \ge 1\}$ be a sequence of NA, identically distributed nonnegative r.v.s with $EX_1 = \mu < \infty$. Then, for any $v > 0$ and $x > 0$,*

$$P(S_n > x) \le n\overline{F}(xv^{-1}) + (e\mu n x^{-1})^v. \quad (2.18)$$

Proof For any fixed $x > 0$ and $v > 0$, let

$$\widetilde{X}_k = X_k I(X_k \le xv^{-1}) + xv^{-1} I(X_k > xv^{-1}), \ k \ge 1 \ \text{ and } \ \widetilde{S}_n = \sum_{k=1}^n \widetilde{X}_k, \ n \ge 1.$$

It is not hard to show that

$$P(S_n > x) \le n\overline{F}(xv^{-1}) + P(\widetilde{S}_n > x). \quad (2.19)$$

And, by Property P_6 of Joag-Dev and Proschan [17], we know that $\{\widetilde{X}_k : k \ge 1\}$ is also NA. Since for any $h > 0$, $(e^{ht} - 1)t^{-1}$ is nondecreasing for $t > 0$, we obtain by Property P_2 of Joag-Dev and Proschan [17] and an elementary inequality $1 + u \le e^u$ that

$$\begin{aligned}
P(\widetilde{S}_n > x) &\le e^{-hx}(Ee^{h\widetilde{X}_1})^n = e^{-hx}\left(\int_0^{xv^{-1}} e^{ht}\, F(dt) + e^{hxv^{-1}}\overline{F}(xv^{-1})\right)^n \\
&= e^{-hx}\left(1 + \int_0^{xv^{-1}} (e^{ht} - 1)\, F(dt) + (e^{hxv^{-1}} - 1)\overline{F}(xv^{-1})\right)^n \\
&\le e^{-hx}\left(1 + (e^{hxv^{-1}} - 1)x^{-1}v\left(\int_0^{xv^{-1}} t\, F(dt) + xv^{-1}\overline{F}(xv^{-1})\right)\right)^n \\
&\le e^{-hx}(1 + (e^{hxv^{-1}} - 1)x^{-1}v\mu)^n \\
&\le \exp\{n(e^{hxv^{-1}} - 1)x^{-1}v\mu - hx\}. \qquad (2.20)
\end{aligned}$$

Letting $h = vx^{-1}\log(xn^{-1}\mu^{-1} + 1) (> 0)$ in (2.20), we obtain that

$$P(\widetilde{S}_n > x) \le (en\mu x^{-1})^v. \quad (2.21)$$

By (2.19) and (2.21), we know that (2.18) holds.

Now we continue to estimate K_{13}. Let $v = p > \gamma_F (\ge 1)$ in (2.18). By Lemma 2.1, (1.2), Assumption 1.3 and Lemma 2.1 of Ng, Tang, Yan and Yang [15], we obtain

$$\begin{aligned}
K_{13} &\le \sum_{n \ge (1+\delta)\lambda(t)} \left(n\overline{F}(xp^{-1}) + (e\mu n x^{-1})^p\right) P(N(t) = n) \\
&= O(1)\left(\overline{F}(x)EN(t)I(N(t) \ge (1+\delta)\lambda(t)) + x^{-p}E(N(t))^p I(N(t) \ge (1+\delta)\lambda(t))\right) \\
&= o(\lambda(t)\overline{F}(x)). \qquad (2.22)
\end{aligned}$$

By (2.14), (2.15), (2.16), (2.17), (2.22) and the arbitrariness of ε and δ, we know that (2.3) holds.

Next, we prove (2.4). For any $0 < \delta < 1$, we also divide $P(S_{N(t)} - \lambda(t)\mu > x)$ into three parts as

$$\begin{aligned}
P(S_{N(t)} - \lambda(t)\mu > x) &= \sum_{n=1}^\infty P(S_n - \lambda(t)\mu > x)P(N(t) = n) \\
&= \left(\sum_{n < (1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t) \le n \le (1+\delta)\lambda(t)} + \sum_{n > (1+\delta)\lambda(t)}\right) P(S_n - \lambda(t)\mu > x)P(N(t) = n) \\
&=: K_{21} + K_{22} + K_{23}. \qquad (2.23)
\end{aligned}$$

Clearly,

$$K_{2i} \le K_{1i} = o(\lambda(t)\overline{F}(x)) \quad \text{as} \;\; t \to \infty, \quad i = 1, 3. \tag{2.24}$$

And for any $\varepsilon > 0$ and all large t,

$$K_{22} \le K_{12} \le (1+\varepsilon)(1+\delta)\lambda(t)\overline{F}(x)L_F^{-1}. \tag{2.25}$$

For any $0 < \varepsilon < \frac{1}{2}$, by the left-hand inequality of (2.2) of Theorem 2.1 and $x \ge \gamma\lambda(t)$, for all large t,

$$\begin{aligned} K_{22} &\ge P(S_{[(1-\delta)\lambda(t)]} > x + \lambda(t)\mu)P((1-\delta)\lambda(t) \le N(t) \le (1+\delta)\lambda(t)) \\ &\ge (1-\varepsilon)[(1-\delta)\lambda(t)]\overline{F}(x + \lambda(t)\mu - [(1-\delta)\lambda(t)]\mu) \max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\} \\ &\ge (1-2\varepsilon)(1-\delta)\lambda(t)\overline{F}(x(1+2\delta\gamma^{-1}\mu)) \max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\}. \end{aligned} \tag{2.26}$$

By (2.23), (2.24), (2.25), (2.26) and the arbitrariness of ε and δ, we know that (2.4) holds.

Finally, we prove (2.5). By Lemma 2.4 of Ng, Tang, Yan and Yang [15], we easily see that Assumption 1.4 implies (1.2). For any $0 < \delta < 1$, we also divide $P(S_{N(t)} - N(t)\mu > x)$ into three parts as

$$\begin{aligned} P(S_{N(t)} - N(t)\mu > x) &= \sum_{n=1}^{\infty} P(S_n - n\mu > x)P(N(t) = n) \\ &= \Bigg(\sum_{n<(1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t) \le n \le (1+\delta)\lambda(t)} + \sum_{n>(1+\delta)\lambda(t)} \Bigg) P(S_n - n\mu > x)P(N(t) = n) \\ &=: K_{31} + K_{32} + K_{33}. \end{aligned} \tag{2.27}$$

Clearly,

$$K_{31} \le K_{11} = o(\lambda(t)\overline{F}(x)) \quad \text{as} \;\; t \to \infty. \tag{2.28}$$

And for any $\varepsilon > 0$ and all large t,

$$K_{32} \le K_{12} \le (1+\varepsilon)(1+\delta)\lambda(t)\overline{F}(x)L_F^{-1}. \tag{2.29}$$

Furthermore, for any $0 < \varepsilon < \frac{1}{2}$, by (1.2) and the left-hand inequality of (2.2) of Theorem 2.1, for all large t,

$$K_{32} \ge (1-2\varepsilon)(1-\delta)\lambda(t)\overline{F}(x) \max\{(1-\varphi(1))L_F, \overline{F}_*(1+\mu\gamma^{-1})\}. \tag{2.30}$$

For any $\varepsilon > 0$, by (1.2), Assumption 1.4 and the right-hand inequality of (2.1) of Theorem 2.1, for all large t,

$$\begin{aligned} K_{33} &\le (1+\varepsilon) \sum_{n>(1+\delta)\lambda(t)} n\overline{F}(x)L_F^{-1}P(N(t) = n) \\ &= (1+\varepsilon)L_F^{-1}\overline{F}(x)EN(t)I(N(t) > (1+\delta)\lambda(t)) \\ &= o(\lambda(t)\overline{F}(x)). \end{aligned} \tag{2.31}$$

By (2.27), (2.28), (2.29), (2.30), (2.31) and the arbitrariness of ε and δ, we get (2.5). This completes the proof of Theorem 2.2.

3 Precise Large Deviations of Sums with Occurrences

In this section, we use Theorems 2.1 and 2.2 to investigate the precise large deviations for sums with certain negatively dependent claim occurrences. In this aspect, Ng, Tang, Yan and Yang [15] ingeniously used the method of Shao [20] and presented the following result.

Theorem 3.A *Let $\{X_k : k \ge 1\}$ be a sequence of i.i.d. nonnegative r.v.s with finite expectation μ and d.f. $F \in \mathscr{C}$, let $\{I_k : k \ge 1\}$ be an NA sequence of Bernoulli variables with common expectation $q \in (0, 1]$ and let $\{N(t) : t \ge 0\}$ be an ordinary renewal counting process driven by a sequence of i.i.d. nonnegative r.v.s $\{Y_k : k \ge 1\}$ which have finite expectation. Suppose that the sequences $\{X_k : k \ge 1\}$, $\{I_k : k \ge 1\}$ and the process $\{N(t) : t \ge 0\}$ are mutually independent. Then, for any $\gamma > 0$ and $x \ge \gamma\lambda(t)$,*

$$P\Bigg(\sum_{k=1}^{N(t)} X_k I_k - \mu q\lambda(t) > x \Bigg) \sim q\lambda(t)\overline{F}(x). \tag{3.1}$$

Let $S_{1n} = \sum_{k=1}^{n} X_k I_k$ for all $n \geq 1$ and $S_1(t) = \sum_{k=1}^{N(t)} X_k I_k$ for all $t \geq 0$. Then S_{1n} and $S_1(t)$ denote the nonstandard non-random and random sums with NA occurrences respectively. We will extend and improve Theorem 3.A in the following three aspects.

1) We may assume that $\{X_k : k \geq 1\}$ and $\{I_k : k \geq 1\}$ are sequences of NA, identically distributed nonnegative r.v.s with common d.f. $F \in \mathscr{D}$ and $\{I_k : k \geq 1\}$ may not be a sequence of Bernoulli variables.

2) In particular, when $\{I_k : k \geq 1\}$ is a sequence of identically distributed Bernoulli variables, the sequence $\{I_k : k \geq 1\}$ may be assumed to be pairwise negatively quadrant dependent. The notion of negatively quadrant dependent was introduced by Lehmann [21]. By definition, a sequence $\{X_k : k \geq 1\}$ is said to be pairwise negatively quadrant dependent (*Pairwise NQD*) if, for any $i, j \in \{1, 2, \ldots\}, i \neq j$ and any $x_i, x_j \in \mathscr{R}$, the inequality

$$P(X_i > x_i, X_j > x_j) \leq P(X_i > x_i)P(X_j > x_j) \tag{3.2}$$

holds. It is will known that any NA sequence is *Pairwise NQD*.

3) We may assume that $N(\cdot)$ is a nonnegative and integer-valued counting process satisfying Assumption 1.3 or 1.4, and not necessarily be a renewal counting process. Meanwhile, it can also be driven by a sequence of positively dependent nonnegative r.v.s.

Let $Z_k = X_k I_k, k \geq 1$, $\varphi_1(1)$ be generated by $\{Z_k : k \geq 1\}$, F_1 be the d.f. of $X_1 I_1$ and the definitions of $\overline{F}_1, \overline{F}_{1*}$ and L_{F_1} be the same as in Section 1.

Theorem 3.1 *Let $\{X_k : k \geq 1\}$ be a sequence of NA, identically distributed nonnegative r.v.s with d.f. $F \in \mathscr{D}$ and finite expectation μ, let $\{I_k : k \geq 1\}$ be a sequence of NA, identically distributed nonnegative r.v.s with d.f. W and finite expectation q and let $\{N(t) : t \geq 0\}$ be a nonnegative and integer-valued counting process. Suppose that the sequence $\{X_k : k \geq 1\}$, $\{I_k : k \geq 1\}$ and the process $\{N(t) : t \geq 0\}$ are mutually independent. Then, for any fixed $\gamma > 0$,*

$$1 \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_{1n} > x)}{n\overline{F}_1(x)} \leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_{1n} > x)}{n\overline{F}_1(x)} \leq L_{F_1}^{-1} \tag{3.3}$$

and

$$\max\{(1-\varphi_1(1))L_{F_1}, \overline{F}_{1*}(1+\mu q\gamma^{-1})\} \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_{1n} - n\mu q > x)}{n\overline{F}_1(x)} \leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_{1n} - n\mu q > x)}{n\overline{F}_1(x)} \leq L_{F_1}^{-1}. \tag{3.4}$$

If $N(\cdot)$ satisfies Assumption 1.3, then for any fixed $\gamma > 0$,

$$1 \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) > x)}{\lambda(t)\overline{F}_1(x)} \leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) > x)}{\lambda(t)\overline{F}_1(x)} \leq L_{F_1}^{-1} \tag{3.5}$$

and

$$\max\{(1-\varphi_1(1))L_{F_1}, \overline{F}_{1*}(1+\mu q\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - \lambda(t)\mu q > x)}{\lambda(t)\overline{F}_1(x)} \leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - \lambda(t)\mu q > x)}{\lambda(t)\overline{F}_1(x)} \leq L_{F_1}^{-1}. \tag{3.6}$$

If $N(\cdot)$ only satisfies Assumption 1.4, then for any fixed $\gamma > 0$,

$$\max\{(1-\varphi_1(1))L_{F_1}, \overline{F}_{1*}(1+\mu q\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - N(t)\mu q > x)}{\lambda(t)\overline{F}_1(x)} \leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - N(t)\mu q > x)}{\lambda(t)\overline{F}_1(x)} \leq L_{F_1}^{-1}. \tag{3.7}$$

Theorem 3.2 *Let $\{X_k : k \geq 1\}$ be a sequence of NA, identically distributed nonnegative r.v.s with d.f. $F \in \mathscr{D}$ and finite expectation μ, let $\{I_k : k \geq 1\}$ be a sequence of Pairwise NQD, identically distributed Bernoulli variables with finite expectation q and let $\{N(t) : t \geq 0\}$ be a nonnegative and integer-valued counting process. Suppose that the sequence $\{X_k : k \geq 1\}$,*

$\{I_k : k \geq 1\}$ *and the process* $\{N(t) : t \geq 0\}$ *are mutually independent. Then, for any fixed* $\gamma > 0$,

$$1 \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_{1n} > x)}{nq\overline{F}(x)} \leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_{1n} > x)}{nq\overline{F}(x)} \leq L_F^{-1} \tag{3.8}$$

and

$$\max\{(1-\varphi_1(1))L_F, \overline{F}_*(1+\mu q\gamma^{-1})\} \leq \liminf_{n\to\infty} \inf_{x\geq\gamma n} \frac{P(S_{1n} - n\mu q > x)}{nq\overline{F}(x)}$$

$$\leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \frac{P(S_{1n} - n\mu q > x)}{nq\overline{F}(x)} \leq L_F^{-1}. \tag{3.9}$$

If $N(\cdot)$ *satisfies Assumption* 1.3, *then for any fixed* $\gamma > 0$,

$$1 \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) > x)}{\lambda(t)q\overline{F}(x)} \leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) > x)}{\lambda(t)q\overline{F}(x)} \leq L_F^{-1} \tag{3.10}$$

and

$$\max\{(1-\varphi_1(1))L_F, \overline{F}_*(1+\mu q\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - \lambda(t)\mu q > x)}{\lambda(t)q\overline{F}(x)}$$

$$\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - \lambda(t)\mu q > x)}{\lambda(t)q\overline{F}(x)} \leq L_F^{-1}. \tag{3.11}$$

If $N(\cdot)$ *only satisfies Assumption* 1.4, *then for any fixed* $\gamma > 0$,

$$\max\{(1-\varphi_1(1))L_F, \overline{F}_*(1+\mu q\gamma^{-1})\} \leq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - N(t)\mu q > x)}{\lambda(t)q\overline{F}(x)}$$

$$\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{P(S_1(t) - N(t)\mu q > x)}{\lambda(t)q\overline{F}(x)} \leq L_F^{-1}. \tag{3.12}$$

In Theorems 3.1 and 3.2, if we further assume that $\{X_k : k \geq 1\}$ is a sequence of i.i.d. nonnegative r.v.s with finite expectation μ and d.f. $F \in \mathscr{C}$, and $\{I_k : k \geq 1\}$ is an NA sequence of Bernoulli variables with common expectation $q \in (0, 1]$, then we can obtain Theorem 3.A easily.

Proof of Theorem 3.1 By (ii) of Theorem 3.3 of Cline and Samorodnitsky [2] and $F \in \mathscr{D}$, we know that $F_1 \in \mathscr{D}$. Thus $L_{F_1} > 0$ and $EZ_1 = \mu q$. Since $\{X_k : k \geq 1\}$ and $\{I_k : k \geq 1\}$ are mutually independent, by Property P_7 of Joag-Dev and Proschan [17], $\{X_k, I_k : k \geq 1\}$ is an NA sequence. Thus $\{Z_k : k \geq 1\}$ is also NA by Property P_6 of Joag-Dev and Proschan [17].

Hence, by Theorems 2.1 and 2.2, Theorem 4.1 is verified. This completes the proof of Theorem 4.1.

Proof of Theorem 3.2 Since $\{I_k : k \geq 1\}$ is a sequence of *Pairwise NQD*, identically distributed Bernoulli variables and $\{X_k : k \geq 1\}$ is an NA, identically distributed sequence, it is not hard to show that $\{Z_k : k \geq 1\}$is also a *Pairwise NQD* sequence, $\overline{F}_1(x) = q\overline{F}(x)$ for all $x \geq 0$, $EZ_1 = \mu q$, $\overline{F}_{1*}(\lambda) = \overline{F}_*(\lambda)$ for all $\lambda > 1$ and $L_{F_1} = L_F$. For any $\theta \in (0, 1)$, let

$$\widetilde{Z}_k = Z_k I(Z_k \leq \theta x) + \theta x I(Z_k > \theta x),\ k \geq 1 \text{ and } \widetilde{S}_{1n} = \sum_{k=1}^{n} \widetilde{Z}_k,\ n \geq 1.$$

It is easy to show that

$$P(\widetilde{S}_{1n} > x) \leq P(\widetilde{S}_n > x) \quad \text{for all } n \geq 1 \text{ and any } x > 0. \tag{3.13}$$

By (3.13), we know that the right-hand inequalities of Theorem 3.2 hold.

Obviously,

$$P(S_{1n} > x) \geq P(\max_{1\leq k\leq n} Z_k > x) \geq nq\overline{F}(x)(1 - nq\overline{F}(x)). \tag{3.14}$$

Set $S_{1n}^{(k)} = S_{1n} - Z_k, k = 1, \ldots, n, n \geq 1$. For any $\lambda > 1$, since $\{Z_k : k \geq 1\}$is also a *Pairwise NQD*, identically distributed sequence, we obtain that

$$P(S_{1n} - nq\mu > x) \geq P(S_{1n} - nq\mu > x, \max_{1\leq k\leq n} Z_k > \lambda x)$$

$$\geq \sum_{k=1}^{n} P(S_{1n} - nq\mu > x, Z_k > \lambda x) - \sum_{1\leq k<l\leq n} P(S_{1n} - nq\mu > x, Z_k > \lambda x, Z_l > \lambda x)$$

$$\geq \sum_{k=1}^{n} P(S_{1n}^{(k)} - nq\mu > (1-\lambda)x, Z_k > \lambda x) - (nq\overline{F}(\lambda x))^2$$

$$\geq \sum_{k=1}^{n} (P(S_{1n}^{(k)} - nq\mu > (1-\lambda)x)P(Z_k > \lambda x) - \varphi_1(1)P(Z_k > \lambda x)) - (nq\overline{F}(\lambda x))^2$$

$$= nq\overline{F}(\lambda x)\left(n^{-1}\sum_{k=1}^{n} P(S_{1n}^{(k)} - nq\mu > (1-\lambda)x) - \varphi_1(1) - nq\overline{F}(\lambda x)\right). \quad (3.15)$$

By (3.14) and (3.15), we can verify the left-hand inequalities of Theorem 3.2 along the same line of proof as that of Theorems 2.1 and 2.2. This completes the proof of Theorem 3.2.

Acknowledgements The authors would like to express their deep gratitude to the referees and editor for the detailed and critical comments on the previous version of this paper.

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