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Global Existence of Solutions for the Cauchy Problem of the Kawahara Equation with L^2 Initial Data

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Abstract In this paper we study solvability of the Cauchy problem of the Kawahara equation $\partial_t u + au\partial_x u + \beta\partial_x^3 u + \gamma\partial_x^5 u = 0$ with L^2 initial data. By working on the Bourgain space $X^{r,s}(R^2)$ associated with this equation, we prove that the Cauchy problem of the Kawahara equation is locally solvable if initial data belong to $H^r(R)$ and $-1 < r \le 0$. This result combined with the energy conservation law of the Kawahara equation yields that global solutions exist if initial data belong to $L^2(R)$.

Keywords Kawahara equation, Cauchy problem, global solution
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1 Introduction

In this paper we study the existence of a solution for the initial value problem

$$\partial_t u + au\partial_x u + \beta\partial_x^3 u + \gamma\partial_x^5 u = 0 \quad \text{in } R^2, \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in R, \tag{1.2}$$

where a, β, γ are real constants (a, $\gamma \ne 0$), and u_0 is a given function.

Equation (1.1) is an important dispersive equation that was first proposed by Kawahara in 1972 [1], as a model equation describing solitary wave propagation in media in which the first order dispersion coefficient β is anomalously small (see also [2–4]). A more specific physical background of this equation was introduced by Hunter and Scheurle in [5], where they used it to describe the evolution of solitary waves in fluids in which the Bond number is less than but close to 1/3 and the Froude number is close to 1. In the literature this equation is also referred to as the fifth order KdV equation or singularly perturbed KdV equation [6, 7]. The fifth order term $\partial_x^5 u$ is called the Kawahara term, and its coefficient, γ, is called the next order dispersion coefficient [2–4].

There has been a great deal of work on solitory wave solutions of the Kawahara equation over the past thirty years [1–10]. It is found that, similarly to the KdV equation, the Kawahara equation also has solitary wave solutions which decay rapidly to zero as $t \to \infty$, but unlike the KdV equation whose solitary wave solutions are non-oscillating, the solitary wave solutions of the Kawahara equation have oscillatory trails. This shows that the Kawahara equation is not only similar but also different from the KdV equation in the properties of solutions, like what happens between the formulations of this equation and the KdV equation. The strong physical background of the Kawahara equation and such similarities and differences between it and

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the KdV equation in both the form and the behavior of the solution render the mathematical treatment of this equation particularly interesting.

In [11] Cui and Tao established a series of Strichartz estimates for general linear dispersive equations in one space variable, and used such estimates to prove that the problem (1.1)–(1.2) has a local solution $u \in C([-T,T], H^r(R))$ (for some $T > 0$) if $u_0 \in H^r(R)$ and $r > 1/4$. This local result combined with the second and the third conservation laws implies that (1.1)–(1.2) has a global solution $u \in (C \cap L^\infty)(R, H^2(R))$ if $u_0 \in H^2(R)$. In this paper, we shall use the method initiated by Bourgain [12–14] and developed by Kenig, Ponce and Vega [15,16] to improve these results, and establish the global existence for this problem for L^2 initial data. More precisely, we shall first study the Bourgain space $X^{r,s}(R^2)$ $(r, s \in R)$ associated with a general class of dispersive equations including the Kawahara equation. By working in this space, we prove that the problem (1.1)–(1.2) has a local solution $u \in C([-T,T], H^r(R))$ (for some $T > 0$) if $u_0 \in H^r(R)$ and $-1 < r \le 0$. This result combined with the energy conservation law of the Kawahara equation then yields the existence of a global solution $u \in (C \cap L^\infty)(R, L^2(R))$ if $u_0 \in L^2(R)$. Exact statement of these results is presented in Theorem 4.1.

In the next section, we work for general linear dispersive equations in one space variable and study the Bourgain spaces $X^{r,s}(R^2)$ and the fundamental solution operator $W(t)$ associated with them. In Section 3 we concentrate attention on the Kawahara equation (1.1) and make an estimate for the bilinear mapping $B(u,v) = \partial_x(uv)$ on the Bourgain space associated with this equation. In Section 4 we use the results of Sections 2, 3 to establish the solvability of the problem (1.1)–(1.2).

2 The Bourgain Space $X^{r,s}(R^2)$ and the Fundamental Solution Operator $W(t)$

In this section we consider the general linear dispersive equation

$$\partial_t u - iP(D_x)u = f(x,t), \qquad (x,t) \in R^2, \tag{2.1}$$

where $P(D_x)$ is a linear partial differential operator with symbol $P(\xi)$ being a non-constant real polynomial in one variable, $D_x = -i\partial_x$, f is a complex-valued given function, and u is a complex-valued unknown function. We shall derive some general results on the Bourgain space associated with the equation (2.1) and the fundamental solution operator of this equation.

Clearly, the symbol of the equation (2.1) is $\tau - P(\xi)$. Hence we introduce

Definition 2.1 *For reals r and s, the Bourgain space $X^{r,s}(R^2)$ associated with the equation (2.1) is defined as the completion of the Schwartz space $S(R^2)$ in the norm*

$$\|u\|_{X^{r,s}(R^2)} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 + |\xi|^2\right)^r \left(1 + \left|\tau - P(\xi)\right|^2\right)^s \left|\widetilde{u}(\xi,\tau)\right|^2 d\xi d\tau \right)^{\frac{1}{2}}, \tag{2.2}$$

where "$\sim$" represents the Fourier transformation in two variables.

Definition 2.2 *The fundamental solution operator $W(t)$ $(t \in R)$ of the equation (2.1) is defined by*

$$W(t)\varphi(x) = c \int_{-\infty}^{\infty} e^{i(x\xi + tP(\xi))} \widehat{\varphi}(\xi) d\xi, \tag{2.3}$$

whenever the right-hand side makes sense in distribution sense, where "$\wedge$" represents the Fourier transformation in one variable, and $c = (2\pi)^{-1}$.

Since $P(\xi)$ is real, it is clear that for every real r the operator $W(t)$ $(t \in R)$ is well-defined on the Sobolev space $H^r(R)$, and

$$\left\|W(t)\varphi\right\|_{H^r(R)} = \left\|\varphi\right\|_{H^r(R)}, \qquad \forall\, \varphi \in H^r(R), \tag{2.4}$$

so that $\{W(t)\colon t \in R\}$ forms an one-parameter group of unitary operators on every $H^r(R)$.

Lemma 2.1 *Let $u(x,t) \in S(R^2)$ and $v(x,t) = W(-t)u(x,t)$. Then*

$$\widetilde{v}(\xi,\tau) = \widetilde{u}\left(\xi, \tau + P(\xi)\right), \qquad \forall\, (\xi,\tau) \in R^2. \tag{2.5}$$

For any real r, we denote by J^r the Bessel potential of order $-r$, i.e.

$$J_x^r \varphi = c \int_{-\infty}^{\infty} e^{ix\xi} \left((1 + |\xi|^2)^{\frac{r}{2}} \widehat{\varphi}(\xi)\right) d\xi,$$

whenever the right-hand side makes sense in distribution sense.

Lemma 2.2 *For reals r and s, $u(x,t) \in X^{r,s}(R^2)$ if and only if $J_x^r J_t^s W(-t)u(x,t) \in L^2(R^2)$, and*

$$\|u\|_{X^{r,s}(R^2)} = \|J_x^r J_t^s W(-t)u(x,t)\|_{L^2(R^2)}. \tag{2.6}$$

Lemma 2.3 *If $s > 1/2$ then $X^{r,s}(R^2) \subset C(R, H^r(R)) \cap L^\infty(R, H^r(R))$, and*

$$\sup_{t\in R} \|u(\cdot,t)\|_{H^r(R)} \le C_s \|u\|_{X^{r,s}(R^2)}, \qquad \forall\, u \in X^{r,s}(R^2). \tag{2.7}$$

Proof Let $v(x,t) = W(-t)u(x,t)$. Then by (2.4) we have

$$\begin{aligned}\sup_{t\in R}\|u(\cdot,t)\|_{H^r(R)} &= \sup_{t\in R}\|v(\cdot,t)\|_{H^r(R)} = \sup_{t\in R}\Big(\int_{-\infty}^{\infty}\big|J_x^r v(x,t)\big|^2 dx\Big)^{\frac12} \\ &\le \Big(\int_{-\infty}^{\infty}\Big(\sup_{t\in R}\big|J_x^r v(x,t)\big|\Big)^2 dx\Big)^{\frac12}.\end{aligned} \tag{2.8}$$

Since $s > 1/2$, by the Sobolev embedding inequality we have

$$\sup_{t\in R}|J_x^r v(x,t)| \le C_s \Big(\int_{-\infty}^{\infty}\big|J_t^s J_x^r v(x,t)\big|^2 dt\Big)^{\frac12} = C_s\Big(\int_{-\infty}^{\infty}\big|J_t^s J_x^r W(-t)u(x,t)\big|^2 dt\Big)^{\frac12}, \quad \forall\, x \in R. \tag{2.9}$$

Substituting (2.9) into (2.8) and using (2.6), we immediately get (2.7).

As in [11, 17, 18], we denote by $L_x^p L_T^q$ ($T > 0$, $1 \le p \le \infty$, $1 \le q < \infty$) the function space consisting of measurable functions $f(x,t)$ defined on $R \times [-T,T]$, satisfying $\|f\|_{L_x^p L_T^q} < \infty$, where the norm $\|\cdot\|_{L_x^p L_T^q}$ is defined as follows:

$$\|f\|_{L_x^p L_T^q} = \Big(\int_{-\infty}^{\infty}\Big(\int_{-T}^{T}|f(x,t)|^q dt\Big)^{\frac{p}{q}} dx\Big)^{\frac1p}\ (1 \le p < \infty);\ \|f\|_{L_x^\infty L_T^q} = \sup_{x\in R}\Big(\int_{-T}^{T}|f(x,t)|^q dt\Big)^{\frac1q}.$$

As usual, the notation D^α ($\alpha \ge 0$) denotes the absolute derivative of order α, i.e.,

$$D_x^\alpha \varphi(x) = c\int_{-\infty}^{\infty} e^{ix\xi}|\xi|^\alpha \widehat{\varphi}(\xi)d\xi,$$

whenever the right-hand side makes sense in distribution sense.

Lemma 2.4 *Let $m = \deg P$, and suppose that $m \ge 3$. Let $u \in X^{r,s}(R^2)$ and assume that $-(m-1)/2 < r \le 0$, $s > 1/2$. Then for any $T > 0$ and $0 \le \alpha \le (m-1+2r)/2$ there holds $D_x^\alpha u \in L_x^p L_T^2$, where $p = 2(m-1)/(m-1+2r-2\alpha)$. In particular, $u \in L^2_{\rm loc}(R^2)$.*

Proof Let $\widetilde{v}(\xi,\tau) = (1+|\xi|^2)^{\frac{r}{2}}\widetilde{u}(\xi,\tau)$, $(\xi,\tau) \in R^2$,

$$\widetilde{w}(\xi,\tau) = \big(1+|\tau-P(\xi)|^2\big)^{\frac{s}{2}}\widetilde{v}(\xi,\tau) = \big(1+|\tau-P(\xi)|^2\big)^{\frac{s}{2}}(1+|\xi|^2)^{\frac{r}{2}}\widetilde{u}(\xi,\tau), \quad (\xi,\tau) \in R^2.$$

Then, since $u \in X^{r,s}(R^2)$, we have $w \in L^2(R^2)$. We first prove that, for any $0 \le \theta \le 1$,

$$D_x^{\frac{\theta(m-1)}{2}} v(x,t) \in L_x^{2/(1-\theta)} L_T^2. \tag{2.10}$$

For this purpose we define, for every $\tau \in R$, a function $\varphi_\tau(x)$ by $\widehat{\varphi}_\tau(\xi) = \widetilde{w}(\xi, \tau+P(\xi))$, and let

$$g(x,t,\tau) = W(t)\varphi_\tau(x) = c\int_{-\infty}^{\infty} e^{i(x\xi+tP(\xi))}\widehat{\varphi}_\tau(\xi)d\xi = c\int_{-\infty}^{\infty} e^{i(x\xi+tP(\xi))}\widetilde{w}(\xi,\tau+P(\xi))d\xi.$$

Then, since $\widetilde{v}(\xi,\tau) = \big(1+|\tau-P(\xi)|^2\big)^{-\frac{s}{2}}\widetilde{w}(\xi,\tau)$, it is easy to see that

$$v(x,t) = c\int_{-\infty}^{\infty} e^{it\tau} g(x,t,\tau)(1+|\tau|^2)^{-\frac{s}{2}} d\tau. \tag{2.11}$$

Since $w \in L^2(R^2)$ so that also $\widetilde{w}(\xi,\tau+P(\xi)) \in L^2(R^2)$, by the Fubini theorem we infer that $\widehat{\varphi}_\tau \in L^2(R)$ for a.e. $\tau \in R$, which implies that $\varphi_\tau \in L^2(R)$ for a.e. $\tau \in R$. Hence, since $g(x,t,\tau) = W(t)\varphi_\tau(x)$, it follows from [11, Theorem 2.6] that $D_x^{\frac{m-1}{2}} g(x,t,\tau) \in L_x^\infty L_T^2$ for a.e. $\tau \in R$, and there exists a constant $C_T > 0$ depending only on T and P such that

$$\big\|D_x^{\frac{m-1}{2}} g(\cdot,\cdot,\tau)\big\|_{L_x^\infty L_T^2} \le C_T\|\varphi_\tau\|_{L^2(R)} = C_T\|\widetilde{w}(\cdot,\tau+P(\cdot))\|_{L^2(R)}, \quad \text{a.e. } \tau \in R.$$

Since $s > 1/2$, this implies, by (2.11), that $D_x^{\frac{m-1}{2}} v \in L_x^\infty L_T^2$ and

$$\begin{aligned}\big\|D_x^{\frac{m-1}{2}} v\big\|_{L_x^\infty L_T^2} &\le c\int_{-\infty}^{\infty} \big\|D_x^{\frac{m-1}{2}} g(\cdot,\cdot,\tau)\big\|_{L_x^\infty L_T^2} \cdot (1+|\tau|^2)^{-\frac{s}{2}} d\tau \\ &\le C\Big(\int_{-\infty}^{\infty} \big\|D_x^{\frac{m-1}{2}} g(\cdot,\cdot,\tau)\big\|^2_{L_x^\infty L_T^2} d\tau\Big)^{\frac12} \\ &\le C_T\Big(\int_{-\infty}^{\infty} \|\widetilde{w}(\cdot,\tau+P(\cdot))\|^2_{L^2(R)} d\tau\Big)^{\frac12} = C_T\|w\|_{L^2(R^2)}.\end{aligned}$$

Combining this result and the obvious assertion that $v \in L^2(R^2)$ we conclude, by using the interpolation theory, that (2.10) holds for any $0 \le \theta \le 1$.

We now take $\varphi \in C_0^\infty(R)$ such that $\varphi(\xi) = 1$ for $|\xi| \le 1$, and write

$$\begin{aligned}\widetilde{u}(\xi,\tau) &= (1+|\xi|^2)^{-\frac{r}{2}}\widetilde{v}(\xi,\tau) \\ &= \varphi(\xi)(1+|\xi|^2)^{-\frac{r}{2}}\widetilde{v}(\xi,\tau) + \big(1-\varphi(\xi)\big)\cdot|\xi|^r(1+|\xi|^2)^{-\frac{r}{2}}\cdot|\xi|^{-r}\widetilde{v}(\xi,\tau) \\ &\equiv \widetilde{u}_1(\xi,\tau) + \widetilde{u}_2(\xi,\tau).\end{aligned}$$

Since $v \in L^2(R^2)$ and $\varphi \in C_0^\infty(R)$, it is clear that $D_x^\alpha u_1 \in L^2(R^2)$ for any $\alpha \ge 0$, so that by also embedding $D_x^\alpha u_1 \in L_x^p L_T^2$ for any $\alpha \ge 0$, $2 \le p \le \infty$ and $T > 0$. Next, since $0 \le \alpha \le (m-1+2r)/2$ and $r \le 0$ imply that $0 \le \alpha - r \le (m-1)/2$, by (2.10) we have $h(x,t) \equiv F^{-1}\big(|\xi|^{\alpha-r}\widetilde{v}(\xi,\tau)\big) = D_x^{\alpha-r}v(x,t) \in L_x^p L_T^2$, where $p = 2(m-1)/(m-1+2(r-\alpha)) \ge 2$. We denote $\psi(\xi) = \big(1-\varphi(\xi)\big)|\xi|^r(1+|\xi|^2)^{-\frac{r}{2}}$. Then $\widehat{D_x^\alpha u_2}(\xi,t) = \psi(\xi)\widehat{h}(\xi,t)$. Since the function ψ is smooth and satisfies $|\partial_\xi^k\psi(\xi)| \le C_k(1+|\xi|)^{-k}$, $\forall\,\xi \in R$, $k = 0,1,2,\dots$, using the Mihlin multiplier theorem we conclude that $D_x^\alpha u_2 \in L_x^p L_T^2$. Hence, the desired assertion holds.

Lemma 2.5 *Let $\psi \in C_0^\infty(R)$, $r \in R$ and $s > 1/2$. Then there exist constants $C_j > 0$ ($j = 1,2,3$) depending only on ψ and s such that*

$$\big\|\psi(t)W(t)\varphi(x)\big\|_{X^{r,s}(R^2)} \le C_1\|\varphi\|_{H^r(R)}, \tag{2.12}$$

$$\big\|\psi(t)u(x,t)\big\|_{X^{r,s}(R^2)} \le C_2\|u\|_{X^{r,s}(R^2)}, \tag{2.13}$$

$$\Big\|\psi(t)\int_0^t W(t-t')u(x,t')dt'\Big\|_{X^{r,s}(R^2)} \le C_3\|u\|_{X^{r,s-1}(R^2)}, \tag{2.14}$$

whenever the right-hand sides make sense. In particular, the constants C_1, C_2 and C_3 are independent of the operator $W(t)$ or the polynomial P.

Proof (i) By Lemma 2.2 we have

$$\begin{aligned}\big\|\psi(t)W(t)\varphi(x)\big\|_{X^{r,s}(R^2)} &= \big\|J_x^rJ_t^sW(-t)\big(\psi(t)W(t)\varphi(x)\big)\big\|_{L^2(R^2)} \\ &= \big\|J_t^s\psi(t)\cdot J_x^r\varphi(x)\big\|_{L^2(R^2)} = \big\|J_t^s\psi(t)\big\|_{L^2(R)}\cdot\big\|J_x^r\varphi(x)\big\|_{L^2(R)}.\end{aligned}$$

Hence the inequality (2.12) follows from the fact that $J_t^s\psi(t) \in L^2(R)$, which is obvious.

(ii) Similarly, by Lemma 2.2 we have

$$\big\|\psi(t)u(x,t)\big\|_{X^{r,s}(R^2)} = \big\|J_x^rJ_t^sW(-t)\big(\psi(t)u(x,t)\big)\big\|_{L^2(R^2)} = \big\|J_t^s\big(\psi(t)J_t^{-s}v(x,t)\big)\big\|_{L^2(R^2)},$$

where $v(x,t) = J_x^rJ_t^sW(-t)u(x,t)$. Hence the inequality (2.13) easily follows from the following one: There exists a constant $C > 0$ depending only on ψ and $s > 1/2$ such that

$$\big\|J_t^s\big(\psi(t)J_t^{-s}\varphi(t)\big)\big\|_{L^2(R)} \le C\|\varphi\|_{L^2(R)} \tag{2.15}$$

for any $\varphi \in L^2(R)$, which is immediate.

(iii) Again, by Lemma 2.2 we have

$$\begin{aligned}\Big\|\psi(t)\int_0^t W(t-t')u(x,t')dt'\Big\|_{X^{r,s}(R^2)} &= \Big\|J_x^rJ_t^sW(-t)\Big(\psi(t)\int_0^t W(t-t')u(x,t')dt'\Big)\Big\|_{L^2(R^2)} \\ &= \Big\|J_t^s\Big(\psi(t)\int_0^t J_{t'}^{1-s}w(x,t')dt'\Big)\Big\|_{L^2(R^2)},\end{aligned}$$

where $w(x,t') = J_x^r J_{t'}^{s-1} W(-t')u(x,t')$. Hence the inequality (2.14) follows from the following one: There exists a constant $C > 0$ depending only on ψ and $s > 1/2$ such that

$$\left(\int_{-\infty}^{\infty} \left| J_t^s \left(\psi(t) \int_0^t J_{t'}^{1-s} \varphi(t')dt'\right)\right|^2 dt\right)^{\frac{1}{2}} \le C\|\varphi\|_{L^2(R)} \tag{2.16}$$

for any $\varphi \in L^2(R)$, which is also immediate.

3 A Bilinear Estimate

In this section we particularly take the polynomial $P(\xi) = -\gamma\xi^5 + \beta\xi^3$. The notations $X^{r,s}(R^2)$ and $W(t)$ respectively denote the Bourgain space and the fundamental solution operator related to the operator $\partial_t - iP(-i\partial_x) = \partial_t + \gamma\partial_x^5 + \beta\partial_x^3$. We assume that

$$\frac{1}{A} \le |\gamma| \le A, \qquad |\beta| \le B, \tag{3.1}$$

where A and B represent positive constants. The estimates to be established below will depend on the constants A and B, and not on specific β and γ.

By Lemma 2.4 we see that if u, $v \in X^{r,s}(R^2)$ and $r > -2$, $s > 1/2$, then u, $v \in L^2_{\rm loc}(R^2)$, so that $uv \in L^1_{\rm loc}(R^2)$. In this section we shall prove the following result.

Theorem 3.1 *Assume that $-1 < r \le 0$ and $\max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\} < s \le \frac{3}{4}$. Then u, $v \in X^{r,s}(R^2)$ implies that $\partial_x(uv) \in X^{r,s-1}(R^2)$, and there exists constant $C > 0$ depending only on r, s, A and B such that*

$$\|\partial_x(uv)\|_{X^{r,s-1}(R^2)} \le C\|u\|_{X^{r,s}(R^2)}\|v\|_{X^{r,s}(R^2)}. \tag{3.2}$$

We shall establish this result by a series of preliminary lemmas.

Lemma 3.2 *The inequality* (3.2) *is implied by the following one*:

$$\|\partial_x(u^2)\|_{X^{r,s-1}(R^2)} \le C\|u\|^2_{X^{r,s}(R^2)}. \tag{3.3}$$

Lemma 3.3 *The inequality* (3.3) *is implied by the following one*:

$$\left\{\iint_{R^2} (1+|\xi|^2)^r (1+|\tau - P(\xi)|^2)^{s-1}|\xi|^2 \left(\iint_{R^2} (1+|\eta|^2)^{-\frac{r}{2}}(1+|\rho - P(\eta)|^2)^{-\frac{s}{2}}|f(\eta,\rho)| \right.\right.$$
$$\left.\left. \cdot (1+|\xi-\eta|^2)^{-\frac{r}{2}}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-\frac{s}{2}}\big|f(\xi-\eta,\tau-\rho)\big|d\eta d\rho\right)^2 d\xi d\tau\right\}^{\frac{1}{2}} \le C\|f\|^2_{L^2(R^2)}. \tag{3.4}$$

Therefore, the problem of proving Theorem 3.1 is reduced to proving (3.4) for arbitrary $-1 < r \le 0$ and $\max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\} < s \le \frac{3}{4}$.

3.1 Some Basic Inequalities

The proof of (3.4) is based on the following calculus inequalities.

Lemma 3.4 (1) *Assume that $r > 0$, $s > 0$, $r + s > 1$ and $r \ne 1$, $s \ne 1$. Then*

$$\int_{-\infty}^{\infty} (1+|x-a|)^{-r}(1+|x-b|)^{-s}dx \le \begin{cases} C(1+|a-b|)^{-\min(r,s)}, & \text{if either } r > 1 \text{ or } s > 1, \\ C(1+|a-b|)^{1-r-s}, & \text{if both } r < 1 \text{ and } s < 1. \end{cases}$$

(2) *Assume that $0 < \alpha < 1$, $r > 1 - \alpha$ and $r \ne 1$. Then*

$$\int_{-\infty}^{\infty} |x|^{-\alpha}(1+|x-a|)^{-r}dx \le \begin{cases} C(1+|a|)^{-\alpha}, & \text{if } r > 1, \\ C(1+|a|)^{1-r-\alpha}, & \text{if } r < 1. \end{cases}$$

(3) *Assume that $\mu \ge 0$ and $r > 1 + \mu$. Then $\int_{-\infty}^{\infty} |x|^\mu (1+|x-a|)^{-r}dx \le C(1+|a|)^\mu$.*

(4) *Assume that $\mu \ge 0$, $0 < \alpha < 1$, $r > 1$ and $r + \alpha - \mu > 1$. Then*

$$\int_{-\infty}^{\infty} |x|^\mu |x-a|^{-\alpha}(1+|x-b|)^{-r}dx \le C(1+|a|)^{\mu-\alpha}(1+|b|)^{\max\{\alpha,\mu-\alpha\}}.$$

3.2 The Case $r = 0$

In the following we denote

$$Q(\xi,\eta) = 5\gamma\xi\eta(\xi-\eta)(\xi^2 - \xi\eta + \eta^2) - 3\beta\xi\eta(\xi-\eta), \tag{3.5}$$

$$Q_0(\xi) = Q\left(\xi, \frac{\xi}{2}\right) = \frac{15\gamma}{16}\xi^3\left(\xi^2 - \frac{4\beta}{5\gamma}\right). \tag{3.6}$$

It is obvious that

$$P(\xi-\eta)+P(\eta)=P(\xi)-Q(\xi,\eta),\quad \forall\,\xi,\,\eta\in R, \tag{3.7}$$

and

$$\frac{\partial Q(\xi,\eta)}{\partial\eta}=5\gamma\xi(\xi-2\eta)\Big(2\eta^2-2\xi\eta+\xi^2-\frac{3\beta}{5\gamma}\Big),\quad \forall\,\xi,\,\eta\in R. \tag{3.8}$$

From (3.8) we easily see that if either $\beta\gamma\le 0$ or $\beta\gamma>0$ and $|\xi|\ge\sqrt{(6/5)AB}$ then the polynomial $Q(\xi,\eta)$ in η has a unique stationary point $\eta=\xi/2$ for $\xi\ne 0$.

Lemma 3.5 *Let $M=1+\sqrt{(6/5)AB}$. Then for all $|\xi|\ge M$ and $\eta\in R$ there holds*

$$\Big|\frac{\partial Q(\xi,\eta)}{\partial\eta}\Big|\ge c_0|\xi|^{\frac12}(1+\xi^2+\eta^2)^{\frac12}|Q(\xi,\eta)-Q_0(\xi)|^{\frac12}, \tag{3.9}$$

where c_0 is a positive number depending only on A and B. For $|\xi|\le M$ and $|\eta|\ge M$ there holds

$$\Big|\frac{\partial Q(\xi,\eta)}{\partial\eta}\Big|\ge c_0|\xi|^{\frac14}|Q(\xi,\eta)|^{\frac34}, \tag{3.10}$$

where c_0 is as before.

Proof From (3.8) it is obvious that if either $|\xi|\ge M$ or $|\eta|\ge M$, then

$$\Big|\frac{\partial Q(\xi,\eta)}{\partial\eta}\Big|\ge c_0|\xi||2\eta-\xi|\big(1+\xi^2+\eta^2\big). \tag{3.11}$$

Consider first the case $|\xi|\le M$, $|\eta|\ge M$. Then clearly $|Q(\xi,\eta)|\le C|\xi||\eta|^4$, or $|\eta|\ge c_0|\xi|^{-\frac14}|Q(\xi,\eta)|^{\frac14}$. Hence, by (3.11), $|\frac{\partial Q(\xi,\eta)}{\partial\eta}|\ge c_0|\xi||\eta|^3\ge c_0|\xi|^{\frac14}|Q(\xi,\eta)|^{\frac34}$, and (3.10) immediately follows.

Next we consider the case $|\xi|\ge M$. A direct computation shows that

$$Q(\xi,\eta)-Q_0(\xi)=-\frac{5}{16}\gamma\xi(2\eta-\xi)^2\Big(3\xi^2-4\xi\eta+4\eta^2-\frac{12\beta}{5\gamma}\Big), \tag{3.12}$$

so that

$$|Q(\xi,\eta)-Q_0(\xi)|\le C|\xi|(2\eta-\xi)^2\big(1+\xi^2+\eta^2\big),\quad \forall\,\xi,\,\eta\in R. \tag{3.13}$$

This implies that

$$|2\eta-\xi|\ge c_0|\xi|^{-\frac12}\big(1+\xi^2+\eta^2\big)^{-\frac12}|Q(\xi,\eta)-Q_0(\xi)|^{\frac12},\quad \forall\,\xi,\,\eta\in R. \tag{3.14}$$

Substituting (3.14) into (3.11) we see that (3.9) holds.

Lemma 3.6 *Assume that $s>1/2$. Then for all $\xi\in R\backslash\{0\}$ and $\tau\in R$,*

$$\iint_{R^2}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho d\eta\le C|\xi|^{-\frac14}(1+|\xi|)^{-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac12}. \tag{3.15}$$

Proof By Lemma 3.4 (1) we have

$$\int_{-\infty}^{\infty}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho\le C(1+|\tau-P(\xi-\eta)-P(\eta)|)^{-2s}.$$

By (3.7), this implies that

$$\int_{-\infty}^{\infty}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho\le C(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}. \tag{3.16}$$

Hence

$$\iint_{R^2}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho d\eta\le C\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta, \tag{3.17}$$

so that (3.15) is implied by the following estimate:

$$\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta\le C|\xi|^{-\frac14}(1+|\xi|)^{-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac12}. \tag{3.18}$$

In the following we give the proof of (3.18).

Let M be as in Lemma 3.5. We first assume that $|\xi| \geq M$. Then by (3.9) and Lemma 3.4 we have

$$\begin{aligned}\int_{-\infty}^{\infty}&(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta\\ &\leq C|\xi|^{-\frac{1}{2}}(1+|\xi|)^{-1}\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}|Q(\xi,\eta)-Q_0(\xi)|^{-\frac{1}{2}}\left|\frac{\partial Q(\xi,\eta)}{\partial\eta}\right|d\eta\\ &\leq C(1+|\xi|)^{-\frac{3}{2}}\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+\zeta|)^{-2s}|\zeta-Q_0(\xi)|^{-\frac{1}{2}}d\zeta\\ &\leq C(1+|\xi|)^{-\frac{3}{2}}(1+|Q_0(\xi)|)^{-\frac{1}{2}}(1+|\tau-P(\xi)|)^{\frac{1}{2}} \quad \text{(by Lemma 3.4 (4))}\\ &\leq C(1+|\xi|)^{-4}(1+|\tau-P(\xi)|)^{\frac{1}{2}}.\end{aligned}$$

This proves (3.18) in the case $|\xi| \geq M$.

Next we assume that $|\xi| \leq M$. We split the integral on the left-hand side of (3.18) into two parts: $|\eta| \leq M$ and $|\eta| \geq M$. Clearly,

$$\int_{|\eta|\leq M}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta \leq C. \tag{3.19}$$

Using (3.10) we deduce that

$$\begin{aligned}\int_{|\eta|\geq M}&(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta\\ &\leq C|\xi|^{-\frac{1}{4}}\int_{|\eta|\geq M}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}|Q(\xi,\eta)|^{-\frac{3}{4}}\left|\frac{\partial Q(\xi,\eta)}{\partial\eta}\right|d\eta\\ &\leq C|\xi|^{-\frac{1}{4}}\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+\zeta|)^{-2s}|\zeta|^{-\frac{3}{4}}d\zeta\\ &\leq C|\xi|^{-\frac{1}{4}}(1+|\tau-P(\xi)|)^{-\frac{3}{4}} \quad \text{(by Lemma 3.4 (2))}.\end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we easily see that (3.18) also holds in the case $|\xi| \leq M$.

Lemma 3.7 *The estimate* (3.4) *holds for* $r = 0$ *and* $1/2 < s \leq 3/4$.

Proof By the Cauchy inequality and Lemma 3.6 we have, for $s > 1/2$,

$$\begin{aligned}\iint_{R^2}&(1+|\rho-P(\eta)|^2)^{-\frac{s}{2}}|f(\eta,\rho)|\cdot(1+|\tau-\rho-P(\xi-\eta)|^2)^{-\frac{s}{2}}|f(\xi-\eta,\tau-\rho)|d\eta d\rho\\ &\leq\left(\iint_{R^2}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\eta d\rho\right)^{\frac{1}{2}}\\ &\quad\cdot\left(\iint_{R^2}|f(\eta,\rho)|^2|f(\xi-\eta,\tau-\rho)|^2d\eta d\rho\right)^{\frac{1}{2}}\\ &\leq C|\xi|^{-\frac{1}{8}}(1+|\xi|)^{-\frac{15}{8}}(1+|\tau-P(\xi)|)^{\frac{1}{4}}\cdot\left(\iint_{R^2}|f(\eta,\rho)|^2|f(\xi-\eta,\tau-\rho)|^2d\eta d\rho\right)^{\frac{1}{2}}.\end{aligned}$$

Substituting this estimate into the left-hand side of (3.4), and noting that, when $s \leq 3/4$,

$$(1+|\tau-P(\xi)|^2)^{s-1}|\xi|^2\cdot|\xi|^{-\frac{1}{4}}(1+|\xi|)^{-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac{1}{2}} \leq C,$$

we see that (3.4) holds in the case $r = 0$ and $1/2 < s \leq 3/4$.

3.3 The Case $-1 < r < 0$

Lemma 3.8 *Let* $-1 < r < 0$ *and* $s > \max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\}$. *Then for all* $\xi \in R\backslash\{0\}$ *and* $\tau \in R$,

$$\begin{aligned}\iint_{R^2}&(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho d\eta\\ &\leq C|\xi|^{-|r|-\frac{1}{4}}(1+|\xi|)^{5|r|-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac{1}{2}}.\end{aligned} \tag{3.21}$$

Proof Let M be as before. Using (3.16) we get

$$\iint_{R^2}(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\rho-P(\eta)|^2)^{-s}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\rho d\eta$$
$$\leq C\int_{-\infty}^{\infty}(1+|\eta|)^{-2r}(1+|\xi-\eta|)^{-2r}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta. \tag{3.22}$$

To estimate the last integral we denote

$$\Omega_1(\xi)=\{\eta\in R:\ \text{either } |\eta|\leq M \text{ or } |\xi-\eta|\leq M\},$$
$$\Omega_2(\xi)=\{\eta\in R:\ |\eta|\geq M \text{ and } |\xi-\eta|\geq M\}.$$

Since $-r=|r|>0$, it is clear that if $\eta\in\Omega_1(\xi)$ then $(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}\leq C(1+|\xi|)^{2|r|}$. Hence, using (3.18) we get

$$\int_{\Omega_1(\xi)}(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$
$$\leq C(1+|\xi|)^{2|r|}\int_{-\infty}^{\infty}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta$$
$$\leq C|\xi|^{-\frac14}(1+|\xi|)^{2|r|-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac12}. \tag{3.23}$$

Next, since $-r=|r|>0$, it is clear that if $\eta\in\Omega_2(\xi)$ then $(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}\leq C|\eta(\xi-\eta)|^{2|r|}$. Hence

$$\int_{\Omega_2(\xi)}(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$
$$\leq C\int_{\Omega_2(\xi)}|\eta(\xi-\eta)|^{2|r|}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta.$$

We denote $\mu=\eta(\xi-\eta)$. From the fact that $\xi^2-\mu=\xi^2-\xi\eta+\eta^2\geq\frac12(\xi^2+\eta^2)\geq\frac13|\mu|$ and the relation $Q(\xi,\eta)=5\gamma\xi\mu(\xi^2-\mu)-3\beta\xi\mu$, we easily deduce that if $|\eta|\geq M$ then $|\mu|\leq C|\xi|^{-\frac12}|Q(\xi,\eta)|^{\frac12}$. Hence

$$\int_{\Omega_2(\xi)}(1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$
$$\leq C|\xi|^{-|r|}\int_{|\eta|\geq M}|Q(\xi,\eta)|^{|r|}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}d\eta. \tag{3.24}$$

For $|\xi|\geq M$, we use (3.9) to get

$$\int_{|\eta|\geq M}|Q(\xi,\eta)|^{|r|}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$
$$\leq C(1+|\xi|)^{-\frac32}\int_{-\infty}^{\infty}|Q(\xi,\eta)|^{|r|}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}|Q(\xi,\eta)-Q_0(\xi)|^{-\frac12}\left|\frac{\partial Q(\xi,\eta)}{\partial\eta}\right|d\eta$$
$$\leq C(1+|\xi|)^{-\frac32}\int_{-\infty}^{\infty}|\zeta|^{|r|}|\zeta-Q_0(\xi)|^{-\frac12}(1+|\tau-P(\xi)+\zeta|)^{-2s}d\zeta$$
$$\leq C(1+|\xi|)^{-\frac32}(1+|Q_0(\xi)|)^{|r|-\frac12}(1+|\tau-P(\xi)|)^{\frac12}\quad\text{(by Lemma 3.4 (4))}$$
$$\leq C(1+|\xi|)^{5|r|-4}(1+|\tau-P(\xi)|)^{\frac12}. \tag{3.25}$$

For $|\xi|\leq M$ we use (3.10) to get

$$\int_{|\eta|\geq M}|Q(\xi,\eta)|^{|r|}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$
$$\leq C|\xi|^{-\frac14}\int_{|\eta|\geq M}|Q(\xi,\eta)|^{|r|-\frac34}(1+|\tau-P(\xi)+Q(\xi,\eta)|)^{-2s}\left|\frac{\partial Q(\xi,\eta)}{\partial\eta}\right|d\eta$$
$$\leq C|\xi|^{-\frac14}\int_{-\infty}^{\infty}|\zeta|^{|r|-\frac34}(1+|\tau-P(\xi)+\zeta|)^{-2s}d\zeta$$
$$\leq C|\xi|^{-\frac14}(1+|\tau-P(\xi)|)^{|r|-\frac34}\quad\text{(by Lemma 3.4 (2), (3))}. \tag{3.26}$$

From (3.24)–(3.26) we see that, for all $\xi \in R\backslash\{0\}$,

$$\int_{\Omega_2(\xi)} (1+|\eta|^2)^{-r}(1+|\xi-\eta|^2)^{-r}(1+|\tau-P(\xi)+Q(\xi,\eta)|^2)^{-s}d\eta$$

$$\leq C|\xi|^{-|r|-\frac{1}{4}}(1+|\xi|)^{5|r|-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac{1}{2}}. \tag{3.27}$$

Substituting (3.23) and (3.27) into (3.22), we see that (3.21) holds.

Lemma 3.9 *The estimate* (3.4) *holds for* $-1 < r < 0$ *and* $\max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\} < s \leq \frac{3}{4}$.

Proof By the Cauchy inequality and Lemma 3.10 we have, for $s > \max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\}$,

$$\iint_{R^2} (1+|\eta|^2)^{-\frac{r}{2}}(1+|\rho-P(\eta)|^2)^{-\frac{s}{2}}|f(\eta,\rho)| \cdot (1+|\xi-\eta|^2)^{-\frac{r}{2}}$$

$$\cdot (1+|\tau-\rho-P(\xi-\eta)|^2)^{-\frac{s}{2}}|f(\xi-\eta,\tau-\rho)|d\eta d\rho$$

$$\leq \Big(\iint_{R^2}(1+|\eta|^2)^{-r}(1+|\rho-P(\eta)|^2)^{-s}(1+|\xi-\eta|^2)^{-r}(1+|\tau-\rho-P(\xi-\eta)|^2)^{-s}d\eta d\rho\Big)^{\frac{1}{2}}$$

$$\cdot \Big(\iint_{R^2}|f(\eta,\rho)|^2|f(\xi-\eta,\tau-\rho)|^2 d\eta d\rho\Big)^{\frac{1}{2}}$$

$$\leq C|\xi|^{-\frac{1}{2}|r|-\frac{1}{8}}(1+|\xi|)^{\frac{5}{2}|r|-\frac{15}{8}}(1+|\tau-P(\xi)|)^{\frac{1}{4}} \cdot \Big(\iint_{R^2}|f(\eta,\rho)|^2|f(\xi-\eta,\tau-\rho)|^2 d\eta d\rho\Big)^{\frac{1}{2}}.$$

Substituting this estimate into the left-hand side of (3.4), and noticing that, when $r \geq -1$ and $s \leq \frac{3}{4}$, $(1+|\xi|^2)^r(1+|\tau-P(\xi)|^2)^{s-1}|\xi|^2 \cdot |\xi|^{-|r|-\frac{1}{4}}(1+|\xi|)^{5|r|-\frac{15}{4}}(1+|\tau-P(\xi)|)^{\frac{1}{2}} \leq C$, we see that (3.4) holds when $-1 < r < 0$ and $\max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\} < s \leq \frac{3}{4}$.

4 Solvability of the Problems (1.1)–(1.2)

For $\varepsilon > 0$ we introduce

$$v(x,t) = \varepsilon^4 u(\varepsilon x, \varepsilon^5 t), \qquad v_0(x) = \varepsilon^4 u_0(\varepsilon x). \tag{4.1}$$

One can easily verify that the problem (1.1)–(1.2) is equivalent to the following problem:

$$\partial_t v + av\partial_x v + \varepsilon^2\beta\partial_x^3 v + \gamma\partial_x^5 v = 0 \quad \text{in } R^2, \tag{4.2}$$

$$v(x,0) = v_0(x) \quad \text{for } x \in R. \tag{4.3}$$

In the following we consider the local solvability of this problem.

Let $P_\varepsilon(\xi) = -\gamma\xi^5 + \varepsilon^2\beta\xi^3$, and let $W_\varepsilon(t)$ be the operator defined by (2.3) with $P(\xi)$ replaced by $P_\varepsilon(\xi)$. Take a function $\psi \in C_0^\infty(R)$ such that $\psi(t) = 1$ for $|t| \leq 1$, and consider the equation

$$v(\cdot,t) = \psi(t)W_\varepsilon(t)v_0 - \frac{1}{2}a\psi(t)\int_0^t W_\varepsilon(t-t')\partial_x\big(v^2(\cdot,t')\big)dt', \quad t \in R. \tag{4.4}$$

It is easy to see that the equation (4.4) is equivalent to the problem (4.2)–(4.3) when $|t| < 1$.

To solve the equation (4.4) we denote by $X_\varepsilon^{r,s}(R^2)$ the Bourgain space related to the equation (4.2), i.e., the space defined by (2.2) with $P(\xi)$ replaced by $P_\varepsilon(\xi)$, where $-1 < r \leq 0$ and $\max\{\frac{1}{2}, \frac{1}{4} - \frac{1}{2}r\} < s \leq 3/4$. We assume that $u_0 \in H^r(R)$ (so that also $v_0 \in H^r(R)$), and introduce a mapping S_ε: $v \to w$ by

$$w(\cdot,t) = \psi(t)W_\varepsilon(t)v_0 - \frac{1}{2}a\psi(t)\int_0^t W_\varepsilon(t-t')\partial_x\big(v^2(\cdot,t')\big)dt', \quad t \in R.$$

By Lemma 2.5 and Theorem 3.1 we see that S_ε is well-defined and it maps $X_\varepsilon^{r,s}(R^2)$ into itself. Furthermore, Lemma 2.5 and Theorem 3.1 also ensure that there exists a constant $C > 0$ independent of ε such that

$$\|w\|_{X_\varepsilon^{r,s}(R^2)} \leq C\|v_0\|_{H^r(R)} + C\|\partial_x(v^2)\|_{X_\varepsilon^{r,s-1}(R^2)} \leq C\|v_0\|_{H^r(R)} + C\|v\|^2_{X_\varepsilon^{r,s}(R^2)}.$$

From the relation $v_0(x) = \varepsilon^4 u_0(\varepsilon x)$ one can easily verify that

$$\|v_0\|_{H^r(R)} \leq \varepsilon^{\frac{7}{2}+r}\|u_0\|_{H^r(R)}. \tag{4.5}$$

Thus

$$\|w\|_{X_\varepsilon^{r,s}(R^2)} \leq C\varepsilon^{\frac{7}{2}+r}\|u_0\|_{H^r(R)} + C\|v\|^2_{X_\varepsilon^{r,s}(R^2)}. \tag{4.6}$$

From this inequality we easily see that if we take ε sufficiently small such that

$$4C^2\varepsilon^{\frac{7}{2}+r}\|u_0\|_{H^r(R)} < 1, \tag{4.7}$$

then S_ε maps the closed ball in $X_\varepsilon^{r,s}(R^2)$

$$\{v \in X_\varepsilon^{r,s}(R^2) : \|v\|_{X_\varepsilon^{r,s}(R^2)} \le 2C\varepsilon^{\frac{7}{2}+r}\|u_0\|_{H^r(R)}\} \tag{4.8}$$

into itself. Moreover, by Lemma 2.5 and Theorem 3.1 we see that

$$\|S_\varepsilon v_1 - S_\varepsilon v_2\|_{X_\varepsilon^{r,s}(R^2)} \le C\|\partial_x(v_1^2 - v_2^2)\|_{X_\varepsilon^{r,s-1}(R^2)}$$

$$\le C\big(\|v_1\|_{X_\varepsilon^{r,s}(R^2)} + \|v_2\|_{X_\varepsilon^{r,s}(R^2)}\big)\|v_1 - v_2\|_{X_\varepsilon^{r,s}(R^2)} \le 4C^2\varepsilon^{\frac{7}{2}+r}\|u_0\|_{H^r(R)}\|v_1 - v_2\|_{X_\varepsilon^{r,s}(R^2)}.$$

Hence, by (4.7), S_ε is a contraction mapping on the ball (4.8).

It follows from the Banach fixed point theorem that for small $\varepsilon > 0$ the equation (4.4) has a unique solution for all $t \in R$, and, consequently, the problem (4.2)–(4.3) has a solution for $|t| < 1$. By (4.1), this implies that the problem (1.1)–(1.2) has a solution for $|t| < \varepsilon^5$. The local solvability of the problem (1.1)–(1.2) is thus established for $u_0 \in H^r(R)$ with $-1 < r \le 0$.

Using the above result for the case $r = 0$ and the energy conservation law of the equation (1.1), we can resort to a standard argument to show that the problem (1.1)–(1.2) is globally solvable for $u_0 \in L^2(R)$. More precisely, we have proved the following result:

Theorem 4.1 *If $u_0 \in H^r(R)$ and $-1 < r \le 0$, then there exists $T > 0$ such that the problem (1.1)–(1.2) has a solution on $R \times [-T, T]$, satisfying $u \in C([-T, T], H^r(R))$, $\partial_x^\alpha u \in L_x^p L_T^2$, $\forall\, \alpha \in [0, 2 + r]$, where $p = 4/(2 + r - \alpha)$. For $r = 0$ the above assertion holds for all $T > 0$.*

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