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A Weierstrass Representation Formula for Minimal Surfaces in $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$

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Abstract We give a general setting for constructing a Weierstrass representation formula for simply connected minimal surfaces in a Riemannian manifold. Then, we construct examples of minimal surfaces in the three dimensional Heisenberg group and in the product of the hyperbolic plane with the real line.

Keywords Minimal immersions, Weierstrass representation, Heisenberg group, Hyperbolic space
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1 Introduction

The classical Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$ with its generalizations to $\mathbb{R}^n$ has been proved to be an extremely useful tool for the study of minimal surfaces in those spaces (see, for example, [1, 2]). In this paper we describe a method to derive a Weierstrass-type representation formula for simply connected immersed minimal surfaces in the three dimensional Heisenberg group $\mathbb{H}_3$ and in the product $\mathbb{H}^2 \times \mathbb{R}$ of the hyperbolic plane with the real line. Our work is motivated by the interest, in the last decade, in the theory of minimal surfaces in $\mathbb{H}_3$, see, for example, [3–8], and by the recent work on minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [9]. First, using the standard harmonic map equation, we give a Weierstrass-type representation formula for simply connected immersed minimal surfaces in a Riemannian manifold. This general setting is applied to the case of 3-dimensional Lie groups endowed with a left invariant metric. We then discuss, using this setting, some examples of minimal surfaces both in $\mathbb{H}_3$ and in $\mathbb{H}^2 \times \mathbb{R}$. By means of similar methods a Weierstrass-type formula for minimal surfaces in the hyperbolic n-space has been derived by Kokubu in [10].

2 The General Setting

Let (M^n, g) be an n-dimensional Riemannian manifold, Σ a Riemann surface and $f : \Sigma \to M^n$ a smooth map. The pull-back bundle $f^*(TM)$ has a (fiber) metric and a compatible connection, the pull-back connection, induced by the Riemannian metric and the Levi–Civita connection of M. Consider the complexified bundle $\mathbb{E} = f^*(TM) \otimes \mathbb{C}$. The metric g may be extended to $\mathbb{E}$ as:

- a complex bilinear form $(.,.) : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$;

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- a hermitian metric $\langle\langle .,. \rangle\rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$;

and the two extensions are related by: $\langle\langle V, W\rangle\rangle = (V, \overline{W})$. Let (u, v) be local coordinates on Σ, and $z = u + iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right); \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

The pull-back connection extends to a complex connection on $\mathbb{E}$, hermitian with respect to $\langle\langle .,. \rangle\rangle$, and it is well known (see, for example, [11]) that $\mathbb{E}$ has a unique holomorphic structure such that a section $W : \Sigma \to \mathbb{E}$ is holomorphic if and only if:

$$\widetilde{\nabla}_{\frac{\partial}{\partial \overline{z}}} W = 0, \tag{1}$$

where $\widetilde{\nabla}$ is the pull-back connection.

Setting

$$\left.\frac{\partial f}{\partial u}\right|_p = f_{*p}\left(\left.\frac{\partial}{\partial u}\right|_p\right), \qquad \left.\frac{\partial f}{\partial v}\right|_p = f_{*p}\left(\left.\frac{\partial}{\partial v}\right|_p\right),$$

we can regard $\phi = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right)$ as a section of $\mathbb{E}$ and we have the following properties:

- The map f is an immersion if and only if $\langle\langle \phi, \phi\rangle\rangle \neq 0$;
- If f is an immersion then f is conformal if and only if $(\phi, \phi) = 0$.

Let now $f : \Sigma \to M$ be a conformal immersion and $z = u + iv$ a local conformal parameter. Then the induced metric is $ds^2 = \lambda^2(du^2 + dv^2) = \lambda^2|dz|^2$, and the Beltrami–Laplace operator on Σ, with respect to the induced metric, is given by

$$\Delta = \lambda^{-2}\left(\frac{\partial}{\partial u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v}\right) = 4\lambda^{-2}\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}.$$

We recall that a map $f : \Sigma \to M$ is *harmonic* if its *tension field* $\tau(f) = \mathrm{trace}\nabla df$ vanishes. Let $\{x_1, \ldots, x_n\}$ be a system of local coordinates in a neighborhood U of M^n such that $U \cap f(\Sigma) \neq \emptyset$. Then, in an open set $\Omega \subset \Sigma$, $\phi = \sum_{j=1}^n \phi_j \frac{\partial}{\partial x_j}$ for some complex-valued functions ϕ_j defined on Ω. With respect to the local decomposition of ϕ, the tension field can be written as:

$$\tau(f) = \sum_i \left\{\Delta f_i + 4\lambda^{-2}\sum_{j,k=1}^n \Gamma^i_{jk}\frac{\partial f_j}{\partial \overline{z}}\frac{\partial f_k}{\partial z}\right\}\frac{\partial}{\partial x_i} = 4\lambda^{-2}\sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k=1}^n \Gamma^i_{jk}\overline{\phi_j}\phi_k\right\}\frac{\partial}{\partial x_i},$$

where Γ^i_{jk} are the Christoffel symbols of M (see, for example [12]). The section ϕ is holomorphic, according to (1), if and only if

$$\begin{aligned}
\widetilde{\nabla}_{\frac{\partial}{\partial \overline{z}}}\left(\sum_i \phi_i \frac{\partial}{\partial x_i}\right) &= \sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\frac{\partial f}{\partial \overline{z}}}\frac{\partial}{\partial x_i}\right\} \\
&= \sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}}\frac{\partial}{\partial x_i} + \phi_i \nabla_{\sum_j \overline{\phi_j}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right\} \\
&= \sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k}\Gamma^i_{jk}\overline{\phi_j}\phi_k\right\}\frac{\partial}{\partial x_i} = 0,
\end{aligned}$$

thus if and only if

$$\frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k=1}^n \Gamma^i_{jk}\overline{\phi_j}\phi_k = 0, \quad i = 1, \ldots, n. \tag{2}$$

From (2) and the expression of the tension field we have that $4\lambda^{-2}\big(\widetilde{\nabla}_{\frac{\partial}{\partial \overline{z}}}\phi\big) = \tau(f)$, and thus $f : \Sigma \to M$ is harmonic if and only if ϕ is a holomorphic section of $\mathbb{E}$. Now, since a conformal map from a surface to a Riemannian manifold is harmonic if and only if it is a minimal immersion (see, for example [13]), we conclude that a conformal immersion is minimal if and only if ϕ is a holomorphic section of $\mathbb{E}$.

Equations (2) can be seen as second order equations in the f_i, and from this point of view they are formally very similar to the equations of geodesics. If we look at (2) as a system of first order differential equations in the ϕ_i, where Γ's are computed in $f_i(z) = 2\mathrm{Re}\int_{z_0}^{z} \phi_i\, dz$, it can be written as:

$$\frac{\partial \phi_i}{\partial \bar{z}} + 2\sum_{j>k} \Gamma^i_{jk} \mathrm{Re}(\overline{\phi_j}\phi_k) + \sum_j \Gamma^i_{jj}|\phi_j|^2 = 0, \quad i = 1, \ldots, n.$$

This implies that $\frac{\partial \phi_i}{\partial \bar{z}} \in \mathbb{R}$, and from

$$\frac{\partial \phi_i}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \mathrm{Re}(\phi_i)}{\partial u} - \frac{\partial \mathrm{Im}(\phi_i)}{\partial v}\right) + \frac{i}{2}\left(\frac{\partial \mathrm{Re}(\phi_i)}{\partial v} + \frac{\partial \mathrm{Im}(\phi_i)}{\partial u}\right),$$

we conclude that

$$\frac{\partial \mathrm{Re}(\phi_i)}{\partial v} + \frac{\partial \mathrm{Im}(\phi_i)}{\partial u} = 0.$$

The latter equation ensures that (locally) the 1-forms $\phi_i dz$ don't have real periods. Therefore we have the following:

Theorem 2.1 (Weierstrass representation) *Let* (M^n, g) *be a Riemannian manifold and* $\{x_1 \ldots, x_n\}$ *local coordinates. Let* $\phi_j, j = 1, \ldots, n$, *be complex-valued functions in an open simply connected domain* $\Omega \subset \mathbb{C}$ *which are solutions of* (2). *Then the map*

$$f_j(u, v) = 2\mathrm{Re}\left(\int_{z_o}^{z} \phi_j\, dz\right)$$

is well defined and defines a minimal conformal immersion if and only if the following conditions are satisfied:

- $\sum_{j,k=1}^{n} g_{jk}\phi_j\overline{\phi_k} \neq 0$;
- $\sum_{j,k=1}^{n} g_{jk}\phi_j\phi_k = 0$.

If $M = \mathbb{R}^n$ with the flat metric, Equations (2) are just the Cauchy–Riemann equations. But, in general, it is quite hard to find explicit solutions. One way of finding solutions is to look at manifolds where (2) reduces to a system of partial differential equations with constant coefficients. The first try is, naturally, the case where M is a Lie group, and this will be considered in the next section. In any case, the existence of a Weierstrass representation may be useful for theoretical results. For example, at least if the metric is analytic, the Cauchy problem for (2) has a solution. In particular, for any point $p \in M$ and any 2-plane $\pi \subseteq T_pM$, there is a minimal surface through p with tangent plane π.

3 Weierstrass Representation on Lie Groups

In this section we will discuss the case of maps $f : \Sigma \to G$, where G is a Lie group endowed with a left invariant metric g. Let E_i, $i = 1, \ldots, n$, be a basis of left invariant vector fields, and let $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, be the coordinates vector fields in some chart U of G. Then in some open set $\Omega \subset \Sigma$ the section $\phi = \frac{\partial f}{\partial z} \in \Gamma(f^*(TG) \otimes \mathbb{C})$ can be decomposed *either* with respect to the coordinates vector fields *or* with respect to the left invariant vector fields:

$$\phi = \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \psi_i E_i,$$

for some complex functions $\phi_i, \psi_i : \Omega \to \mathbb{C}$. Moreover, there exists an invertible matrix $A = (A_{ij})$, with function entries $A_{ij} : f(\Omega) \cap U \to \mathbb{R}$, $i, j = 1, \ldots, n$, such that

$$\phi_i = \sum_j A_{ij}\psi_j. \tag{3}$$

Now let C^k_{ij} be the structure's constants of the Lie algebra $\mathfrak{g}$ of G, that is, $[E_i, E_j] = C^k_{ij}E_k$. The Kozul formula for the Levi–Civita connection is:

$$2g(\nabla_{E_i}E_j, E_k) = C^k_{ij} - C^i_{jk} + C^j_{ki} := L^k_{ij}. \tag{4}$$

Using the expression of ϕ with respect to the left invariant vector fields and (4) we get

$$\widetilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}\left(\sum_i \psi_i E_i\right) = \sum_i \left\{\frac{\partial \psi_i}{\partial \bar{z}}E_i + \psi_i \nabla_{\sum_j \overline{\psi_j}E_j} E_i\right\} = \sum_i \left\{\frac{\partial \psi_i}{\partial \bar{z}} + \frac{1}{2}\sum_{j,k} L^i_{jk}\overline{\psi_j}\psi_k\right\}E_i.$$

This means that the section ϕ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \bar{z}} + \frac{1}{2}\sum_{j,k} L^i_{jk}\overline{\psi_j}\psi_k = 0, \quad i = 1, \ldots, n. \tag{5}$$

From Theorem 2.1 we have:

Theorem 3.1 *Let $\psi_j, j = 1, \ldots, n$, be complex-valued functions defined in a open simply connected set $\Omega \subset \mathbb{C}$, such that the following conditions are satisfied*:

- $\sum_i \psi_i \overline{\psi_i} \neq 0$;
- $\sum_i {\psi_i}^2 = 0$;
- ψ_j *are solutions of* (5).

Then, the map $f : \Omega \to G$, defined by

$$f_i(u, v) = 2\mathrm{Re}\Big(\int_{z_o}^{z} \sum_j A_{ij}\psi_j \, dz\Big),$$

is a conformal minimal immersion.

If the dimension of M is three, as in the case of minimal surfaces in $\mathbb{R}^3$, we can give a simple geometric description of almost all solutions of the equation $\sum_{i=1}^{3} {\psi_i}^2 = 0$. The idea is the following. From $\sum_{i=1}^{3} \psi_i^2 = 0$ we have that

$$(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -\psi_3^2,$$

which suggests the definition of two new complex functions

$$G := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}, \quad H := \sqrt{-\frac{1}{2}(\psi_1 + i\psi_2)}. \tag{6}$$

The functions G and H are single-valued complex functions which, for suitably chosen square roots, satisfy

$$\begin{cases} \psi_1 = G^2 - H^2, \\ \psi_2 = i(G^2 + H^2), \\ \psi_3 = 2GH. \end{cases}$$

The induced metric is then

$$ds^2 = 2(|H|^2 + |G|^2)^2(du^2 + dv^2),$$

and the Gauss map takes the form

$$N = \frac{1}{|H/G|^2 + 1}[2\mathrm{Re}(H/G)E_1 + 2\mathrm{Im}(H/G)E_2 + (|H/G|^2 - 1)E_3].$$

If $\pi : \mathbb{S}^2(1) \setminus \{0, 0, 1\} \to \mathbf{R}^2$ is the stereographic projection from the north pole, then $\pi \circ N = (\mathrm{Re}H/G, \mathrm{Im}H/G)$. If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and extend the π to a map $\tilde{\pi} : \mathbb{S}^2(1) \to \mathbb{C} \cup \{\infty\}$, with $\tilde{\pi}(0, 0, 1) = \infty$, then

$$\pi \circ N = H/G.$$

This means that the map $g = H/G$ can be identified with the Gauss map of f.

Remark 3.2 Equations (5) have the advantage, with respect to (2), to be partial differential equations with constant coefficients. However, this just shifts the difficulties, because after finding explicit solutions of (5), we have to compute the ϕ_i, and, for this, we have to compute the A_{ij} along the f_i. In the next sections we will study two cases where these difficulties may be overcome.

4 Minimal Surfaces in the Heisenberg Group $\mathbb{H}_3$

The 3-dimensional Heisenberg group $\mathbb{H}_3$ is the two-step nilpotent Lie group standardly represented in $Gl_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

with $x_i \in \mathbb{R}$. Endowed with the left-invariant metric

$$g = dx_1^2 + dx_2^2 + \left(dx_3 + \frac{1}{2}x_2 dx_1 - \frac{1}{2}x_1 dx_2 \right)^2, \tag{7}$$

$(\mathbb{H}_3, g)$ has a rich geometric structure. In fact its group of isometries is of dimension 4, which is the maximal possible dimension for a non-constant curvature metric on a 3-manifold. Also, from the algebraic point of view, $(\mathbb{H}_3, g)$ is a 2-step nilpotent Lie group, i.e. "almost Abelian". An orthonormal basis of left-invariant vector fields is given, with respect to the coordinates vector fields, by

$$\begin{cases} E_1 &= \dfrac{\partial}{\partial x_1} - \dfrac{x_2}{2}\dfrac{\partial}{\partial x_3} \\ E_2 &= \dfrac{\partial}{\partial x_2} + \frac{x_1}{2}\dfrac{\partial}{\partial x_3} \\ E_3 &= \dfrac{\partial}{\partial x_3} \end{cases}, \tag{8}$$

and the non zero L_{ij}^k's are

$$L_{12}^3 = 1, \quad L_{21}^3 = -1, \quad L_{13}^2 = -1, \quad L_{31}^2 = -1, \quad L_{23}^1 = 1, \quad L_{32}^1 = 1.$$

Let $f : \Omega \subset \mathbb{C} \to \mathbb{H}_3$ be a smooth immersion and let $\phi = \frac{\partial f}{\partial z} = \sum \phi_i \frac{\partial}{\partial x_i} = \sum \psi_i E_i$. From the expressions of the left invariant vector fields (8), we have

$$\phi_i = \sum_j A_{ij}\psi_j,$$

where $A = (A_{ij})$ is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{x_2}{2} & \dfrac{x_1}{2} & 1 \end{bmatrix}.$$

According to (5) the section $\phi = \psi_1 E_1 + \psi_2 E_2 + \psi_3 E_3$ is holomorphic if and only if

$$\begin{cases} \dfrac{\partial \psi_1}{\partial \bar{z}} + \mathrm{Re}(\psi_2 \overline{\psi_3}) = 0, \\ \dfrac{\partial \psi_2}{\partial \bar{z}} - \mathrm{Re}(\psi_1 \overline{\psi_3}) = 0, \\ \dfrac{\partial \psi_3}{\partial \bar{z}} - i\mathrm{Im}(\psi_1 \overline{\psi_2}) = 0. \end{cases} \tag{9}$$

Remark 4.1 System (9) was found also in [14] using the Lie-algebra formulation of the harmonic map equation.

Therefore, in this context, Theorem 3.1 assumes the following form:

Theorem 4.2 *Let* $\psi_j, j = 1, \ldots, 3$, *be complex-valued functions defined in a simply connected domain* $\Omega \subset \mathbb{C}$, *such that the following conditions are satisfied*:

- $\sum_{i=1} \psi_i \overline{\psi_i} \neq 0$;
- $\sum_{i=1} {\psi_i}^2 = 0$;
- ψ_j *are solutions of* (9).

Then, the map $f : \Omega \to \mathbb{H}_3$, *defined by*

$$f_i(u, v) = 2\mathrm{Re}\left(\int_{z_o}^{z} \sum_j A_{ij}\psi_j \, dz \right),$$

is a conformal minimal immersion.

Let us now write Equations (9), which ensure that ϕ is a holomorphic section, in terms of the functions G and H defined by (6), that is,

$$\begin{cases} G\dfrac{\partial G}{\partial \overline{z}} - H\dfrac{\partial H}{\partial \overline{z}} = \mathrm{Im}(G\overline{H})(|G|^2 - |H|^2), \\ i\left(G\dfrac{\partial G}{\partial \overline{z}} + H\dfrac{\partial H}{\partial \overline{z}}\right) = \mathrm{Re}(G\overline{H})(|G|^2 - |H|^2), \\ H\dfrac{\partial G}{\partial \overline{z}} + G\dfrac{\partial H}{\partial \overline{z}} = -\dfrac{i}{2}(|G|^4 - |H|^4). \end{cases}$$

Note that the third equation is a combination of the first two. In fact, multiplying the first times i and adding the second we get

$$2i\frac{\partial G}{\partial \overline{z}} = (|G|^2 - |H|^2)\overline{H}, \tag{10}$$

while, subtracting i times the first from the second yields

$$2i\frac{\partial H}{\partial \overline{z}} = (|G|^2 - |H|^2)\overline{G}. \tag{11}$$

It is now a straightforward computation to verify that the third equation is given by (10) multiplied by H plus (11) multiplied by G. Therefore, Theorem 4.2 can be written as:

Theorem 4.3 *Let G and H be complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$ such that:*

- *G and H are not identically zero;*
- *G and H are solutions of (10) and (11).*

Then the map $f : \Omega \to \mathbb{H}_3$, defined by

$$\begin{cases} f_1 = 2\mathrm{Re}\left(\displaystyle\int_{z_o}^{z} (G^2 - H^2)\, dz\right), \\ f_2 = 2\mathrm{Re}\left(i\displaystyle\int_{z_o}^{z} (G^2 + H^2)\, dz\right), \\ f_3 = 2\mathrm{Re}\left(\displaystyle\int_{z_o}^{z} \left(2GH - \dfrac{f_2}{2}(G^2 - H^2) + i\dfrac{f_1}{2}(G^2 + H^2)\right) dz\right), \end{cases}$$

is a conformal minimal immersion.

We observe that in this case we do not have the problem discussed in Remark 3.2, since we obtain the first two functions by direct integration and substituting in the third equation we obtain the third function by a direct integration.

We shall now discuss some examples.

Example 4.4 (Vertical planes) Let G and H be two holomorphic solutions of (10) and (11), i.e. $\frac{\partial G}{\partial \overline{z}} = 0$ and $\frac{\partial H}{\partial \overline{z}} = 0$. Then we must have $|G| = |H|$. Thus, from standard arguments of complex analysis, H and G differ by a unitary complex number, that is, $H = e^{i\theta}G$, $\theta \in \mathbb{R}$. The corresponding minimal immersion $f : \mathbb{C} \to \mathbb{H}_3$ is given by

$$\begin{cases} f_1 = 2\mathrm{Re}((1 - e^{2i\theta})\tilde{G}) \\ f_2 = 2\mathrm{Re}(i(1 + e^{2i\theta})\tilde{G}) \\ f_3 = 2\mathrm{Re}\left(\displaystyle\int_{z_o}^{z} \left[2e^{i\theta}G^2 - \dfrac{f_2}{2}(1 - e^{2i\theta})G^2 + i\dfrac{f_1}{2}G^2(1 + e^{2i\theta})\right] dz\right), \end{cases}$$

where $\tilde{G} = \int_{z_o}^{z} G^2$. A simple calculation gives $\cos\theta f_1 + \sin\theta f_2 = 0$, thus the image of the immersion lies in a "plane" parallel to the x_3-axis, which forms an angle θ with the plane $x_1 = 0$. In this example the Gauss map $g = H/G = e^{i\theta}$ has rank zero. These are the only minimal surfaces of $\mathbb{H}_3$ with Gauss map of rank zero.

Example 4.5 (The saddle-type surface) We shall now discuss the case where G and H are two purely imaginary functions depending only on one variable, that is $G(u,v) = il(v)$, $H(u,v) =$

$ih(v)$, where l and h are two smooth real-valued functions defined in a open subset of $\mathbb{R}$. Since we can assume that $|H| \neq |G|$, otherwise we are in the previous example, from (10) and (11), we get

$$\frac{\partial G}{\partial \overline{z}}\,\overline{G} = \frac{\partial H}{\partial \overline{z}}\,\overline{H}. \tag{12}$$

Replacing the expressions of G and H in (12) gives $ll' = hh'$, where with $'$ we denote the derivative with respect to v. So there exists a function $q(v)$ such that $l = \sqrt{q+a}$ and $h = \sqrt{q-a}$ for some constant $a \in \mathbb{R}$. The function q is determined by the condition that G and H satisfy (10) and (11), that is,

$$q' = 4a\sqrt{q^2 - a^2}. \tag{13}$$

Then $\phi_1 = (G^2 - H^2) = -2a$, $\phi_2 = i(G^2 + H^2) = -2iq$ and, by integration, we get:

$$\begin{cases} f_1 = -4au, \\ f_2 = 4Q(v), \end{cases}$$

where $Q(v)$ is a primitive function of $q(v)$. To calculate f_3, first note that

$$\phi_3 = 2GH - \frac{f_2}{2}(G^2 - H^2) + i\frac{f_1}{2}(G^2 + H^2) = -2\sqrt{q^2 - a^2} + 4aQ + i4auq. \tag{14}$$

Now, by integration of (13), we find that $\sqrt{q^2 - a^2} = 4aQ$. Substituting this in (14) gives

$$\phi_3 = -4aQ + i4auq.$$

Then, the corresponding immersion $f : \mathbb{C} \to \mathbb{H}_3$ is given by

$$\begin{cases} f_1(u,v) = -4au, \\ f_2(u,v) = 4Q(v), \\ f_3(u,v) = -8auQ(v). \end{cases}$$

It follows that the image of the immersion lies on the graph of the function $x_3 = \frac{1}{2}x_1x_2$. This is a ruled surface of saddle type and the rank of its Gauss map is 1. In fact $g = H/G = h(v)/l(v)$ depends only on one parameter. It is interesting to note that the only minimal surfaces with Gauss map of rank 1 in the Heisenberg group are the ruled minimal surfaces and that they can be explicitly described (see [15]).

Example 4.6 (Helicoids) In this example we describe a minimal surface with Gauss map of rank 2. Since the Gauss map is $g = H/G$ we can choose for G and H two functions which are the product of a function depending on one variable times a complex unit function depending on the other variable, that is $G(u,v) = e^{-iv/2}l(u)$, $H(u,v) = ie^{iv/2}h(u)$, where l and h are two smooth functions defined in a open subset of $\mathbb{R}$. Equation (12) gives:

$$l^2 + (l^2)' = -h^2 + (h^2)'. \tag{15}$$

Then there exists a function $\rho(u)$ such that $l = \sqrt{\rho' - \rho}/2$ and $h = \sqrt{\rho' + \rho}/2$ are solutions of (15). Using (10) and (11) we obtain that ρ is a solution of the following differential equation:

$$\rho'' - \rho = \rho\sqrt{(\rho')^2 - \rho^2},$$

which is equivalent to

$$\sqrt{(\rho')^2 - \rho^2} = \frac{1}{2}\rho^2 + c \quad c \in \mathbb{R}.$$

The corresponding minimal immersion $f : \mathbb{C} \setminus \{0\} \to \mathbb{H}_3$ is given by

$$\begin{cases} f_1(u,v) = \rho(u)\cos v, \\ f_2(u,v) = \rho(u)\sin v, \\ f_3(u,v) = cv + b, \quad b \in \mathbb{R}. \end{cases} \tag{16}$$

If $c \neq 0$, (16) gives a minimal parametrization of a helicoid, while, if $c = 0$, (16) gives the minimal parametrization of the horizontal plane $x_3 = b$.

Example 4.7 (Catenoid-type surface) In this example, we give the Weierstrass functions G and H for the Catenoid-type surface described in [6]. Let

$$h = \sqrt{\frac{g^2+4}{g^2-4}}, \quad g^2 > 4,$$

where $g = g(u)$ is a real-valued function which is a solutions of the ordinary differential equation

$$g'^2 = \frac{g^2(g^4-16)-4}{g^2-4}.$$

Then the functions

$$H = \frac{1}{2}e^{i(v+l/2)}\sqrt{g' + 2g(1+il'/2)}, \quad G = \frac{1}{2}e^{-i(v+l/2)}\sqrt{g' - 2g(1+il'/2)},$$

with $l = l(u)$ a real-valued function, are solutions of (10) and (11) if $l' = \frac{2h}{g^2+4}$. The corresponding ϕ's are

$$\begin{cases} \phi_1(u,v) = 1/2[g'\cos(l+2v) - gl'\sin(l+2v) + 2ig\sin(l+2v)], \\ \phi_2(u,v) = 1/2[g'\sin(l+2v) + gl'\cos(l+2v) - 2ig\cos(l+2v)], \\ \phi_3(u,v) = h/2. \end{cases}$$

Finally, after integration, we get the minimal parametrisation

$$\begin{cases} f_1(u,v) = g\cos(l+2v), \\ f_2(u,v) = g\sin(l+2v), \\ f_3(u,v) = \tilde{h}, \end{cases} \tag{17}$$

where $\tilde{h}$ is a primitive of h. The parametrization (17) is exactly, up to a change of coordinates, the parametrisation of the minimal Catenoid of revolution (about the x_3-axes) described in [6].

5 Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Let $\mathbb{H}^2$ be the hyperbolic plane $\{(x_1,x_2) \in \mathbb{R}^2 \, : \, x_2 > 0\}$ endowed with the metric, of constant Gauss curvature -1, $g_{\mathbb{H}} = (dx_1^2 + dx_2^2)/x_2^2$. The hyperbolic plane $\mathbb{H}^2$, with the group structures derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. We can then consider $\mathbb{H}^2 \times \mathbb{R}$ as a Lie group with the product structure and the left invariant metric

$$g = \frac{dx_1^2 + dx_2^2}{x_2^2} + dx_3^2.$$

With respect to the left invariant metric g, an orthonormal basis of left invariant vector fields is

$$E_1 = x_2\,\frac{\partial}{\partial x_1}, \quad E_2 = x_2\,\frac{\partial}{\partial x_2}, \quad E_3 = \frac{\partial}{\partial x_3}.$$

In this case the matrix defined in (3) takes the form

$$A = \begin{pmatrix} x_2 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

while the non zero L_{ij}^k are $L_{12}^1 = -2,\ L_{11}^2 = 2$. We then have the following:

Theorem 5.1 *Let $\psi_j, j = 1,\dots,3$, be three complex-valued functions defined in a simply connected domain $\Omega \subset \mathbb{C}$, such that the following conditions are satisfied:*

- $\sum_{i=1}\psi_i\overline{\psi_i} \neq 0$;
- $\sum_{i=1}{\psi_i}^2 = 0$;

- *ψ_3 is holomorphic and ψ_1,ψ_2 are solutions of the system*

$$\begin{cases} \dfrac{\partial\psi_1}{\partial\bar z} - \overline{\psi_1}\psi_2 = 0, \\ \dfrac{\partial\psi_2}{\partial\bar z} + \|\psi_1\|^2 = 0. \end{cases} \tag{18}$$

Then the map $f:\Omega\to\mathbb{H}^2\times\mathbb{R}$ given by

$$f_i(u,v) = 2\mathrm{Re}\Big(\int_{z_o}^{z}\sum_j A_{ij}\psi_j\,dz\Big)$$

defines a conformal minimal immersion.

Proof According to (5) the section $\phi=\psi_1E_1+\psi_2E_2+\psi_3E_3$ is holomorphic if and only if

$$\begin{cases} \dfrac{\partial\psi_1}{\partial\bar z} - \overline{\psi_1}\psi_2 = 0, \\ \dfrac{\partial\psi_2}{\partial\bar z} + \|\psi_1\|^2 = 0, \\ \dfrac{\partial\psi_3}{\partial\bar z} = 0. \end{cases}$$

Then the theorem follows from Theorem 3.1.

If we put $\tilde f_2 = 2\mathrm{Re}\int_{z_o}^{z}\psi_2\,dz$, the minimal immersion becomes

$$f(u,v) = \left(2\mathrm{Re}\int_{z_o}^{z} e^{\tilde f_2}\,\psi_1\,dz,\ e^{\tilde f_2},\ 2\mathrm{Re}\int_{z_o}^{z}\psi_3\,dz\right).$$

Example 5.2 If ψ_2 is a holomorphic function, then from (18) we have that ψ_1 must be identically zero and the corresponding immersion is the minimal parametrization of the vertical planes x_1 = constant. If ψ_1 and ψ_2 are not holomorphic, then (18) implies that $\psi_1^2+\psi_2^2$ is holomorphic. The latter condition is surely satisfied if $\psi_1^2+\psi_2^2=a$ for some constant $a\in\mathbb{R}$. Let examine the following cases:

$a=0$: Then, from $\psi_1^2+\psi_2^2+\psi_3^2=0$, we have that $\psi_3=0$ and the corresponding immersion is the minimal parametrization of the horizontal planes x_3 = constant. These are, in fact, totally geodesic surfaces.

$a=-1$: In this case, decomposing $\psi_1(u,v)=a_1(u,v)+ia_2(u,v)$ and $\psi_2(u,v)=a_3(u,v)+ia_4(u,v)$ with respect to their real and imaginary parts, the condition $\psi_1^2+\psi_2^2=-1$ reduces to the following system:

$$\begin{cases} a_1^2-a_2^2+a_3^2-a_4^2=-1, \\ a_1a_2+a_3a_4=0. \end{cases} \tag{19}$$

Choosing the solution

$$\begin{cases} a_1(u,v)=\sin(2v)a_4(u,v), \\ a_3(u,v)=-\sin(2v)a_2(u,v), \end{cases}$$

of the second equation of (19), then, from (18), the functions a_1 and a_2 are solutions of

$$\left(\frac{\partial a_1}{\partial u}-\frac{\partial a_2}{\partial v}\right)\left(\frac{\partial a_1}{\partial v}+\frac{\partial a_2}{\partial u}\right)=4a_1a_2.$$

A solution of the latter equation is

$$a_1=\frac{2\,(\cos(2\,u)+\sin(2\,v))\,\tan(2\,v)}{2-\sin(2\,u-2\,v)+\sin(2\,(u+v))};\quad a_2=\frac{2\,\sin(2\,u)}{-2+\sin(2\,u-2\,v)-\sin(2\,(u+v))}.$$

Finally, after integration, the immersion $f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$ becomes:

$$\begin{aligned} f_1(u,v) &= \frac{2\sin(2u)\tan(v)}{\sin(2u)^2\tan(v)^2 + (1+\cos(2u)\tan(v))^2}, \\ f_2(u,v) &= \frac{1-\tan(v)^2}{\sin(2u)^2\tan(v)^2 + (1+\cos(2u)\tan(v))^2}, \\ f_3(u,v) &= 2u. \end{aligned}$$

Let $\tilde{\alpha} : \mathbb{I} \to \mathbb{H}^2$ be the standard isometry from the hyperbolic disc $\mathbb{I} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ to $\mathbb{H}^2$, and define $\alpha : \mathbb{I} \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ by $\alpha(x,y,z) = (\tilde{\alpha}(x,y), z)$. Then the immersion f is the composition $\alpha \circ \tilde{f}$ where $\tilde{f}(u,v) = (\tan v \cos(2u), \tan v \sin(2u), 2u)$ is the immersion of a minimal helicoid in $\mathbb{I} \times \mathbb{R}$ described in [9].

Remark 5.3 (Futher investigations) We would like to point out that some of the basic questions that have been solved for the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$ are still without an answer for the Weierstrass representation in $\mathbb{H}_3$ and $\mathbb{H}^2 \times \mathbb{R}$. Among these questions are:

- An analysis of the non simply connected case;
- The application of the Weierstrass representation formula to the Berstein problem;
- The research of new examples.

Moreover, we believe that our method can be applied to other three-dimensional manifolds. In particular, using the recent study of Harold Rosenberg of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ [16], we shall deal, in a forthcoming paper, with the Weierstrass representation formula of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

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