Acta Mathematica Sinica, English Series
May, 2006, Vol. 22, No. 3, pp. 693–700
Published online: Oct. 17, 2005
DOI: 10.1007/s10114-005-0601-x
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Inequalities of Maximum of Partial Sums and Weak Convergence for a Class of Weak Dependent Random Variables

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Abstract In this paper, we establish a Rosenthal-type inequality of the maximum of partial sums for ρ^--mixing random fields. As its applications we get the Hájeck–Rènyi inequality and weak convergence of sums of ρ^--mixing sequence. These results extend related results for NA sequence and ρ^*-mixing random fields.

Keywords ρ^--mixing, ρ^*-mixing, NA, rosenthal type inequalities, weak convergence
MR(2000) Subject Classification 60E15, 60F16

1 Introduction

Following the introductory concept of ρ^--mixing random variables in 1999 (see refs. [1]), Zhang got moment inequalities of partial sums, central limit theorems, complete convergence and the strong law of large numbers (see refs [1–3]). Since ρ^--mixing random variables include NA and ρ^*-mixing random variables, which have a lot of applications, their limit properties have aroused wide interest recently. In this paper, we obtain a Rosenthal-type inequality of the maximum of partial sums, the Hájeck–Rènyi inequality and weak convergence of sums of ρ^--mixing sequence under the Lindeberg condition, which develop the results in refs [1–3]. Now we introduce some definitions as follows:

Definition 1.1 [4] *A sequence $\{X_k; k \in N\}$ is called negatively associated (NA) if for every pair of disjoint subsets S, T of N,*

$$\mathrm{Cov}\{f(X_i; i \in S), g(X_j; j \in T)\} \leq 0,$$

whenever $f, g \in \mathscr{C}, \mathscr{C}$ is a class of functions which are coordinatewise increasing.

Definition 1.2 [1] *A sequence $\{X_k; k \in N\}$ is called ρ^*-mixing if*

$$\rho^*(s) = \sup\{\rho(S,T); S, T \subset N, \mathrm{dist}(S,T) \geq s\} \to 0\,(s \to \infty),$$

where

$$\rho(S,T) = \sup\{|E(f-Ef)(g-Eg)/(\| f-Ef \|_2 \| g-Eg \|_2)|; f \in L_2(\sigma(S)), g \in L_2(\sigma(T))\}.$$

Definition 1.3 [1] *A sequence $\{X_k; k \in N\}$ is called ρ^--mixing if*

$$\rho^-(s) = \sup\{\rho^-(S,T); S, T \subset N, \mathrm{dist}(S,T) \geq s\} \to 0\,(s \to \infty),$$

where

$$\rho^-(S,T) = 0 \vee \sup\left\{\frac{\mathrm{Cov}\{f(X_i; i \in S), g(X_j; j \in T)\}}{\sqrt{\mathrm{Var}\{f(X_i; i \in S)\}\mathrm{Var}\{g(X_j; j \in T)\}}}; f, g \in \mathscr{C}\right\}.$$

Received September 28, 2003, Accepted July 28, 2004
Supported by Scientific research project of education department of Zhejiang Province (No. 20051897)

It is easy to see that $\{X_k; k \in N\}$ is negatively associated if and only if $\rho^-(s) = 0$ for $s \geq 1$. It is obvious that $\rho^-(s) \leq \rho^*(s)$, so ρ^--mixing is weaker than ρ^*-mixing. In the past several years, many limit results for NA sequences and ρ^*-mixing fields were obtained (see refs [5–12]). Peligrad [6] studied the importance of the condition $\lim_{n\to\infty} \rho^*(n) < 1$ in estimating the moments of partial sums or the maximum of partial sums for ρ^*-mixing fields. In this paper, we consider the condition

$$\lim_{n\to\infty} \rho^-(n) \leq r, \quad 0 \leq r < \left(\frac{1}{6p}\right)^{\frac{p}{2}}, \quad p \geq 2. \tag{1}$$

The following two properties of ρ^--mixing are used in the next sections:

Property P1 [1] A subset of a ρ^--mixing field $\{X_i\}_{i\geq 1}$ with mixing coefficients $\rho^-(s)$ is also ρ^--mixing with coefficients not greater than $\rho^-(s)$.

Property P2 [1] Increasing functions defined on disjoint subsets of a ρ^--mixing field $\{X_i\}_{i\geq 1}$ with mixing coefficients $\rho^-(s)$ are also ρ^--mixing with coefficients not greater than $\rho^-(s)$.

2 Result

Theorem 2.1 *For a positive integer $N \geq 1$, positive real numbers $p \geq 2$ and $0 \leq r < (\frac{1}{6p})^{\frac{p}{2}}$, if $\{X_i\}_{i\geq 1}$ is a sequence of random variables with $\rho_N^- \leq r$, with $EX_i = 0$ and $E|X_i|^p < \infty$ for every $i \geq 1$, then for all $n \geq 1$, there is a positive constant $D = D(p, N, r)$ such that*

$$E \max_{1\leq i\leq n} \left|\sum_{j=1}^{i} X_j\right|^p \leq D\left(\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{\frac{p}{2}}\right).$$

This theorem is a basic tool in establishing various almost-sure results for sums of dependent random variables.

Corollary 2.1 *For a positive integer $N \geq 1$ and $0 \leq r < \frac{1}{12}$, $\{X_i\}_{i\geq 1}$ is a sequence of random variables with $\rho_N^- \leq r$, with $EX_i = 0$ and $E|X_i|^2 < \infty$ for every $i \geq 1$. If $\{b_n, n \geq 1\}$ is a positive nondecreasing real number sequence, then for $\varepsilon > 0$, there is a positive constant C such that*

$$P\left(\max_{k\leq n} \left|\frac{1}{b_k} \sum_{i=1}^{k} (X_i - EX_i)\right| \geq \varepsilon\right) \leq C \sum_{i=1}^{n} \frac{\sigma_i^2}{b_i^2},$$

where $\sigma_i^2 = \mathrm{Var} X_i$ for $i \geq 1$.

In this paper we shall use our maximal inequality to establish the functional form of the central limit theorem.

For sequence $\{X_i\}_{i\geq 1}$ is a sequence of square integrable ($EX_i^2 < \infty$) centered ($EX_i = 0$) random variables, we denote $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n X_i)$ and by $\mathrm{Cov}(X_i, X_j)^- = (\mathrm{Cov}(X_i, X_j))^-$. And define, for $0 \leq t \leq 1$,

$$\nu_n(t) = \inf\left\{m : 1 \leq m \leq n, \frac{\sigma_m^2}{\sigma_n^2} \geq t\right\} \quad \text{and} \quad W_n(t) = \frac{\sum_{i=1}^{\nu_n(t)} X_i}{\sigma_n}. \tag{2}$$

Theorem 2.2 *Suppose that $\{X_i\}_{i\geq 1}$ is a sequence of ρ^--mixing centered and not all degenerated random variables with $EX_i^2 < \infty, i \geq 1$, which satisfies:*

(i) $\limsup_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{|i-j|\geq r, i,j\leq n} \mathrm{Cov}(X_i, X_j)^- \longrightarrow 0$, *as $r \to \infty$;*

(ii) *For every $\epsilon > 0$, $\frac{1}{\sigma_n^2} \sum_{i=1}^n EX_i^2 I(|X_i| > \epsilon\sigma_n) \longrightarrow 0$, as $n \to \infty$;*

(iii) $\limsup_{n\to\infty} \frac{\sum_{\nu_n(t)}^{\nu_n(t+\delta)} \mathrm{Var}(X_i)}{\mathrm{Var}\left(\sum_{\nu_n(t)}^{\nu_n(t+\delta)} X_i\right)} < \infty$, *for all $0 \leq t < t + \delta \leq 1$.*

Then $W_n(t) \overset{\mathscr{D}}{\Rightarrow} W(t)$, where $W_n(t)$ is defined by (2) and $W(t)$ denotes the standard Brownian process on $[0, 1]$.

Corollary 2.2 *Suppose that $\{X_i\}_{i\geq 1}$ is a weakly stationary ρ^--mixing sequence with $EX_1 = 0$ and $0 < EX_1^2 < \infty$. If $\sigma^2 = \mathrm{Var}(X_1)+2\sum_{k=2}^{\infty}\mathrm{Cov}(X_1, X_k) > 0$ and $\sum_{k=2}^{\infty}|\mathrm{Cov}(X_1, X_k)| < \infty$, then $\sigma_n^2/n \to \sigma^2$ and $W_n \overset{\mathscr{D}}{\Rightarrow} W$, where W_n and W are defined as in Theorem 2.2.*

3 Proof

The proof of Theorem 2.1 uses the following lemmas. The first lemma is Lemma 1 in Bradley [5].

Lemma 3.1 *Suppose $0 \leq r < 1$ and $\{X_1, X_2, \dots, X_n\}$ is a family of square integrable centered random variables such that for any nonempty subset,*

$$S \subset \{1, 2, \dots, n\}, \quad S^* = \{1, 2, \dots, n\} - S, \quad \mathrm{corr}\Big(\sum_{k\in S} X_k, \sum_{k\in S^*} X_k\Big) \leq r.$$

Then $E(\sum_{k=1}^n X_k)^2 \leq \frac{1+r}{1-r}\sum_{k=1}^n EX_k^2$.

Lemma 3.2 *Assume $\{X_i\}_{i\geq 1}$ is a sequence of random variables with $EX_i = 0, E|X_i|^p < \infty$, for some $p \geq 2$ and every $i \geq 1$. For some $0 \leq r < (\frac{1}{6p})^{\frac{p}{2}}$, assume $\rho_1^- \leq r$ (or for a positive integer N, $\rho_N^- \leq r$). Then for all $n \geq 1$, there is a positive constant $D' = D'(p, r)$ such that*

$$E|S_n|^p \leq D'\bigg(\sum_{i=1}^n E|X_i|^p + \Big(\sum_{i=1}^n EX_i^2\Big)^{\frac{p}{2}}\bigg)$$

and

$$E\max_{1\leq i\leq n}|S_i|^p \leq D'\bigg[\Big(E\max_{1\leq i\leq n}|S_i|\Big)^p + \sum_{i=1}^n E|X_i|^p + \Big(\sum_{i=1}^n EX_i^2\Big)^{\frac{p}{2}}\bigg].$$

Proof of Lemma 3.2 The proof of the first inequality is similar to the proofs of Lemma 3.3 in [2] and Theorem 2.1 in [1] with only some changes. The proof of the second inequality is similar to the proof of Lemma A.2 in [11].

Lemma 3.3 *For some $p \geq 2$, let $0 \leq r < (\frac{1}{6p})^{\frac{p}{2}}$ be fixed. Suppose $\{Y_i\}_{i\geq 1}$ is a sequence of square integrable centered random variables with $\rho_1^- \leq r$. Then for all $n \geq 1$, there is a positive constant $K = K(p, r)$ such that $E\max_{1\leq i\leq n}|\sum_{j=1}^i Y_j| \leq K\sqrt{\sum_{j=1}^n EY_j^2}$.*

Proof of Lemma 3.3 When $\sum_{j=1}^n EY_j^2 = \infty$ or 0, the inequality is obvious.

Define

$$a_n = \sup_Y\bigg(E\max_{1\leq i\leq n}\Big|\sum_{j=1}^i Y_j\Big| \Big/ \Big[\sum_{j=1}^n EY_j^2\Big]^{\frac{1}{2}}\bigg),$$

where the supremum is taken over all sequences $Y := \{Y_i\}$ of square integrable centered random variables with $\rho_1^-(\{Y_i\}) \leq r$ and $\sum_{j=1}^n EY_j^2 < \infty$.

Fix such a random sequence $\{Y_i\}$ and with loss of generality assume that $\sum_{j=1}^n EY_j^2 = 1$.

By using Lemma 3.1, we have

$$\begin{aligned} a_n &= E\max_{1\leq i\leq n}\Big|\sum_{j=1}^i Y_j\Big| \leq \sum_{i=1}^n E\Big|\sum_{j=1}^i Y_j\Big| \leq \sum_{i=1}^n\bigg(E\Big|\sum_{j=1}^i Y_j\Big|^2\bigg)^{1/2} \\ &\leq C\sum_{i=1}^n\Big(\sum_{j=1}^i EY_j^2\Big)^{1/2} \leq Cn\Big(\sum_{j=1}^n EY_j^2\Big)^{1/2} < \infty. \end{aligned}$$

Let M be a positive integer that will be specified later. For $1 \leq j \leq n$, define

$$Y_j^{(n)} = (-M^{-1/2}) \vee (Y_j \wedge M^{-1/2}), \quad Y_j^{(n1)} = Y_j^{(n)} - EY_j^{(n)}, \quad Y_j^{(n2)} = Y_j - Y_j^{(n1)},$$

so

$$\sum_{j=1}^i Y_j = \sum_{j=1}^i Y_j^{(n1)} + \sum_{j=1}^i Y_j^{(n2)}.$$

Since

$$E\max_{1\le i\le n}\left|\sum_{j=1}^{i}Y_j^{(n2)}\right|\le\sum_{j=1}^{n}E|Y_j^{(n2)}|\le 3M^{1/2}\sum_{j=1}^{n}EY_j^2=3M^{1/2},$$

we get

$$E\max_{1\le i\le n}\left|\sum_{j=1}^{i}Y_j\right|\le E\max_{1\le i\le n}\left|\sum_{j=1}^{i}Y_j^{(n1)}\right|+3M^{1/2}.$$

To estimate $E\max_{1\le i\le n}|\sum_{j=1}^{i}Y_j^{(n1)}|$, we shall use a blocking procedure.

Take $m_0=0$ and define the integers m_k recursively by

$$m_k=\min\left\{m:m>m_{k-1},\sum_{j=m_{k-1}}^{m}E(Y_j^{(n1)})^2>\frac{1}{M}\right\}.$$

Note that, if we denote by l the number of integers produced by this procedure, i.e.: $m_0,m_1,\dots,m_{l-1}$, we have

$$1\ge\sum_{k=1}^{l-1}\sum_{j=m_{k-1}+1}^{m_k}E(Y_j^{(n1)})^2>\frac{l-1}{M},$$

so that $l\le M$.

We partition the sum $\sum_{j=1}^{n}Y_j^{(n1)}$ into partial sums $\sum_{j=1}^{n}Y_j^{(n1)}=\sum_{k=1}^{l}X_k$, where $X_k=\sum_{j=m_{k-1}+1}^{m_k}Y_j^{(n1)}$, for $1\le k\le l-1$ and $X_l=\sum_{j=m_{l-1}+1}^{n}Y_j^{(n1)}$, for $m_l\ge n$.

Obviously

$$E\max_{1\le i\le n}\left|\sum_{j=1}^{i}Y_j^{(n1)}\right|\le E\max_{1\le k\le l}\left|\sum_{j=1}^{k}X_j\right|+E\max_{1\le k\le l}\left(\max_{m_{k-1}<j<m_k}\left|\sum_{t=m_{k-1}+1}^{j}Y_t^{(n1)}\right|\right)=I_1+I_2.$$

We evaluate the I_1 and I_2 above.

$$I_1\le\sum_{k=1}^{l}E|X_k|\le\sum_{k=1}^{l}(EX_k^2)^{1/2}.$$

Since $\rho_1^-(\{Y_i\})\le r<1$, by Property P2 we have $\rho_1^-(\{Y_i^{(n1)}\})\le r<1$. By Lemma 3.1, let $K_1\ge(1+r)/(1-r)$, for all $1\le k\le l$, we have

$$EX_k^2=E\left(\sum_{j=m_{k-1}+1}^{m_k}Y_j^{(n1)}\right)^2\le K_1\sum_{j=m_{k-1}+1}^{m_k}E(Y_j^{(n1)})^2.$$

By Cauchy–Schwarz inequality for sequences, we obtain

$$I_1\le K_1^{1/2}\sum_{k=1}^{l}\left(\sum_{j=m_{k-1}+1}^{m_k}E(Y_j^{(n1)})^2\right)^{1/2}\le(K_1M)^{1/2}\left[\sum_{j=1}^{n}E(Y_j^{(n1)})^2\right]^{1/2}\le(K_1M)^{1/2}.$$

To estimate I_2 we notice that

$$(I_2)^4\le\sum_{k=1}^{l}E\max_{m_{k-1}<j<m_k}\left|\sum_{t=m_{k-1}+1}^{j}Y_t^{(n1)}\right|^4.$$

By Lemma 3.2, we obtain

$$\begin{aligned}&E\max_{m_{k-1}<j<m_k}\left|\sum_{t=m_{k-1}+1}^{j}Y_t^{(n1)}\right|^4\\&\quad\le K_2\left[E^4\max_{m_{k-1}<j<m_k}\left|\sum_{t=m_{k-1}+1}^{j}Y_t^{(n1)}\right|+\sum_{j=m_{k-1}+1}^{m_k-1}E(Y_j^{(n1)})^4+\left(\sum_{j=m_{k-1}+1}^{m_k-1}E(Y_j^{(n1)})^2\right)^2\right].\end{aligned}$$

We estimate each term on the right-hand side of above inequality.

By the definition of m_k, we have

$$\sum_{j=m_{k-1}+1}^{m_k-1} E(Y_j^{(n1)})^2 \leq \frac{1}{M}.$$

By using the definition of $Y_k^{(n1)}$, we obtain

$$\sum_{j=m_{k-1}+1}^{m_k-1} E(Y_j^{(n1)})^4 \leq \frac{4}{M} \sum_{j=m_{k-1}+1}^{m_k-1} E(Y_j^{(n1)})^2 \leq \frac{4}{M^2}.$$

To fixed $\{Y_i\}_{i\geq 1}$, let $\{Y_i^{'}\}_{i\geq 1}$ be a new random variable so that for $i = m_{k-1}+1, \dots, m_k-1, Y_i^{'} = Y_i$, otherwise $Y_i^{'} = 0$; so by Property P1 we have $\rho_1\{Y_1^{'}\} \leq r$. By the definition of a_n, we get

$$E^4 \max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} Y_t^{(n1)} \right| \leq a_n^4 \left(\sum_{j=m_{k-1}+1}^{m_k-1} E(Y_j^{(n1)})^2 \right)^2 \leq \frac{a_n^4}{M^2}.$$

Overall, by the above considerations, we obtain

$$(I_2)^4 \leq K_2 \left[\frac{a_n^4}{M} + \frac{4}{M} + \frac{1}{M} \right].$$

Now let $K_3 = \max\{K_1, K_2\}$; by our estimates for I_1 and I_2, we get

$$\begin{aligned} E \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Y_j \right| &\leq E \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Y_j^{(n1)} \right| + 3M^{1/2} \\ &\leq [(K_3 M)^{1/2} + (K_3/M)^{1/4} a_n + (5K_3/M)^{1/4}] + 3M^{1/2}. \end{aligned}$$

Therefore, by the definition of a_n, we obtain

$$a_n \leq (K_3/M)^{1/4} a_n + (K_3^{1/2} + 3) M^{1/2} + (5K_3/M)^{1/4}.$$

Now we select $M = M(r)$ to be $M = M(r) = [16K_3] + 1$; and since $K_3 \geq 1$, we obtain

$$a_n \leq \frac{a_n}{2} + K_3^{1/2}(1 + 3K_3^{-1/2}) 4K_3^{1/2}(1 + K_3^{-1/2}/4) + 1 \leq \frac{a_n}{2} + 21K_3,$$

which implies the desired result.

Proof of Theorem 2.1 The conclusion of Theorem 2.1 is an easy consequence of Lemma 3.1 after a standard reduction procedure. Let M be the integer mentioned in Theorem 2.1 such that $\rho_N^- \leq r$. We consider now N sequences of a random variable $\{Y_{ij} : i \geq 0\}, 1 \leq j \leq N$, defined by $Y_{ij} = X_{iN+j}$.

Notice that for each i, the first interlaced mixing coefficient $\rho_1^-(Y)$ for the sequence $\{Y_{ij} : i \geq 0\}$ is smaller than $\rho_N^- \leq r$.

It is easy to see that

$$\max_{1\leq m\leq n} |S_m| \leq \sum_{j=1}^{N} \max_{1\leq k\leq n/N} \left| \sum_{i=0}^{k} X_{iN+j} \right|.$$

By using Lemma 3.3 and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} E \max_{1\leq m\leq n} |S_m| &\leq \sum_{j=1}^{N} E \max_{1\leq k\leq n/N} \left| \sum_{i=0}^{k} X_{iN+j} \right| \leq K \sum_{j=1}^{N} \left(\sum_{i=0}^{n/N} EX_{iN+j}^2 \right)^{1/2} \\ &\leq KN^{1/2} \left(\sum_{j=1}^{N} \sum_{i=0}^{n/N} EX_{iN+j}^2 \right)^{1/2} = KN^{1/2} \left(\sum_{m=1}^{n} X_m^2 \right)^{1/2}, \end{aligned}$$

which implies the desired result in Theorem 2.1 by Lemma 3.2.

Proof of Corollary 2.1 Let $S_n = \sum_{i=1}^{n}(X_i - EX_i)$ for $n \geq 1$. Without loss of generality, setting $b_0 = 0$, we have

$$S_k = \sum_{i=1}^{k} b_i \frac{X_i - EX_i}{b_i} = \sum_{i=1}^{k} \left(\sum_{j=1}^{i} (b_j - b_{j-1}) \frac{X_i - EX_i}{b_i} \right) = \sum_{j=1}^{k} (b_j - b_{j-1}) \sum_{i=j}^{k} \frac{X_i - EX_i}{b_i}.$$

Note that $\frac{1}{b_k}\sum_{j=1}^{k}(b_j-b_{j-1})=1$, so

$$\Big\{\Big|\frac{S_k}{b_k}\Big|\ge\varepsilon\Big\}\subset\Big\{\max_{1\le j\le k}\Big|\sum_{i=j}^{k}\frac{X_i-EX_i}{b_i}\Big|\ge\varepsilon\Big\}.$$

Therefore

$$\begin{aligned}\Big\{\max_{1\le k\le n}\Big|\frac{S_k}{b_k}\Big|\ge\varepsilon\Big\}&\subset\Big\{\max_{1\le k\le n}\max_{1\le j\le k}\Big|\sum_{i=j}^{k}\frac{X_i-EX_i}{b_i}\Big|\ge\varepsilon\Big\}\\&=\Big\{\max_{1\le j\le k\le n}\Big|\sum_{i=1}^{k}\frac{X_i-EX_i}{b_i}-\sum_{i=1}^{j-1}\frac{X_i-EX_i}{b_i}\Big|\ge\varepsilon\Big\}\\&\subset\Big\{\max_{1\le j\le k}\Big|\sum_{i=1}^{j}\frac{X_i-EX_i}{b_i}\Big|\ge\varepsilon/2\Big\}.\end{aligned}$$

By Theorem 2.1 we get

$$P\Big(\max_{1\le k\le n}\Big|\frac{S_k}{b_k}\Big|\ge\varepsilon\Big)\le C\sum_{i=1}^{n}\frac{\sigma_i^2}{b_i^2},$$

which implies the result in Corollary 2.1.

Proof of Theorem 2.2 By condition (ii), there exists a sequence of positive numbers $\epsilon_n\to 0$ such that $\frac{1}{\sigma_n^2}\sum_{i=1}^{n}EX_i^2I(|X_i|>\epsilon_n\sigma_n)\to 0$, as $n\to\infty$.

In the following proving process we take C as a positive number and define

$$\eta_{ni}=(-\epsilon_n)\vee\Big(\frac{X_i}{\sigma_n}\wedge\epsilon_n\Big)-E\Big[(-\epsilon_n)\vee\Big(\frac{X_i}{\sigma_n}\wedge\epsilon_n\Big)\Big]\quad\text{and}\quad\varphi_{ni}=\frac{X_i}{\sigma_n}-\eta_{ni}.$$

Denote $W_n'(t)=\sum_{i=1}^{\nu_n(t)}\eta_{ni}$ and $W_n''(t)=\sum_{i=1}^{\nu_n(t)}\varphi_{ni}$, where $\nu_n(t)$ is defined as above. Notice that $W_n(t)=W_n'(t)+W_n''(t)$. By Property P2, we can find that $\{\eta_{ni}:i\le n\}_{n\ge1}$, $\{\varphi_{ni}:i\le n\}_{n\ge1}$ are also sequences of ρ^--mixing centered square integrable random variables, so by Theorem 2.1 and condition (ii) of Theorem 2.2, we can get

$$E\big[\sup_t|W_n''(t)|\big]^2\le E\max_{1\le i\le n}\Big(\sum_{j=1}^{i}\varphi_{ni}\Big)^2\le C\sum_{i=1}^{n}\frac{X_i^2}{\sigma_n^2}I\Big(\frac{|X_i|}{\sigma_n}>\epsilon_n\Big)\to 0,\ \text{ as }\ n\to\infty.$$

Therefore $W_n''(t)$ converges weakly to 0 and the limiting distribution of $W_n(t)$ is the same as the limiting distribution of $W_n'(t)$ if the last one exists (i.e., $W_n(t)-W_n'(t)\overset{\mathscr{D}}{\Rightarrow}0$ as $n\to\infty$).

It's easy to verify that:

(i)$'$ $\limsup_{n\to\infty}\sum_{|i-j|\ge r,i,j\le n}\mathrm{Cov}(\eta_{ni},\eta_{nj})^-\longrightarrow 0$, as $r\to\infty$;

(ii)$'$ For every $\epsilon>0$, $\sum_{i=1}^{n}E\eta_{ni}^2I(|\eta_{ni}|>\epsilon)\longrightarrow 0$, $n\to\infty$;

(iii)$'$ $\sup_n\sum_{i=1}^{n}\mathrm{Var}(\eta_{ni})<\infty$.

By $\lim_{n\to\infty}\mathrm{Var}(\sum_{i=1}^{n}\eta_{ni})=\lim_{n\to\infty}\mathrm{Var}(\sum_{i=1}^{n}\frac{Xi}{\sigma_n})=1$ and Theorem 3.2 of Reference [3], we get $W'(1)\to N(0,1)$.

In order to prove that $W'(t)\to N(0,t)$, from the above analysis, we have to show only that $\lim_{n\to\infty}E\big(\sum_{i=1}^{\nu_n(t)}\eta_{ni}\big)^2=t$, which is equivalent to $\lim_{n\to\infty}E(\sum_{i=1}^{\nu_n(t)}\frac{X_i}{\sigma_n})^2=t$. Let $||X||=(EX^2)^{1/2}$. Since

$$\Big\|\sum_{i=1}^{\nu_n(t)}\frac{X_i}{\sigma_n}\Big\|\le\Big\|\sum_{i=1}^{\nu_n(t)-1}\frac{X_i}{\sigma_n}\Big\|+\Big\|\frac{X_{\nu_n(t)}}{\sigma_n}\Big\|,$$

by the definition of $\nu_n(t)$, this relation gives $\sqrt{t}\le\|\sum_{i=1}^{\nu_n(t)}\frac{X_i}{\sigma_n}\|\le\sqrt{t}+\|\frac{X_{\nu_n(t)}}{\sigma_n}\|$. By the condition (ii), we get $\lim_{n\to\infty}||\frac{X_{\nu_n(t)}}{\sigma_n}||=0$. Then we get $\lim_{n\to\infty}E(\sum_{i=1}^{\nu_n(t)}\frac{X_i}{\sigma_n})^2=t$.

Next, we have to show that $W_n'(t)$ has asymptotically uncorrelated increments, that is,

$$\lim_{n\to\infty}\mathrm{Cov}(W_n'(t),W_n'(t+\delta)-W_n'(t))=0.$$

By condition (iii)′ and Theorem 2.1,

$$\mathrm{Var}\Big(\sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj}\Big) \le C \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} E\eta_{nj}^2 \le C \sum_{j=1}^{n} E\eta_{nj}^2 \le C.$$

Since $\epsilon_n \to 0$, we can take $i_n \to \infty$, such that $\epsilon_n i_n \to 0$. Then

$$\begin{aligned}
&|\mathrm{Cov}(W_n'(t), W_n'(t+\delta) - W_n'(t))| \\
&\le \Big|\mathrm{Cov}\Big(\sum_{i=1}^{\nu_n(t)-i_n} \eta_{ni}, \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj}\Big)\Big| + \Big|\mathrm{Cov}\Big(\sum_{i=\nu_n(t)-i_n+1}^{\nu_n(t)} \eta_{ni}, \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj}\Big)\Big| \\
&\le C\rho^-(i_n) + \mathrm{Cov}\Big(\sum_{i=1}^{\nu_n(t)-i_n} \eta_{ni}, \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj}\Big)^- + \sum_{i=\nu_n(t)-i_n+1}^{\nu_n(t)} ||\eta_{ni}|| \Big\| \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj})\Big\| \\
&\le C\rho^-(i_n) + \sum_{|i-j|\ge i_n, i,j\le n} \mathrm{Cov}(\eta_{ni}, \eta_{nj})^- + \epsilon_n i_n \Big\| \sum_{j=\nu_n(t)+1}^{\nu_n(t+\delta)} \eta_{nj}\Big\| \\
&\le C\rho^-(i_n) + \frac{C}{\sigma_n^2} \sum_{|i-j|\ge i_n, i,j\le n} \mathrm{Cov}(X_i, X_j)^- + C\epsilon_n i_n \to 0,
\end{aligned}$$

as $n \to \infty$ uniformly in t, δ, with $0 \le t \le t+\delta \le 1$. So it's easy to know that, uniformly in t, δ, with $0 \le t \le t+\delta \le 1$,

$$\lim_{n\to\infty} E\Big(\sum_{i=\nu_n(t)}^{\nu_n(t+\delta)} \eta_{ni}\Big)^2 = \lim_{n\to\infty} E\Big(\sum_{i=\nu_n(t)}^{\nu_n(t+\delta)} \frac{X_i}{\sigma_n}\Big)^2 = \delta.$$

Then, by condition (iii), we can get

$$\lim_{n\to\infty} \sum_{i=\nu_n(t)}^{\nu_n(t+\delta)} E\eta_{ni}^2 = \lim_{n\to\infty} \frac{\sum_{i=\nu_n(t)}^{\nu_n(t+\delta)} EX_i^2}{\sigma_n^2} \le C \lim_{n\to\infty} \frac{E(\sum_{i=\nu_n(t)}^{\nu_n(t+\delta)} X_i)^2}{\sigma_n^2} \le C\delta.$$

Finally, we prove that $W_n', n \ge 1$ is tight to finish the proof. According to [13, Theorem 7.3], we have to prove only that

$$I_1 = \lim_{\delta\to 0} \limsup_{n\to\infty} P\Big(\max_{|s-t|\le\delta} |W_n'(s) - W_n'(t)| > \epsilon\Big) \to 0.$$

In order to prove it, we shall use Theorem 2.1 with $p = 4$ and the property of η_{ni}. Then

$$\begin{aligned}
I_1 &= \lim_{\delta\to 0} \limsup_{n\to\infty} P\Big(\bigcup_{i=0}^{[\frac{1}{\delta}]} \max_{i\delta\le s\le(i+1)\delta} |W_n'(s) - W_n'(i\delta)| > \epsilon\Big) \\
&\le \lim_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[\frac{1}{\delta}]} P\Big(\max_{i\delta\le s\le(i+1)\delta} |W_n'(s) - W_n'(i\delta)| > \epsilon\Big) \\
&\le C \lim_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[\frac{1}{\delta}]} E\Big(\max_{k\le\nu_n((i+1)\delta)} \sum_{j=\nu_n(i\delta)}^{k} \eta_{nj}\Big)^4 \\
&\le C \lim_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[\frac{1}{\delta}]} \Big[\sum_{j=\nu_n(i\delta)}^{\nu_n((i+1)\delta)} E\eta_{nj}^4 + \Big(\sum_{j=\nu_n(i\delta)}^{\nu_n((i+1)\delta)} E\eta_{nj}^2\Big)^2\Big] \\
&\le C \lim_{\delta\to 0} \limsup_{n\to\infty} \sum_{i=0}^{[\frac{1}{\delta}]} \Big[\epsilon_n \sum_{j=\nu_n(i\delta)}^{\nu_n((i+1)\delta)} E\eta_{nj}^2 + \Big(\sum_{j=\nu_n(i\delta)}^{\nu_n((i+1)\delta)} E\eta_{nj}^2\Big)^2\Big]
\end{aligned}$$

$$\leq C\lim_{\delta\to 0}\limsup_{n\to\infty}\sum_{i=0}^{[\frac{1}{\delta}]}(\epsilon_n\delta+\delta^2)\leq C\lim_{\delta\to 0}\limsup_{n\to\infty}(\epsilon_n+\delta)=0.$$

Proof of Corollary 2.2 For the sequence is weakly stationary, then

$$\frac{\sigma_n^2}{n}=\frac{1}{n}\Big(\sum_{i=1}^{n}EX_i^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}EX_iX_j\Big)=EX_1^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{n}EX_iX_j$$

$$=EX_1^2+2\sum_{i=2}^{n}\frac{n-i}{n}EX_1X_i=EX_1^2+2\sum_{i=2}^{n}EX_1X_i-2\sum_{i=2}^{n}\frac{i}{n}EX_1X_i.$$

Since $\sum_{k=2}^{\infty}|\mathrm{Cov}(X_1,X_k)|<\infty$, then we have $\sum_{k=2}^{\infty}\mathrm{Cov}(X_1,X_k)<\infty$. By the Kronecker lemma, we get $\sum_{i=2}^{n}\frac{i}{n}EX_1X_i\to 0$, as $n\to\infty$. So $\sigma_n^2/n\to\sigma^2$.

Now, in order to prove this corollary, it's enough to verify that the three conditions of Theorem 2.2 are satisfied.

It's obvious that $\sum_{k=2}^{\infty}\mathrm{Cov}(X_1,X_k)^-<\infty$, then

$$\frac{1}{\sigma_n^2}\sum_{|i-j|\geq r,i,j\leq n}\mathrm{Cov}(X_i,X_j)^-\leq\frac{C}{n}\sum_{k=0}^{n-r}(n-r+1-k)\mathrm{Cov}(X_1,X_{k+r})^-\to 0, as\ \ n\to\infty.$$

For every $\epsilon>0$, by $EX_1^2<\infty$, it's easy to get Condition (ii).

For the sequence is stationary, so it's enough to show $\limsup_{n\to\infty}\frac{\sum_{i=1}^{n}\mathrm{Var}(X_i)}{\mathrm{Var}(\sum_{i=1}^{n}X_i)}<\infty$, which is obvious by $\lim_{n\to\infty}\frac{\sigma_n^2}{n}=\sigma^2$.

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