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Hölder Continuity of Weak Solutions for Parabolic Equations with Nonstandard Growth Conditions

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Abstract In this paper, we investigate the interior regularity including the local boundedness and the interior Hölder continuity of weak solutions for parabolic equations of the $p(x,t)$-Laplacian type. We improve the Moser iteration technique and generalize the known results for the elliptic problem to the corresponding parabolic problem.

Keywords Boundedness, Hölder continuity, Parabolic equations, Nonstandard growth conditions
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1 Introduction

An interest in the equations or the functionals with a variable exponent $p(x)$ arises not only in analyzing variational problems with integrand of the form $|Du|^{p(x)}$ in [1–3] but also in investigating various problems of mathematical physics (an electro-rheological fluid problem, a thermistor problem, a nonlinear Stokes system, etc.) in [4–6].

Many authors have already studied these problems (See [1], [2], [4], [7], etc.) such as

$$\min\left\{\int_\Omega |Du|^{p(x)}\,dx\right\}, \quad \text{or} \quad -\operatorname{div}(|Du|^{p(x)-2}Du)=0.$$

In [1, 2, 7, 8, 9] they get an interior *a priori* estimate for the Hölder norm of solutions to elliptic equations with nonstandard growth conditions.

In this paper, we will study the corresponding problems for parabolic equations. However, our problems are much harder and more complicated.

In the following, $\Omega\subset\mathbb{R}^N$ is an open domain, $N\ge 2$, $T>0$, $Q\equiv\Omega\times[0,T)$, $S\equiv\partial\Omega\times[0,T)$, $\Gamma\equiv S\cup(\Omega\times\{0\})$, and γ will denote a positive constant whose value is not necessarily the same on each occurrence.

We consider the following equation in Q:

$$u_t-\operatorname{div}(|Du|^{p(x,t)-2}Du)=0, \tag{1.1}$$

where $p(x,t)$ is a measurable function in Q satisfying

$$1<p_1\le p(x,t)\le p_2<\infty \tag{1.2}$$

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and

$$|p(x,t)-p(y,s)| \le \frac{C_1}{\log(|x-y|+C_2|t-s|^{p_2})^{-1}}, \tag{1.3}$$

for any $(x,t),(y,s)\in Q$ such that $|x-y|<\frac{1}{2}$ and $|t-s|<\frac{1}{2}$, where p_1, p_2, C_1, C_2 are positive constant numbers.

In §2, we give our main results and some auxiliary lemmas. In §3, we consider the local boundedness of local weak solutions by Moser iteration methods. Finally, in §4 we investigate the local Hölder continuity for local weak solutions by using a similar technique to that in Chapter 3 in [10].

2 Main Results and Lemmas

We denote $v_+ \equiv \max\{v;0\}$; $v_- \equiv \max\{-v;0\}$. The spaces

$$W^{1,p(x)}(\Omega) \equiv \left\{ f\in W^{1,1}(\Omega): \int_\Omega |Df|^{p(x)}dx<\infty \right\}$$

with norm $||u||_{1,p(x)} = \inf\left\{\lambda>0: \int_\Omega \left|\frac{Du(x)}{\lambda}\right|^{p(x)}dx\le 1\right\} + \inf\left\{\lambda>0: \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}$, and

$$W^{1,p(\cdot)}(Q) \equiv \left\{ f\in L^1(0,T;W^{1,1}(\Omega)): \iint_Q |D_xf|^{p(x,t)}dxdt<\infty \right\},$$

with norm $||u||_{1,p(\cdot)} = \inf\left\{\lambda>0: \iint_Q \left|\frac{D_xu}{\lambda}\right|^{p(x,t)}dxdt\le 1\right\} + \inf\left\{\lambda>0: \iint_Q \left|\frac{u}{\lambda}\right|^{p(x,t)}dxdt \le 1\right\}$, are two Banach spaces (See [8]).

Definition 2.1 *A function u is a local weak sub(super)-solution of* (1.1) *in Q if $u\in W^{1,p(\cdot)}_{\rm loc}(Q)\cap C([0,T];L^2_{\rm loc}(\Omega))$, and for every subinterval $[t_1,t_2]$ of $[0,T)$,*

$$\int_\Omega u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \{-u\varphi_t + |Du|^{p(x,\tau)-2}Du\cdot D\varphi\}dxd\tau \le (\ge) 0, \tag{2.1}$$

for any nonnegative function $\varphi\in C^1([t_1,t_2];C_0^\infty(\Omega))$. A function u is a local weak solution if it is both a local weak subsolution and a local weak supersolution of (1.1).

In this paper, our main results are:

Theorem 2.2 *Let $p_1>\max\{1,\frac{2N}{N+2}\}$ hold. Every local weak solution u of* (1.1) *in Q is locally bounded in Q.*

Theorem 2.3 *Let u be a bounded local weak solution of* (1.1) *in Q and $p_1>2$. Then the function $(x,t)\to u(x,t)$ is locally Hölder continuous in Q.*

Let $v\in L^1(Q)$, and for $0<h<T$, we introduce the Steklov averages $v_h(\cdot,t)$ defined for all $0<t<T$ by

$$v_h(\cdot,t) \equiv \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} v(\cdot,\tau)d\tau, & t\in(0,T-h]; \\ \dfrac{1}{h}\displaystyle\int_{T-h}^{T} v(\cdot,\tau)d\tau, & t\ge T-h. \end{cases}$$

Fix $t\in(0,T)$ and let h be a small positive number such that $0<t<t+h<T$. In (2.1) take $t_1=t, t_2=t+h$ and choose a testing function, φ, independent of the variable $\tau\in(t,t+h)$. Dividing by h and recalling the definition of Steklov averages, we obtain

$$\iint_{\Omega\times\{t\}} \left\{\frac{\partial}{\partial t}u_h\cdot\varphi + [|Du|^{p(x,\tau)-2}Du]_h\cdot D\varphi\right\}dxd\tau \le (\ge) 0, \tag{2.2}$$

for any $0<t<T-h$ and any nonnegative function $\varphi\in C_0^\infty(\Omega)$.

In order to prove our main results, we need the following auxiliary lemmas:

Lemma 2.4 ([11, 10]) *Let $\{Y_n\}, n = 0, 1, 2, \ldots,$ be a sequence of positive numbers, satisfying the recursive inequalities $Y_{n+1} \le Cb^n Y_n^{1+\alpha}$, where $C, b > 1$ and $\alpha > 0$ are given numbers. If $Y_0 \le C^{-1/\alpha} b^{-1/\alpha^2}$, then $\{Y_n\}$ converges to zero as $n \to \infty$.*

Lemma 2.5 ([11, 10]) *There exists a constant γ depending only on N, m, r such that for every $v \in V_0^{m,r}(Q) \equiv L^\infty(0, T; L^m(\Omega)) \cap L^r(0, T; W_0^{1,r}(\Omega))$,*

$$\iint_Q |v(x,t)|^q dxdt \le \gamma^q \Big(\iint_Q |Dv(x,t)|^r dxdt \Big) \Big(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v(x,t)|^m dx \Big)^{\frac{r}{N}},$$

where $q = r\frac{N+m}{N}$. Moveover, $||v||_{q,Q} \le \gamma ||v||_{V^{m,r}(Q)}$.

Lemma 2.6 ([10]) *Let $r > 1$. There exists a constant γ depending only on N, r such that for every $v \in V_0^{m,r}(Q)$,*

$$||v||_{r,Q}^r \le \gamma ||v| > 0|^{\frac{r}{r+N}} ||v||_{V^{r,r}(Q)}^r.$$

Lemma 2.7 ([11, 10]) *Let $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$ for some $\rho > 0$ and some $x_0 \in \mathbb{R}^N$, and let k and l be any pair of real numbers such that $k < l$. Then, there exists a constant γ depending only on N and independent of k, l, v, x_0, ρ such that*

$$(l-k)|A(l)| \le \gamma \frac{\rho^{N+1}}{|B_\rho(x_0) \setminus A(k)|} \int_{A(k)\setminus A(l)} |Dv| dx,$$

where $A(k) \equiv \{x \in B_\rho(x_0) | v(x) > k\}$, $|A(k)| \equiv \mathrm{meas}(A(k))$.

In [6], if $p(x)$ satisfies (1.2) and $|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}$ for $|x - y| < \frac{1}{2}$, then, for any $\varphi \in W_{\mathrm{loc}}^{1,p(x)}(\Omega)$, there exists $\varphi_n \in C^\infty(\Omega)$ such that $\lim_{n\to\infty} ||\varphi_n - \varphi||_{W^{1,1}(\Omega')} = 0$ and

$$\lim_{n\to\infty} \int_{\Omega'} |D\varphi_n|^{p(x)} dx = \int_{\Omega'} |D\varphi|^{p(x)} dx, \quad \forall\, \Omega' \subset\subset \Omega.$$

This result is true for $W_{\mathrm{loc}}^{1,p(\cdot)}(Q)$ under the condition that (1.2) and (1.3) hold.

Similarly to [8], we have:

Definition 2.8 *A measurable function $p : Q \to \mathbb{R}$ satisfies condition* (H) *in Q if there exists a constant $L > 0$ such that $R^{-\mathrm{osc}\{p,Q_R\}} \le L$ for any cylinder $\overline{Q}_R \subset Q$, where $Q_R \equiv B_R \times (0, R^{p_2})$.*

Clearly, if $p(x,t)$ satisfies Condition (1.3), then $p(x,t)$ satisfies condition (H).

3 Local Boundedness of Weak Solutions

In this section, we will prove Theorem 2.2. Thus, we will firstly show the following result:

Proposition 3.1 *Let $p_1 > \max\{1, \frac{2N}{N+2}\}$ hold. Every nonnegative local weak subsolution u of* (1.1) *in Q is locally bounded in Q. Moreover, there exists a constant $\gamma = \gamma(N, p^+, p^-, \rho)$ such that for any $[(x_0,t_0) + Q(\rho^{p^+}, \rho)] \subset Q$ and any $\sigma \in (0,1)$,*

$$\sup_{[(x_0,t_0)+Q(\sigma\rho^{p^+},\sigma\rho)]} u \le \max\Big\{1,\ \gamma(1-\sigma)^{-\frac{N+p^-}{N}\frac{p^+}{q-\delta}} \times \Big\{ \frac{1}{|[(x_0,t_0)+Q(\rho^{p^+},\rho)]|} \iint_{[(x_0,t_0)+Q(\rho^{p^+},\rho)]} u^\delta dxd\tau \Big\}^{\frac{p^-}{N}\frac{1}{q-\delta}} \Big\}, \tag{3.1}$$

where $Q_\rho \equiv B_\rho \times (0, \rho^{p_2})$, $p^+ = \sup_{Q_\rho} p(x,t)$, $p^- = \inf_{Q_\rho} p(x,t)$ and $\max\{p^+, 2\} \le \delta < q = \frac{N+2}{N} p^-$.

Note: The assumption that u is nonnegative is not essential and is used here only to deduce that u is locally bounded above.

Suppose that u is a local weak solution of (1.1) in Q. Then, there exist $R_0 > 0$ and $\delta(N, p_1, p_2, R_0) > 0$ such that

$$\int_{t_1}^{t_2} \int_{B_R} |u|^\delta dxd\tau < +\infty,$$

for any sufficiently small R satisfying $0 < R \le R_0$, any $0 < \delta < \delta(N, p_1, p_2, R_0)$, $0 < t_1 < t_2 < T$ such that $t_2 - t_1 \le CR^{p_2}$.

Indeed, let $p^- = \min_{Q_R} p(x,t)$, $p^+ = \max_{Q_R} p(x,t)$ and $Q_R \equiv B_R \times (0, R^{p_2})$. Obviously, by Young's inequality

$$\iint_{Q_{2R}} |Du(x,t)|^{p^-} dxdt \le |Q_{2R}| + \iint_{Q_{2R}} |Du(x,t)|^{p(x,t)} dxdt,$$

we have $u \in V^{2,p^-}(Q_{2R}) \equiv L^\infty(0,T; L^2(B_{2R})) \cap L^{p^-}(0,T; W^{1,p^-}(B_{2R}))$. Moreover, $u\psi \in V_0^{2,p^-}(Q_{2R})$ for the cutoff function $\psi \in C_0^\infty(Q_{2R})$. This result, together with Lemma 2.5, implies the integrability of $u(x,t)$ in Q_R with the exponent δ ($\delta = \frac{N+2}{N} p^-$). In order to satisfy $\delta > p^+$, we choose only a sufficiently small R to fix R_0. Thus, the desired result is obtained.

Proof of Proposition 3.1 Fix a point (x_0, t_0) of Q. Let $Q(\theta, \rho) \equiv B_\rho \times (-\theta, 0)$, and let ρ and θ be so small that $[(x_0, t_0) + Q(\theta, \rho)] \subset Q$. Let $k > 0$ and u be a nonnegative weak subsolution of (1.1) in Q.

We may assume $(x_0, t_0) \equiv (0, 0)$ (Otherwise, modulo a translation). Let $Q_\rho \equiv B_\rho \times (0, \rho^{p_2})$, $p^+ = \sup_{Q_\rho} p(x,t)$ and $p^- = \inf_{Q_\rho} p(x,t)$. Fix $\sigma \in (0,1)$. Consider the sequences

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho, \quad \theta_n = \sigma\theta + \frac{1-\sigma}{2^n}\theta$$

and the corresponding cylinders $Q_n \equiv Q(\theta_n, \rho_n)$, and also the sequence of increasing levels $k_n = k - \frac{k}{2^n}$. We also introduce the sequences $\tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2}$, $\tilde{\theta}_n = \frac{\theta_n + \theta_{n+1}}{2}$ and $\tilde{Q}_n \equiv Q(\tilde{\theta}_n, \tilde{\rho}_n)$. It follows from the definitions that $Q_0 = Q(\theta, \rho), Q_\infty = Q(\sigma\theta, \sigma\rho), k_0 = 0, k_\infty = k$ and $Q_{n+1} \subset \tilde{Q}_n \subset Q_n$ for $n = 0, 1, 2, \ldots$.

Then in (2.2) we take $\varphi = (u_h - k_{n+1})_+ \eta^{p^+}$ (over Q_n) as a testing function, and integrate over $(-\theta_n, t)$, where $t \in (-\theta_n, 0)$ and the cutoff function $\eta \in [0,1]$ is to be determined later. First, integrating by parts and letting $h \to 0$, we have

$$\begin{aligned}
\int_{-\theta_n}^t \int_{B_{\rho_n}} \frac{\partial}{\partial\tau} u_h \varphi dxd\tau &= \frac{1}{2} \int_{-\theta_n}^t \int_{B_{\rho_n}} \frac{\partial}{\partial\tau} (u_h - k_{n+1})_+^2 \eta^{p^+} dxd\tau \\
&\to \frac{1}{2} \int_{B_{\rho_n}} (u - k_{n+1})_+^2 \eta^{p^+}(x,t) dx \\
&\quad - \frac{1}{2} \int_{B_{\rho_n}} (u - k_{n+1})_+^2 \eta^{p^+}(x, -\theta_n) dx \\
&\quad - \frac{p^+}{2} \int_{-\theta_n}^t \int_{B_{\rho_n}} (u - k_{n+1})_+^2 \eta^{p^+ - 1} |\eta_t| dxd\tau. \qquad (3.2)
\end{aligned}$$

Then, in estimating the remaining parts, we obtain

$$\begin{aligned}
&\int_{-\theta_n}^t \int_{B_{\rho_n}} [|Du|^{p(x,\tau)-2} Du]_h \cdot D[(u_h - k_{n+1})_+ \eta^{p^+}] dxd\tau \\
&\to \int_{-\theta_n}^t \int_{B_{\rho_n}} [|Du|^{p(x,\tau)-2} Du] \cdot [D(u - k_{n+1})_+ \eta^{p^+} + p^+ (u - k_{n+1})_+ \eta^{p^+ - 1} D\eta] dxd\tau \\
&\ge \int_{-\theta_n}^t \int_{B_{\rho_n}} |D(u - k_{n+1})_+|^{p(x,\tau)} \eta^{p^+} dxd\tau \\
&\quad - p^+ \int_{-\theta_n}^t \int_{B_{\rho_n}} |D(u - k_{n+1})_+|^{p(x,\tau)-1} (u - k_{n+1})_+ \eta^{p^+ - 1} |D\eta| dxd\tau. \qquad (3.3)
\end{aligned}$$

By Young's inequality, we have

$$p^+ \int_{-\theta_n}^t \int_{B_{\rho_n}} |D(u - k_{n+1})_+|^{p(x,\tau)-1} (u - k_{n+1})_+ \eta^{p^+ - 1} |D\eta| dxd\tau$$

$$\leq \epsilon \int_{-\theta_n}^{t}\int_{B_{\rho_n}} |D(u-k_{n+1})_+|^{p(x,\tau)}\eta^{p^+}dxd\tau$$
$$+\gamma(p^+,\epsilon)\int_{-\theta_n}^{t}\int_{B_{\rho_n}} (u-k_{n+1})_+^{p(x,\tau)}|D\eta|^{p(x,\tau)}dxd\tau$$
$$\leq \epsilon \int_{-\theta_n}^{t}\int_{B_{\rho_n}} |D(u-k_{n+1})_+|^{p(x,\tau)}\eta^{p^+}dxd\tau$$
$$+\gamma(p^+,\epsilon)\int_{-\theta_n}^{t}\int_{B_{\rho_n}} (u-k_{n+1})_+^{p^+}|D\eta|^{p^+}dxd\tau$$
$$+\gamma(p^+,\epsilon)\int_{-\theta_n}^{t}\int_{B_{\rho_n}} \chi[(u-k_{n+1})_+>0]dxd\tau.$$

The nonnegative piecewise smooth cutoff function η is taken to satisfy

$$\eta\equiv 0 \quad \text{on} \quad \Gamma(Q_n), \quad \eta\equiv 1 \quad \text{in} \quad \tilde{Q}_n, \quad |D\eta|\leq \frac{2^{n+2}}{(1-\sigma)\rho}, \quad \text{and} \quad 0\leq \eta_t\leq \frac{2^{n+2}}{(1-\sigma)\theta}.$$

Let $|A_{n+1}|\equiv \text{meas}\{(x,t)\in Q_n | u(x,t)>k_{n+1}\}$. Then, by estimating (3.2) and (3.3), we have

$$\sup_{-\theta_n<t<0}\int_{B_{\rho_n}} (u-k_{n+1})_+^2\eta^{p^+}(x,t)dx+\int_{-\theta_n}^{t}\int_{B_{\rho_n}} |D(u-k_{n+1})_+|^{p(x,\tau)}\eta^{p^+}dxd\tau$$
$$\leq \gamma\int_{-\theta_n}^{t}\int_{B_{\rho_n}} (u-k_{n+1})_+^2\eta^{p^+-1}|\eta_t|dxd\tau$$
$$+\gamma\int_{-\theta_n}^{t}\int_{B_{\rho_n}} (u-k_{n+1})_+^{p^+}|D\eta|^{p^+}dxd\tau+\gamma|A_{n+1}|$$
$$\leq \frac{\gamma 2^{n+2}}{(1-\sigma)\theta}\iint_{Q_n}(u-k_{n+1})_+^2dxd\tau$$
$$+\gamma\left[\frac{2^{n+2}}{(1-\sigma)\rho}\right]^{p^+}\iint_{Q_n}(u-k_{n+1})_+^{p_+}dxd\tau+\gamma|A_{n+1}|. \tag{3.4}$$

Moreover, we observe that for all $s>0$,

$$\begin{aligned}\iint_{Q_n}(u-k_n)_+^s dxd\tau &\geq \iint_{Q_n}(u-k_n)_+^s\chi[u>k_{n+1}]dxd\tau\\ &\geq (k_{n+1}-k_n)^s|A_{n+1}|=\frac{k^s}{2^{(n+1)s}}|A_{n+1}|.\end{aligned}$$

Then, we get

$$\begin{aligned}\iint_{Q_n}(u-k_{n+1})_+^{p^+}dxd\tau &\leq \left(\iint_{Q_n}(u-k_{n+1})_+^{\delta}dxd\tau\right)^{\frac{p^+}{\delta}}|A_{n+1}|^{1-\frac{p^+}{\delta}}\\ &\leq \gamma\frac{2^{(\delta-p^+)n}}{k^{\delta-p^+}}\iint_{Q_n}(u-k_n)_+^{\delta}dxd\tau,\end{aligned}$$

and

$$\begin{aligned}\iint_{Q_n}(u-k_{n+1})_+^{2}dxd\tau &\leq \left(\iint_{Q_n}(u-k_{n+1})_+^{\delta}dxd\tau\right)^{\frac{2}{\delta}}|A_{n+1}|^{1-\frac{2}{\delta}}\\ &\leq \gamma\frac{2^{(\delta-2)n}}{k^{\delta-2}}\iint_{Q_n}(u-k_n)_+^{\delta}dxd\tau,\end{aligned}$$

where $\delta>p^+$ is a positive number such that $|u(x,t)|\in L^{\delta}_{\text{loc}}(Q)$.

Then (3.4) yields

$$\sup_{-\theta_n<t<0}\int_{B_{\tilde\rho_n}}(u-k_{n+1})_+^2(x,t)dx+\iint_{\tilde Q_n}|D(u-k_{n+1})_+|^{p(x,\tau)}dxd\tau$$
$$\le\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)\iint_{Q_n}(u-k_n)_+^\delta dxd\tau$$
$$+\gamma\frac{2^{(n+1)\delta}}{k^\delta}\iint_{Q_n}(u-k_n)_+^\delta dxd\tau$$
$$\le\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)\iint_{Q_n}(u-k_n)_+^\delta dxd\tau.$$

By Young's inequality,

$$\iint_{\tilde Q_n}|D(u-k_{n+1})_+|^{p^-}dxd\tau\le\iint_{\tilde Q_n}|D(u-k_{n+1})_+|^{p(x,\tau)}dxd\tau+|A_{n+1}|.$$

Therefore, we obtain

$$\sup_{-\theta_n<t<0}\int_{B_{\tilde\rho_n}}(u-k_{n+1})_+^2(x,t)dx+\iint_{\tilde Q_n}|D(u-k_{n+1})_+|^{p^-}dxd\tau$$
$$\le\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)\iint_{Q_n}(u-k_n)_+^\delta dxd\tau. \tag{3.5}$$

Then, the next step is to construct a nonnegative piecewise smooth cutoff function $\tilde\eta_n$ in $\tilde Q_n$ satisfying

$$\tilde\eta_n=0\quad\text{on}\quad S(\tilde Q_n),\quad \tilde\eta_n\equiv1\quad\text{on}\quad Q_{n+1},\quad |D\tilde\eta_n|\le\frac{2^{n+2}}{(1-\sigma)\rho}.$$

For the function $(u-k_{n+1})\tilde\eta_n$, by (1.3), (3.5) and Lemma 2.5, we have

$$\iint_{\tilde Q_n}(u-k_{n+1})_+^q\tilde\eta_n^q\,dxd\tau\le\gamma^q\Big(\sup_{-\theta_n<t<0}\int_{B_{\tilde\rho_n}}(u-k_{n+1})_+^2dx\Big)^{\frac{p^-}{N}}$$
$$\times\Big(\iint_{\tilde Q_n}|D(u-k_{n+1})_+|^{p^-}dxd\tau+\iint_{\tilde Q_n}|(u-k_{n+1})_+|^{p^-}|D\tilde\eta_n|^{p^-}dxd\tau\Big)$$
$$\le\gamma\Big[\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)\Big]^{\frac{p^-}{N}}\cdot\Big[\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)$$
$$+\frac{\gamma 2^{(n+2)p^-}}{(1-\sigma)^{p^-}\rho^{p^-}}\frac{2^{(\delta-p^-)n}}{k^{\delta-p^-}}\Big]\Big(\iint_{Q_n}(u-k_n)_+^\delta dxd\tau\Big)^{1+\frac{p^-}{N}}$$
$$\le\Big[\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\Big(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\Big)\Big]^{1+\frac{p^-}{N}}\Big(\iint_{Q_n}(u-k_n)_+^\delta dxd\tau\Big)^{1+\frac{p^-}{N}}, \tag{3.6}$$

where $\max\{p^+,2\}\le\delta\le q=\frac{N+2}{N}p^-$.

Introduce the dimensionless quantities

$$Y_n=\frac{1}{|Q_n|}\iint_{Q_n}(u-k_n)_+^\delta dxd\tau,\quad n=0,1,\dots,$$

and let $\theta=\rho^{p^+}$; we will derive an iterative inequality for Y_n, that is,

$$Y_{n+1}\le\gamma\frac{1}{|\tilde Q_n|}\iint_{\tilde Q_n}(u-k_{n+1})_+^\delta\tilde\eta_n^\delta dxd\tau$$
$$\le\gamma\Big(\frac{1}{|\tilde Q_n|}\iint_{\tilde Q_n}(u-k_{n+1})_+^q\tilde\eta_n^q dxd\tau\Big)^{\frac{\delta}{q}}\Big(\frac{|A_{n+1}|}{|Q_n|}\Big)^{1-\frac{\delta}{q}}$$

$$\leq \gamma\left[\frac{\gamma 2^{n\delta}}{(1-\sigma)^{p^+}}\left(\frac{1}{\rho^{p^+}k^{\delta-p^+}}+\frac{1}{\theta k^{\delta-2}}\right)\right]^{\frac{(N+p^-)\delta}{Nq}}\left(\frac{\gamma 2^{n\delta}}{k^{\delta}}\right)^{1-\frac{\delta}{q}}|Q_n|^{\frac{\delta}{q}\frac{p^-}{N}}Y_n^{1+\frac{p^-\delta}{Nq}}$$

$$\leq \frac{\gamma b^n}{k^{\frac{\delta}{q}(q-\delta)}(1-\sigma)^{\frac{\delta}{q}\frac{N+p^-}{N}p^+}}Y_n^{1+\frac{p^-\delta}{Nq}}, \tag{3.7}$$

where $b = 2^{\delta(1+\frac{\delta}{q}\frac{p^-}{N})} > 1$.

It follows from Lemma 2.4 that $Y_n \to 0$ as $n \to \infty$, provided k is chosen to satisfy $k = \max\{\bar{k}, 1\}$, where

$$Y_0 = \frac{1}{|Q(\rho^{p^+},\rho)|}\iint_{Q(\rho^{p^+},\rho)} u^{\delta}dxd\tau = \gamma\bar{k}^{(q-\delta)\frac{N}{p^-}}(1-\sigma)^{\frac{N+p^-}{p^-}p^+}.$$

This in turn implies

$$\sup_{Q(\sigma\rho^{p^+},\sigma\rho)} u \leq \max\left\{1, \gamma(1-\sigma)^{-\frac{p^+(N+p^-)}{N(q-\delta)}}\left(\frac{1}{|Q(\rho^{p^+},\rho)|}\iint_{Q(\rho^{p^+},\rho)} u^{\delta}dxd\tau\right)^{\frac{p^-}{N}\frac{1}{q-\delta}}\right\}.$$

Thus, we end the proof of Proposition 3.1.

Proof of Theorem 2.2 Similarly to Proposition 3.1, we have the corresponding results for the local weak supersolution of (1.1) in Q. Thus, combining with Proposition 3.1, every local weak solution of (1.1) in Q is locally bounded in Q under the same conditions. Therefore, we end the proof.

Remark 3.2 Indeed, we may obtain similar results under weaker assumptions, that is, when the conditions (1.2) and (1.3) are being replaced by the (p,q)-growth conditions.

4 Hölder Continuity of Weak Solutions

Without loss of generality, (by Definition 2.1 and Theorem 2.2) we may assume that the local weak solution $u \in L^\infty(Q)$ of (1.1) in Q to study its interior Hölder continuity. In the following, we begin to prove our second main result (Theorem 2.3).

Fix a point (x_0, t_0) in Q and let ρ be small such that $Q_\rho = B_\rho(x_0) \times (t_0 - \rho^{p^+}, t_0) \subset Q$, where $Q_\rho(x_0,t_0) \equiv B_\rho(x_0) \times (t_0 - \rho^{p_2}, t_0)$, $p^+ = \sup_{Q_\rho(x_0,t_0)} p(x,t)$ and $p^- = \inf_{Q_\rho(x_0,t_0)} p(x,t)$.

Let $\mu^+ = \operatorname{ess\,sup}_{Q_\rho} u$, $\mu^- = \operatorname{ess\,inf}_{Q_\rho} u$, and $\omega = \operatorname{ess\,osc}_{Q_\rho} u = \mu^+ - \mu^-$. We may assume that $(x_0,t_0) = (0,0)$ (Otherwise, modulo a translation).

Fix ω and $\frac{1}{d} = (\frac{\omega}{2})^{p^+-2}$. Then we construct the cylinder $Q_0 = Q(d\rho^{p^+}, \rho)$ such that $Q_0 \subset Q(a\rho^{p^+},\rho)$, where $\frac{1}{a} = (\frac{\omega}{A})^{p^+-2}$.

Lemma 4.1 *There exists a number $\sigma \in (0,1)$ independent of ω, ρ such that if*

$$\left|Q_0 \cap \left[u < \mu^- + \frac{\omega}{2}\right]\right| \leq \sigma|Q_0| \tag{4.1}$$

and

$$\left(\frac{\omega}{A}\right)^{p^+-2} > \rho^{\epsilon}, \tag{4.2}$$

where $A > 2$, $\epsilon \in (0,1)$ are two numbers to be determined later, then

$$u(x,t) > \mu^- + \frac{\omega}{4} \quad a.e. \quad (x,t) \in Q_1,$$

where $Q_n = Q(d(\frac{\rho}{2^n})^{p^+}, \frac{\rho}{2^n})$.

Proof Let $\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}$, $k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}$ and $Q_{\rho_n} = Q(d(\rho_n)^{p^+}, \rho_n)$.

Similarly to the proof of Proposition 3.1, we choose a piecewise smooth cutoff function η_n

in Q_{ρ_n} satisfying

$$\eta_n \equiv 1 \quad \text{in} \quad Q_{\rho_{n+1}}, \quad \eta_n = 0 \quad \text{on} \quad \Gamma(Q_{\rho_n}), \quad |D\eta_n| \le \frac{2^{n+1}}{\rho}, \quad 0 \le \frac{\partial}{\partial t}\eta_n \le \frac{1}{d}\left(\frac{2^{n+1}}{\rho}\right)^{p^+},$$

and take $(u-k_n)_-\eta_n^{p^+}$ as a testing function. Then, we have

$$\begin{aligned}
&\sup_{-d\rho_n^{p^+}<t<0} \int_{B_{\rho_n}} (u-k_n)_-^2 \eta_n^{p^+}(x,t)dx + \iint_{Q_{\rho_n}} |D(u-k_n)_-|^{p(x,\tau)}\eta_n^{p^+} dxd\tau \\
&\le \frac{1}{d}\left(\frac{\gamma 2^{n+1}}{\rho}\right)^{p^+} \iint_{Q_{\rho_n}} (u-k_n)_-^2 dxd\tau + \gamma\left[\frac{2^{n+1}}{\rho}\right]^{p^+} \iint_{Q_{\rho_n}} (u-k_n)_-^{p_+} dxd\tau \\
&\quad + \gamma \iint_{Q_{\rho_n}} \chi[(u-k_n)_- > 0]dxd\tau \\
&\le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2}\right)^{p^+} \iint_{Q_{\rho_n}} \chi[(u-k_n)_- > 0]dxd\tau.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
&\sup_{-d\rho_n^{p^+}<t<0} \int_{B_{\rho_n}} (u-k_n)_-^{p^-} \eta_n^{p^+}(x,t)dx + \frac{1}{d}\iint_{Q_{\rho_n}} |D(u-k_n)_-|^{p^-}\eta_n^{p^+} dxd\tau \\
&\le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2}\right)^{p^+} \frac{1}{d}\iint_{Q_{\rho_n}} \chi[(u-k_n)_- > 0]dxd\tau. \qquad (4.3)
\end{aligned}$$

We introduce a change of the time-variable $z = t/d$ which transforms Q_{ρ_n} into $\tilde{Q}_{\rho_n} \equiv B_{\rho_n} \times (-\rho_n^{p^+}, 0)$, set $v(\cdot, z) = u(\cdot, zd)$, $\tilde{\eta}_n(\cdot, z) = \eta_n(\cdot, zd)$ and $|A_n| = \text{meas}\{(x,z) \in \tilde{Q}_{\rho_n} | \, v(x,z) < k_n\}$, and then get

$$||(v-k_n)_-\tilde{\eta}_n^{p^+}||_{V^{p^-}(\tilde{Q}_{\rho_n})}^{p^-} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2}\right)^{p^+} |A_n|.$$

By Lemma 2.6 and the above inequality, we have

$$\begin{aligned}
\frac{1}{2^{p^-(n+2)}}\left(\frac{\omega}{2}\right)^{p^-}|A_{n+1}| &\le |k_n - k_{n+1}|^{p^-}|A_{n+1}| \le ||(v-k_n)_-||_{p^-,\tilde{Q}_{\rho_{n+1}}}^{p^-} \\
&\le ||(v-k_n)_-\tilde{\eta}_n^{p^+}||_{p^-,\tilde{Q}_{\rho_n}}^{p^-} \le ||(v-k_n)_-\tilde{\eta}_n^{p^+}||_{p^-\frac{N+p^-}{p^-},\tilde{Q}_{\rho_n}}^{p^-} |A_n|^{\frac{p^-}{N+p^-}} \\
&\le ||(v-k_n)_-\tilde{\eta}_n^{p^+}||_{V^{p^-}(\tilde{Q}_{\rho_n})}^{p^-}|A_n|^{\frac{p^-}{N+p^-}} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}|A_n|^{1+\frac{p^-}{N+p^-}}. \qquad (4.4)
\end{aligned}$$

Since (4.2) holds and ϵ is so small, we have $\frac{\omega}{2} \ge \frac{\omega}{A} \ge \rho^{\frac{\epsilon}{p^+-2}} \ge \rho$ only if we choose $\epsilon \le p^+ - 2$. Then, using (1.3) and Definition 2.8, we get $(\frac{\omega}{2})^{p^+-p^-} \le \gamma$. We also introduce the quantity $Y_n = \frac{|A_n|}{|\tilde{Q}_{\rho_n}|}$, use (4.4) and the above result, and then get

$$Y_{n+1} \le \gamma 4^{np^+}\left(\frac{\omega}{2}\right)^{p^+-p^-} Y_n^{1+\frac{p^-}{N+p^-}} \le \gamma 4^{np^+} Y_n^{1+\frac{p^-}{N+p^-}}.$$

By Lemma 2.4, it follows that $Y_n \to 0$ as $n \to \infty$, provided

$$Y_0 \le \gamma^{-\frac{N+p^-}{p^-}} 4^{-p^+(\frac{N+p^-}{p^-})^2} \le \sigma.$$

That is,

$$\iint_{Q_{\rho_n}} \chi[(u-k_n)_- > 0]dxd\tau \to 0.$$

Therefore,

$$\left|\left[u<\mu^-+\frac{\omega}{4}\right]\cap Q_1\right|=0.$$

Thus, we have the desired result.

Next, we will exploit the fact that at the time level $\theta=(\frac{\omega}{2})^{2-p^+}(\frac{\varrho}{2})^{p+}$, the function $x\to u(x,-\theta)$ is strictly above the level $\mu^-+\frac{\omega}{4}$ in $B_{\frac{\varrho}{2}}$.

Lemma 4.2 *Let* (4.1) *and* (4.2) *hold. For every number* $\sigma_1\in(0,1)$*, there exists a positive integer* s *such that*

$$\left|x\in B_{\frac{\varrho}{4}}:u(x,t)<\mu^-+\frac{\omega}{2^s}\right|\le\sigma_1|B_{\frac{\varrho}{4}}|,\quad\forall\, t\in(-\theta,0).\tag{4.5}$$

Proof Let $Q(\theta,\frac{\varrho}{2})=B_{\frac{\varrho}{2}}\times(-\theta,0),k=\mu^-+\frac{\omega}{4},H_k^-=\operatorname{ess\,sup}_{Q(\theta,\frac{\varrho}{2})}(u-k)_-\le\frac{\omega}{4}$ and $\Psi(u)=\ln^+\left\{\frac{H_k^-}{H_k^--(u-k)_-+\frac{\omega}{2^{n+2}}}\right\}$. By virtue of Lemma 4.1, $\Psi(x,-\theta)=0$ for any $x\in B_{\frac{\varrho}{2}}$. As before, taking $\varphi=\frac{\partial}{\partial u_h}[\Psi^2(u_h)]\eta^{p^+}$ as a testing function, we have

$$\begin{aligned}\int_{B_{\frac{\varrho}{2}}}\Psi^2(x,t)\eta^{p^+}(x)dx&\le\gamma\iint_{Q(\theta,\frac{\varrho}{2})}\Psi(\Psi')^{2-p^+}|D\eta|^{p^+}dxd\tau+\gamma\iint_{Q(\theta,\frac{\varrho}{2})}\chi[(u-k)_->0]dxd\tau\\&\le\left[\frac{\gamma}{\rho^{p^+}}\cdot n\ln2\cdot\left(\frac{\omega}{2}\right)^{p^+-2}+\gamma\right]\cdot\theta|B_{\frac{\varrho}{2}}|\le\gamma nA^{p^+-2}|B_{\frac{\varrho}{2}}|,\end{aligned}\tag{4.6}$$

where we use $\Psi\le n\ln2$, $|\Psi'|\le\frac{2}{\omega}$, $|D\eta|\le\frac{4}{\rho}$ and $(\frac{\omega}{2})^{2-p^+}\le\frac{\theta}{(\frac{\varrho}{2})^{p^+}}\le(\frac{\omega}{A})^{2-p^+}2^{p^+}$.

Since $\Psi^2\ge(n-1)^2\ln^2 2$ on the set $\{x\in B_{\frac{\varrho}{4}}|u(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\}$, $t\in(-\theta,0)$, (4.6) gives that for all $t\in(-\theta,0)$,

$$\left|x\in B_{\frac{\varrho}{4}}:u(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\right|\le\gamma A^{p^+-2}\frac{n}{(n-1)^2}|B_{\frac{\varrho}{2}}|.\tag{4.7}$$

To prove the lemma we have only to choose n sufficiently large.

Lemma 4.3 *Let* (4.1) *and* (4.2) *hold. The numbers* $\sigma_1\in(0,1)$ *and* $s\gg1$ *can be chosen a priori such that*

$$u(x,t)>\mu^-+\frac{\omega}{2^{s+1}},\quad a.e.\quad(x,t)\in Q\left(\theta,\frac{\rho}{8}\right).\tag{4.8}$$

Proof Let $\rho_n=\frac{\varrho}{8}+\frac{\rho}{2^{n+3}}$ and $k_n=\mu^-+\frac{\omega}{2^{s+1}}+\frac{\omega}{2^{s+1+n}}$. Owing to Lemma 4.1, $(u-k_n)_-(x,-\theta)=0$ for any $x\in B_{\rho_n}$. As before, taking $(u-k_n)_-\eta_n^{p^+}$ as a testing function, we have

$$\begin{aligned}&\sup_{-\theta<t<0}\int_{B_{\rho_n}}(u-k_n)_-^2\eta_n^{p^+}dx+\frac{1}{\gamma}\iint_{Q(\theta,\rho_n)}|D(u-k_n)_-|^{p(x,\tau)}\eta_n^{p^+}dxd\tau\\&\le\frac{\gamma2^{np^+}}{\rho^{p^+}}\iint_{Q(\theta,\rho_n)}|(u-k_n)_-|^{p^+}dxd\tau+\gamma\iint_{Q(\theta,\rho_n)}\chi[(u-k_n)_->0]dxd\tau\\&\le\frac{\gamma2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\iint_{Q(\theta,\rho_n)}\chi[(u-k_n)_->0]dxd\tau.\end{aligned}\tag{4.9}$$

For all $t\in(-\theta,0)$, by Definition 2.8 and the definition of θ,

$$\int_{B_{\rho_n}}(u-k_n)_-^2\eta_n^{p^+}dx\ge\left(\frac{\omega}{2^s}\right)^{2-p^-}\int_{B_{\rho_n}}(u-k_n)_-^{p^-}\eta_n^{p^+}dx\ge\frac{\theta}{\rho^{p^+}}\int_{B_{\rho_n}}(u-k_n)_-^{p^-}\eta_n^{p^+}dx$$

if s is appropriately closen so large as to satisfy the result of Lemma 4.2.

We put this in (4.9), divide through by $\frac{\theta}{\rho^{p^+}}$ and introduce a change of variable $z=\frac{t\rho^{p^+}}{\theta}$ in $Q(\theta,\rho_n)$. This transforms $Q(\theta,\rho_n)$ into $Q_n\equiv B_{\rho_n}\times(-\rho^{p^+},0)$. Setting $v(x,z)=u(x,\frac{z\theta}{\rho^{p^+}})$,

$\tilde{\eta}_n(x,z) = \eta_n(x, \frac{z\theta}{\rho^{p^+}})$ and $|A_n| = \text{meas}\{x \in B_{\rho_n} | v(x,z) < k_n\}$, by (4.9) and (1.3), we have

$$||(v-k_n)_- \tilde{\eta_n}^{p^+}||^{p^-}_{V^{p^-}(Q_n)} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}} \left(\frac{\omega}{2^s}\right)^{p^+} |A_n|. \tag{4.10}$$

By Lemma 2.6 and (4.10), we have

$$\frac{1}{2^{p^-(n+2)}}\left(\frac{\omega}{2^s}\right)^{p^-}|A_{n+1}| \le \iint_{Q_{n+1}\cap[v<k_{n+1}]} (v-k_n)_-^{p^-}\,dxdz \le \iint_{Q_n} (v-k_n)_-^{p^-}\,\tilde{\eta}_n^{p^+p^-}\,dxdz$$

$$\le \gamma ||(v-k_n)_- \tilde{\eta}_n^{p+}||^{p^-}_{p^-,Q_n} |A_n|^{\frac{p^-}{N+p^-}} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+} |A_n|^{1+\frac{p^-}{N+p^-}}. \tag{4.11}$$

Setting $Y_n = \frac{|A_n|}{|Q_n|}$ and using (1.3), (4.2) and (4.11), we get $Y_{n+1} \le \gamma 4^{np^+} Y_n^{1+\frac{p^-}{N+p^-}}$. By Lemma 2.4, it follows that Y_n tends to zero as $n \to \infty$, provided

$$Y_0 \le \gamma^{-\frac{N+p^-}{p^-}} 4^{-p^+(\frac{N+p^-}{p^-})^2} \equiv \sigma_1. \tag{4.12}$$

To prove the lemma, we fix σ_1 as in (4.12) and pick s according to Lemma 4.2.

We summarize the results obtained so far as in the following.

Proposition 4.4 *There exist numbers $\sigma, \nu_1 \in (0,1)$ and $A_1 \gg 1$ such that if for* ***some*** *cylinder of the type $[(x_0,t_0) + Q(d\rho^{p^+}, \rho)]$,*

$$\left|[(x_0,t_0)+Q_0] \cap \left[u < \mu^- + \frac{\omega}{2}\right]\right| \le \sigma |(x_0,t_0)+Q_0|, \tag{4.13}$$

then either

$$\omega \le A_1 \rho^{\frac{\epsilon}{p^+-2}} \tag{4.14}$$

or

$$\operatorname*{ess\ osc}_{Q(d(\frac{\rho}{8})^{p^+}, \frac{\rho}{8})} u \le \nu_1 \omega. \tag{4.15}$$

Proof Assume that (4.14) is violated. By Lemma 4.3, we can determine a number s such that

$$\operatorname*{ess\,inf}_{Q(\theta,\frac{\rho}{4})} u \ge \mu^- + \frac{\omega}{2^{s+1}}.$$

Change the sign of this inequality and add $\operatorname{ess\,sup}_{Q(\theta,\frac{\rho}{8})} u$ to the left-hand side and μ^+ to the right-hand side. This gives

$$\operatorname*{ess\ osc}_{Q(\theta,\frac{\rho}{8})} u \le \left(1 - \frac{1}{2^{s+1}}\right)\omega.$$

Therefore the proposition follows with $\nu = 1 - \frac{1}{2^{s+1}}$, since $Q(d(\frac{\rho}{8})^{p^+}, \frac{\rho}{8}) \subset Q(\theta, \frac{\rho}{8})$.

If the assumptions of Proposition 4.4 are violated, that is, for **every** subcylinder $[(x_0,t_0) + Q(d\rho^{p^+}, \rho)] \subset Q(a\rho^{p^+}, \rho)$, where $\frac{1}{a} = (\frac{\omega}{A})^{p^+-2}$,

$$\left|[(x_0,t_0)+Q_0] \cap \left[u < \mu^- + \frac{\omega}{2}\right]\right| > \sigma |(x_0,t_0)+Q_0|.$$

Since $\mu^- + \frac{\omega}{2} \le \mu^+ - \frac{\omega}{2}$, we rewrite this as

$$\left|[(x_0,t_0)+Q_0] \cap \left[u > \mu^+ - \frac{\omega}{2}\right]\right| \le (1-\sigma) |(x_0,t_0)+Q_0|. \tag{4.16}$$

As before, we fix Q_0 and will have similar results.

Lemma 4.5 *Let* (4.16) *and*

$$\left(\frac{\omega}{A_2}\right)^{p^+-2} > \rho^{\epsilon} \tag{4.17}$$

hold. There exists a time level $\bar{t}\in[-d\rho^{p^+},-\frac{\sigma}{2}d\rho^{p^+}]$, *such that*

$$\Big|x\in B_\rho: u(x,\bar{t})>\mu^+-\frac{\omega}{2}\Big|\le\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)|B_\rho|.$$

Proof If not, for all $\bar{t}\in[-d\rho^{p^+},-\frac{\sigma}{2}d\rho^{p^+}]$,

$$\Big|x\in B_\rho: u(x,t)>\mu^+-\frac{\omega}{2}\Big|>\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)|B_\rho|,$$

and

$$\Big|(x,t)\in Q_0: u(x,t)>\mu^+-\frac{\omega}{2}\Big|\ge\int_{-d\rho^{p^+}}^{-\frac{\sigma}{2}d\rho^{p^+}}\Big|x\in B_\rho: u(x,\tau)>\mu^+-\frac{\omega}{2}\Big|d\tau>(1-\sigma)|Q_0\Big|,$$

contradicting (4.16).

Lemma 4.6 *Let* (4.16) *and* (4.17) *hold. There exists a positive integer* $\bar{s}>2$ *such that*

$$\Big|x\in B_\rho: u(x,t)>\mu^+-\frac{\omega}{2^{\bar{s}}}\Big|\le\Big(1-\Big(\frac{\sigma}{2}\Big)^2\Big)|B_\rho|,$$

for all $t\in[-\frac{\sigma}{2}d\rho^{p^+},0]$.

Proof Consider the logarithmic inequalities over $B_\rho\times(\bar{t},0)$ for the function $(u-k)_+$ with the level $k=\mu^+-\frac{\omega}{2}$. As in Lemma 4.2, we take $\varphi=\frac{\partial}{\partial u_h}[\Psi^2(u_h)]\eta^{p^+}$ as a testing function, where $H_k^+=\operatorname{ess\,sup}_{Q_0}(u-k)_+$, $\Psi(u)=\ln^+\{\frac{H_k^+}{H_k^+-(u-k)_++\frac{\omega}{2^{n+1}}}\}$ and a cutoff function η is to be determined later, and then get

$$\begin{aligned}\int_{B_{(1-\alpha)\rho}}&\Psi^2(x,t)dx\\ &\le\int_{B_\rho}\Psi^2(x,t)\eta^{p^+}(x)dx\\ &\le\int_{B_\rho}\Psi^2(x,\bar{t})dx+\gamma\int_{\bar{t}}^0\int_{B_\rho}\Psi(\Psi')^{2-p^+}|D\eta|^{p^+}dxd\tau+\gamma\int_{\bar{t}}^0\int_{B_\rho}\chi[(u-k)_->0]dxd\tau\\ &\le n^2\ln^2 2\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)|B_\rho|+\frac{\gamma}{(\alpha\rho)^{p^+}}\cdot n\ln 2\cdot 2^{p^+}\Big(\frac{\omega}{2}\Big)^{p^+-2}\cdot d\rho^{p^+}|B_\rho|+\gamma\cdot d\rho^{p^+}|B_\rho|\\ &\le\Big[n^2\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)+\frac{\gamma n}{(\alpha)^{p^+}}+\gamma\Big]|B_\rho|\le\Big[n^2\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)+\frac{\gamma n}{(\alpha)^{p^+}}\Big]|B_\rho|,\end{aligned}$$

where we use $\Psi\le n\ln 2$, $|\Psi'|\le\frac{2}{\omega}$, $|D\eta|\le\frac{1}{\alpha\rho}$ and $|\bar{t}|\le d\rho^{p^+}$.

Since $\Psi^2\ge(n-1)^2\ln^2 2$ on the small set $\{x\in B_{(1-\alpha)\rho}: u(x,t)>\mu^+-\frac{\omega}{2^{n+1}}\}$, we obtain

$$\Big|x\in B_{(1-\alpha)\rho}: u(x,t)>\mu^+-\frac{\omega}{2^{n+1}}\Big|\le\Big[\Big(\frac{n}{n-1}\Big)^2\Big(\frac{1-\sigma}{1-\frac{\sigma}{2}}\Big)+\frac{\gamma}{(\alpha)^{p^+}n}\Big]|B_\rho|,$$

for all $t\in(\bar{t},0)$. Choose α so small and then n so large that

$$\Big(\frac{n}{n-1}\Big)^2\le\Big(1-\frac{\sigma}{2}\Big)(1+\sigma)\quad\text{and}\quad\frac{\gamma}{(\alpha)^{p^+}n}\le\frac{3}{4}\sigma^2.$$

Then for such a choice of n the lemma follows with $\bar{s}=n+1$.

Since (4.16) holds for **all** cylinders of the type Q_0, the conclusion of Lemma 4.6 holds true for all time levels satisfying $t\ge-(a-d)\rho^{p^+}=-(1-(\frac{2}{A})^{p^+-2})a\rho^{p^+}$. If A is chosen sufficiently large, we deduce:

Corollary 4.7 *Let* (4.16) *and* (4.17) *hold. For all* $t\in[-\frac{a}{2}\rho^{p^+},0]$,

$$\Big|x\in B_\rho: u(x,t)>\mu^+-\frac{\omega}{2^{\bar{s}}}\Big|\le\Big(1-\Big(\frac{\sigma}{2}\Big)^2\Big)|B_\rho|.$$

Lemma 4.8 *Let (4.16) and (4.17) hold. For every $\bar\sigma \in (0,1)$, there exists a positive integer $s^* > \bar s$ such that*

$$\Big|x \in B_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^{s^*}}\Big| \le \bar\sigma \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big|.$$

Proof Consider the local energy estimates over $Q(a\rho^{p^+}, 2\rho)$ for the function $(u-k)_+$ with the levels $k = \mu^+ - \frac{\omega}{2^s}$, where $\bar s \le s \le s^*$ and s^* is to be chosen. We take $(u-k)_+\eta^{p^+}$ as a testing function, and then get

$$\begin{aligned}
\iint_{A_s} |Du|^{p-} dx d\tau &\le \iint_{Q(\frac{a}{2}\rho^{p^+},\rho)} |D(u-k)_+|^{p(x,\tau)} dx d\tau + |A_s| \\
&\le \frac{\gamma}{\rho^{p^+}} \iint_{Q(a\rho^{p^+},2\rho)} (u-k)_+^{p^+} dx d\tau + \frac{\gamma}{a\rho^{p^+}} \iint_{Q(a\rho^{p^+},2\rho)} (u-k)_+^2 dx d\tau + \gamma|A_s| \\
&\le \frac{\gamma}{\rho^{p^+}} \Big(\frac{\omega}{2^s}\Big)^{p^+} \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big|, \qquad (4.18)
\end{aligned}$$

where $A_s = \{(x,t) \in Q(\frac{a}{2}\rho^{p^+}, \rho) : u(x,t) > \mu^+ - \frac{\omega}{2^s}\}$. Let $A_s(t) = \{x \in B_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^s}\}$.

Applying Lemma 2.7 for $k = \mu^+ - \frac{\omega}{2^s}$, $l = \mu^+ - \frac{\omega}{2^{s+1}}$, $l - k = \frac{\omega}{2^{s+1}}$ and Corollary 4.7, we have

$$\frac{\omega}{2^{s+1}} |A_{s+1}(t)| \le \frac{4\gamma}{\sigma^2} \frac{\rho^{N+1}}{|B_\rho|} \int_{A_s(t)\setminus A_{s+1}(t)} |Du| dx,$$

for all $t \in (-\frac{a}{2}\rho^{p^+}, 0)$, where we use $|x \in B_\rho : u(x,t) > \mu^+ - \frac{\omega}{2^s}| = |B_\rho| - |A_s(t)| \ge (\frac{\sigma}{2})^2 |B_\rho|$. From this, integrating over such a time interval, we get

$$\begin{aligned}
\frac{\omega}{2^{s+1}} |A_{s+1}| &\le \frac{\gamma\rho}{\sigma^2} \iint_{A_s\setminus A_{s+1}} |Du| dx d\tau \\
&\le \frac{\gamma\rho}{\sigma^2} \Big(\iint_{A_s} |Du|^{p^-} dx d\tau \Big)^{\frac{1}{p^-}} |A_s \setminus A_{s+1}|^{\frac{p^- - 1}{p^-}} \\
&\le \frac{\gamma\rho}{\sigma^2} \Big[\frac{\gamma}{\rho^{p^+}} \Big(\frac{\omega}{2^s}\Big)^{p^+} \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big| \Big]^{\frac{1}{p^-}} |A_s \setminus A_{s+1}|^{\frac{p^- - 1}{p^-}} \\
&\le \frac{\gamma}{\sigma^2} \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big|^{\frac{1}{p^-}} |A_s \setminus A_{s+1}|^{\frac{p^- - 1}{p^-}},
\end{aligned}$$

where we again use (1.3), (4.17) and (4.18). That is,

$$|A_{s+1}|^{\frac{p^-}{p^- - 1}} \le \gamma \sigma^{-\frac{2p^-}{p^- - 1}} \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big|^{\frac{1}{p^- - 1}} |A_s \setminus A_{s+1}|,$$

for all $\bar s \le s \le s^*$. We add them for

$$s = \bar s, \bar s + 1, \bar s + 2, \dots, s^* - 1.$$

The right-hand side can be majorized by a convergent series bounded above by $|Q(\frac{a}{2}\rho^{p^+}, \rho)|$. Therefore,

$$(s^* - \bar s) |A_{s^*}|^{\frac{p^-}{p^- - 1}} \le \gamma \sigma^{-\frac{2p^-}{p^- - 1}} \Big|Q\Big(\frac{a}{2}\rho^{p^+}, \rho\Big)\Big|^{\frac{p^-}{p^- - 1}}.$$

To prove the lemma we divide by $(s^* - \bar s)$ and take s^* so large that $\frac{\gamma}{\sigma^2 (s^* - \bar s)^{\frac{p^- - 1}{p^-}}} \le \bar\sigma$.

Lemma 4.9 *Let (4.16) and (4.17) hold. The number $\bar\sigma$ (and hence s^* and A) can be chosen so that $u(x,t) \le \mu^+ - \frac{\omega}{2^{s^*+1}}$, a.e. $Q(\frac{a}{2}(\frac{\rho}{2})^{p^+}, \frac{\rho}{2})$.*

Proof As in Lemma 4.1, applying the local energy estimates over $Q(\frac{a}{2}\rho_n^{p^+}, \rho_n)$ to the function $(u-k_n)_+$, where $\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}$ and $k_n = \mu^+ - \frac{\omega}{2^{s^*+1}} - \frac{\omega}{2^{s^*+1+n}}$, we have

$$\sup_{-\frac{a}{2}\rho_n^{p^+}\le t\le 0} \int_{B_{\rho_n}} (u-k_n)_+^2 \eta_n^{p^+}(x,t)dx + \iint_{Q(\frac{a}{2}\rho_n^{p^+},\rho_n)} |D(u-k_n)_+|^{p(x,\tau)} \eta_n^{p^+} dxd\tau$$
$$\le \left(\frac{\gamma 2^{n+1}}{\rho}\right)^{p^+} \iint_{Q(\frac{a}{2}\rho_n^{p^+},\rho_n)} (u-k_n)_+^2 dxd\tau$$
$$+ \gamma\frac{2}{a}\left[\frac{2^{n+1}}{\rho}\right]^{p^+} \iint_{Q(\frac{a}{2}\rho_n^{p^+},\rho_n)} (u-k_n)_+^{p_+} dxd\tau + \gamma \iint_{Q(\frac{a}{2}\rho_n^{p^+},\rho_n)} \chi[(u-k_n)_+ > 0]dxd\tau$$
$$\le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2^{s^*}}\right)^{p^+} \iint_{Q(\frac{a}{2}\rho_n^{p^+},\rho_n)} \chi[(u-k_n)_- > 0]dxd\tau,$$

where we use $|D\eta| \le \frac{2^{n+1}}{\rho}$, $|\frac{\partial}{\partial\tau}\eta| \le \frac{2}{a}(\frac{2^{n+1}}{\rho})^{p^+}, \frac{1}{a} = (\frac{\omega}{A})^{p^+-2}$, $A = 2^{s^*}$. As before, we introduce a change of variable $z = \frac{2t}{a}$ which maps $Q(\frac{a}{2}\rho_n^{p^+}, \rho_n)$ into $Q_n = B_{\rho_n} \times (-\rho_n^{p^+}, 0)$, set $v(\cdot, z) = u(\cdot, \frac{a}{2}z)$, $\tilde{\eta_n}(\cdot, z) = \eta_n(\cdot, \frac{a}{2}z)$ and $|A_n| = \text{meas}\{x \in B_{\rho_n} : v(x,z) > k_n\}$, and then get

$$||(v-k_n)_+\tilde{\eta_n}^{p^+}||_{V^{p^-}(Q_n)}^{p^-} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2^{s^*}}\right)^{p^+} |A_n|.$$

By Lemma 2.6, we have

$$\frac{1}{2^{p^-(n+2)}}\left(\frac{\omega}{2^{s^*}}\right)^{p^-} |A_{n+1}| = |k_n - k_{n+1}|^{p^-} : (x,z) \in Q_{n+1}|v(x,z) > k_{n+1}|$$
$$\le ||(v-k_n)_+||_{Q_{n+1}}^{p^-} \le ||(v-k_n)_+\tilde{\eta}_n^{p+}||_{Q_n}^{p^-}$$
$$\le \gamma||(v-k_n)_+\tilde{\eta}_n^{p+}||_{V^{p^-}(Q_n)}^{p^-}|A_n|^{\frac{p^-}{N+p^-}} \le \frac{\gamma 2^{np^+}}{\rho^{p^+}}\left(\frac{\omega}{2^{s^*}}\right)^{p^+} |A_n|^{1+\frac{p^-}{N+p^-}}.$$

We also introduce the quantity $Y_n = \frac{|A_n|}{|Q_n|}$, using (1.3) and (4.17), and then get

$$Y_{n+1} \le \gamma 4^{np^+}\left(\frac{\omega}{2}\right)^{p^+-p^-} Y_n^{1+\frac{p^-}{N+p^-}} \le \gamma 4^{np^+} Y_n^{1+\frac{p^-}{N+p^-}}.$$

By Lemma 2.4, it follows that $Y_n \to 0$ as $n \to \infty$, provided $Y_0 \le \gamma^{-\frac{N+p^-}{p^-}} 4^{-p^+(\frac{N+p^-}{p^-})^2} \le \bar{\sigma}$. Thus, we have the desired result.

Proposition 4.10 *There exist numbers $\sigma, \nu_2 \in (0,1)$ and $A_2 \gg 1$ such that if for* ***all*** *cylinders of the type $[(x_0,t_0) + Q(d\rho^{p^+}, \rho)]$,*

$$\left|[(x_0,t_0) + Q_0] \cap \left[u > \mu^+ - \frac{\omega}{2}\right]\right| \le (1-\sigma)|(x_0,t_0) + Q_0|, \tag{4.19}$$

then either

$$\omega \le A_2 \rho^{\frac{\epsilon}{p^+-2}} \tag{4.20}$$

or

$$\operatorname*{ess\,osc}_{Q(\frac{a}{2}(\frac{\rho}{2})^{p^+}, \frac{\rho}{2})} u \le \nu_2\omega. \tag{4.21}$$

We combine Proposition 4.4 and Proposition 4.10 into:

Proposition 4.11 *There exist two positive constants $\nu = \max\{\nu_1, \nu_2\}$, $\bar{A} = \max\{A_1, A_2\}$ that can be determined a priori such that either $\omega \le \bar{A}\rho^{\frac{\epsilon}{p^+-2}}$ or $\operatorname{ess\,osc}_{Q(d(\frac{\rho}{8})^{p^+}, \frac{\rho}{8})} u \le \nu\omega$.*

Similarly to proving Proposition 3.1 and Lemma 3.1 in Chapter 3 in [10] by using Proposition 4.11, we have the desired result on Hölder continuity. Therefore, we end the proof of Theorem 2.3.

Remark 4.12 Moreover, if the function p belongs to $C^1(Q)$ and the weak solution u is sufficiently integrable, then Du is locally bounded under suitable assumptions (i.e. the constant number s is so large that $4s+6 > p_2$ and $8(s+1) > 2N + N(p_2 - p_1)$). The assumptions given as above are weaker than those in ([12], Chapter 6, Theorem 3.4) when p is a constant number.

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