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The Self-similar Solution to Some Nonlinear Integro-differential Equations Corresponding to Fractional Order Time Derivative

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Abstract In this paper we study the self-similar solution to a class of nonlinear integro-differential equations which correspond to fractional order time derivative and interpolate nonlinear heat and wave equation. Using the space-time estimates which were established by Hirata and Miao in [11] we prove the global existence of self-similar solution of Cauchy problem for the nonlinear integro-differential equation in $C_*([0,\infty);\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^n))$.

Keywords Self-similar solution, Space-time estimates, Integro-differential equation, Fractional order time derivative, Mittag–Leffler's function, Cauchy problem
MR(2000) Subject Classification 35K05, 35L05, 45K05

1 Introduction

This paper is devoted to the study of the self-similar solution of Cauchy problem for nonlinear integro-differential equation

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \displaystyle\int_0^t R_\alpha(t-s) \triangle u(s)ds + \int_0^t R_\alpha(t-s) f(u(s))ds, \quad 0<\alpha<1, \\ u(0)=\psi, \end{cases} \tag{1.1}$$

where $u(t)=u(x,t)$, $\psi=\psi(x)$, $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$, $R_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and

$$f(u)=\mu|u|^\rho u \quad \text{or} \quad f(u)=\mu u^{\rho+1}, \quad \rho>0. \tag{1.2}$$

Formally, (1.1) is written as a fractional order evolution equation

$$\begin{aligned} \frac{\partial^{1+\alpha}u}{\partial t^{1+\alpha}} &= \triangle u + f(u), \quad && u=u(x,t), \\ u(0) &= \psi, && \psi=\psi(x), \end{aligned} \tag{1.3}$$

and this equation interpolates nonlinear heat and wave equations.

For the linear case ($f(u)=0$) for $n=1$, (1.1) describes the heat conduction with memory [1, 2], and many authors studied the problem ([1–7]). Scheider, Wyss [5] and Fujita [3] obtained the representation formula of fundamental solution for this linear equation for $n=1$. Furthermore, Fujita [3] proved that the fundamental solution of this linear equation has some

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properties which are similar to heat equation's and one and other properties which are similar to wave equation's. For example, the points where the fundamental solution attains its maximum propagate with finite speed. Fujita [4] also obtained the representation of the solution of linear integro-differential equation including wave equation by probability methods. Hirata and Miao in [8] gave the representation formula for linear integro-differential equation with inhomogeneous term in multidimensional spaces and space-time estimates for the solutions of linear integro-differential equations based on Laplace and Fourier transforms, Mittag–Leffler's functions, Mihlin–Hörmander's multiple estimates. Moreover, they proved the well-posedness of local mild solution of Cauchy problem for nonlinear integro-differential equation in $C([0,T);L^r(\mathbb{R}^n))$ or $L^q(0,T;L^p(\mathbb{R}^n))$ and the existence of global small solutions for nonlinear intrgro-differential equations by using the space-time estimates and contraction mapping principle.

Recently, many authors studied the self-similar solution for nonlinear evolution equations. Cannone and Planchon [9–11] developed the Kato's method [12] and established the existence of self-similar solution and asympotic behavior of Navier–Stokes equations by the Littlewood–Paley description of Besov spaces. In [13] Meyer gave an excellent analysis to self-similar solution of nonlinear evolution equations and proved that the similarity results were also valid for the heat equations. Miao and Zhang in [17] found a way based on space-time estimates which were established in [14, 15, 16] and established self-similar solution for the general semi-linear heat equations.

It is well known that

$$\sup_{t\geq 0} t^{-\frac{s}{2}}\|H(t)\psi\|_p = \|\psi\|_{\dot{B}^s_{p,\infty}}, \quad H(t)=e^{t\triangle}, \quad s\geq 0. \tag{1.4}$$

But we can not use Schrödinger semigroup $S(t)=e^{it\triangle}$ to give an equivalent norm such as the above heat semigroup. Cazenave and Weissler [10, 18], Ribaud and Youssfi [19] introduced a class of new function spaces

$$E_{s,p}(\mathbb{R}^n\times\mathbb{R}) = \Big\{u(x,t)\in\mathscr{S}(\mathbb{R}^n\times\mathbb{R}); \|u\|_{E_{s,p}} = \sup_{t>0} t^{\sigma}\|u(t,x)\|_{\dot{H}^{s,p}}<\infty$$
$$2\leq p<\infty, \quad 0\leq s<\frac{n}{p}, \quad \sigma=\sigma(s,p)=\frac{1}{2}\left(\frac{2}{\alpha}-\frac{n}{p}+s\right)\Big\},$$

and established the global self-similar solution of (1.1) under condition $\|\psi;E_{s,p}(\mathbb{R}^n)\|<\varepsilon$, where

$$E_{s,p}(\mathbb{R}^n)=\{\psi(x)\in\dot{H}^{s,p}(\mathbb{R}^n); \quad \|\psi\|_{E_{s,p}}=\sup_{t>0}t^{\sigma}\|S(t)\psi\|_{\dot{H}^{s,p}}<\infty\}.$$

On the other hand, Planchon [20, 21, 22] substituted classical critical space $\dot{H}^{s_2}$ with $\dot{B}^{s_2}_{2,\infty}$, $s_2=\frac{n}{2}-\frac{2}{\alpha}$, and proved the global self-similar solution to Schrödinger equation in $C_*(\mathbb{R};\dot{B}^{s_2}_{2,\infty})$ under condition $\alpha\in 2\mathbb{N}$ by using the generalized Strichartz-type estimates and Littlewood–Paley decompositions theory. Miao and Zhang [17] extended Planchon's results to generalize nonlinear term by devoting to some equivalent description of Besov space. Using the above different spaces, Ribaud and Youssfi [23], Planchon [21, 22] established the global self-similar solution for semilinear wave equations, respectively, under suitable conditions. The purpose of this paper is devoted to the study of the global self-similar solution for nonlinear integro-differential Equation (1.1). Using the space-time estimates which were established by Hirata and Miao in [8] we prove the global existence of self-similar solution for the nonlinear integro-differential equation in $C_*([0,\infty);\dot{B}^{s_p}_{p,\infty}(\mathbb{R}^n))$.

Before stating our main results, we first introduce some notations. According to scaling principle (see the next section), one easily sees that $u_\lambda(t,x)=\lambda^{\frac{2}{\rho}}u(\lambda^{\frac{2}{\alpha+1}}t,\lambda x)$ with $\lambda>0$ is the solution of (1.1) or (1.3) with initial data $\psi(x)$ replaced by $\lambda^{\frac{2}{\rho}}\psi(\lambda x)$ if $u=u(t,x)$ satisfies (1.1) or (1.3).

Definition 1.1 *$u = u(t,x)$ is said to be a self-similar solution of* (1.1) *or* (1.3) *if*

$$u_\lambda(t,x) = \lambda^{\frac{2}{\rho}} u(\lambda^{\frac{2}{\alpha+1}} t, \lambda x) \equiv u(t,x), \quad \forall\, \lambda > 0. \tag{1.5}$$

Remark 1.2 (1) It is well known that the self-similar solution to (1.1) or (1.3) is of the form

$$u(t,x) = t^{-\frac{\alpha+1}{\rho}} U\Big(\frac{x}{t^{\frac{\alpha+1}{2}}}\Big). \tag{1.6}$$

(2) As we know, self-similar solutions to nonlinear evolution equations can be studied by semi-linear elliptic equations. But for (1.1) or (1.3) it seems difficult to study the self-similar solution in this way.

(3) One important approach to the self-similar solutions of (1.1) or (1.3) is the study of small global well-posedness of Cauchy problem (1.1) or (1.3) in Besov spaces or in some nonstandard function spaces. These new global solutions to (1.1) or (1.3) admit a class of self-similar solutions. In fact, let the initial data $\psi(x)$ satisfy

$$\psi(x) = \lambda^{\frac{2}{\rho}} \psi(\lambda x), \quad \lambda > 0, \tag{1.7}$$

for example, we can take

$$\psi(x) = \kappa \frac{\Omega(x')}{|x|^{\frac{2}{\rho}}}, \quad x' = \frac{x}{|x|},$$

where the function $\Omega(x)$ is defined on the unit sphere Σ^{n-1} in $\mathbb{R}^n$. If $u(t,x)$ is the unique solution of (1.1) or (1.3) with the data in (1.7), then $u(t,x)$ is just a self-similar solution of (1.1) or (1.3).

(4) Let $u(t,x)$ be a self-similar solution to (1.1) or (1.3). Then the function $u(0,x) = \psi(x)$ is a homogeneous function with degree $-\frac{2}{\rho}$. To obtain the self-similar solution of (1.1) or (1.3) we have to choose a suitable homogeneous function space X with the degree $-\frac{2}{\rho}$ (see [16], [17] or [24]) and prove that (1.1) or (1.3) generates a global flow in X.

Let E_α and $E_{\alpha,\beta}$ denote Mittag–Leffler's function and generalized Mittag–Leffler's function [25, 26], that is,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{1.8}$$

For the detailed properties of Mittag–Leffler's function, one can see [6–8, 13].

Definition 1.2 *We call (q,r,p) the admissible triplet with respect to α if*

$$\frac{1}{q} = \frac{n(1+\alpha)}{2}\Big(\frac{1}{p} - \frac{1}{r}\Big), \quad 1 < p \le r < \begin{cases} \dfrac{n(1+\alpha)p}{n(1+\alpha)-2}, & \text{if } n \ge 2, \\ \infty, & \text{if } n = 1. \end{cases} \tag{1.9}$$

Definition 1.3 *The triplet (q,r,p) is called a generalized admissible triplet with respect to α if*

$$\frac{1}{q} = \frac{n(1+\alpha)}{2}\Big(\frac{1}{p} - \frac{1}{r}\Big), \quad 1 < p \le r < \begin{cases} \dfrac{n(1+\alpha)p}{n(1+\alpha)-2p}, & n(1+\alpha) > 2p, \\ \infty, & n(1+\alpha) \le 2p. \end{cases} \tag{1.10}$$

Remark 1.4 (1) It is easy to see that (q,r,p) is an admissible triplet or generalized admissible triplet, then q is uniquely determined by r and p. So we may write $q = q(r,p)$.

(2) Clearly, $r < q \le \infty$ if (q,r,p) is an admissible triplet, and $1 < q \le \infty$ if (q,r,p) is a generalized admissible triplet.

Definition 1.5 (1) *Let B be a Banach space and for $\sigma > 0$, $I = [0,T)$ and $\dot{I} = (0,T)$. Define $\mathscr{C}_\sigma(I;B)$ and its homogeneous space $\dot{\mathscr{C}}_\sigma(I;B)$ as follows:*

$$\mathscr{C}_\sigma(I;B) = \{f \in \mathscr{C}(\dot{I};B);\ \|f;\mathscr{C}_\sigma(I;B)\| = \sup_{t\in I} t^{\frac{1}{\sigma}} \|f\|_B < \infty\}, \tag{1.11}$$

$$\dot{\mathscr{C}}_\sigma(I;B) = \{f \in \mathscr{C}_\sigma(I;B);\ \lim_{t\to 0} t^{\frac{1}{\sigma}} \|f(t)\|_B = 0\}.$$

(2) $C_*(I;B)=\{v;\ v\in L^\infty(I;B)$ *and* v *are continuous in the sense of distributions*$\}$.

Remark 1.6 It is easy to see that

(1) $f\in\mathscr{C}_\sigma(I;L^r)$ if and only if $t^{\frac{1}{\sigma}}f\in C_b(I;L^r)$;

(2) if (q,r,p) is an admissible triplet then

$$\mathscr{C}_{q(r,p)}(I;L^r)=\Big\{f\in C(\dot I;L^r),\ \|f\|_{\mathscr{C}_{q(r,p)}(I;L^r)}=\sup_{t\in I}t^{\frac{1}{q}}\|f\|_r<\infty\Big\}. \tag{1.12}$$

In particular, $\mathscr{C}_{q(r,p)}(I;L^r)=C_b(I;L^p)$ if $r=p$.

(3) $v\in C_*(I;B)$ means v is a bounded flow in B.

Definition 1.7 *Let* $\sigma>0$, $1<r<\infty$. *Define Banach space* $\mathscr{E}_{\sigma,r}(\mathbb{R}^n)$ *as*

$$\mathscr{E}_{\sigma,r}(\mathbb{R}^n)=\Big\{\psi(x)\in\mathscr{S}'(\mathbb{R}^n);\ \|\psi\|_{\mathscr{E}_{\sigma,r}}=\sup_{t>0}t^{\frac{1}{\sigma}}\|W(t)\psi\|_r<\infty\Big\}, \tag{1.13}$$

where $W(t)=\mathscr{F}^{-1}E_{\alpha+1}(-|\xi|^2t^{\alpha+1})*$ *denotes the solution operator of free integro-differential equation with respect to* (1.1).

Our main results can be stated as follows.

Theorem 1.1 *Let* $r\ge p>p_0=\frac{n\rho}{2}$, $\rho>\max(1,\frac{2}{n})$. *Let* (q,r,p_0) *be any generalized admissible triplet with*

$$\max(p_0,1+\rho)<r<p_0(1+\rho). \tag{1.14}$$

Let $\psi(x)\in\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\cap\mathscr{E}_{q(r,p_0),r}(\mathbb{R}^n)$ *with* (1.7). *Then there exists* $\eta(p)>0$ *with the following properties*: *If* $\|\psi;\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\|+\|\psi;\mathscr{E}_{q(r,p_0),r}(\mathbb{R}^n)\|<\eta(p)$, *then there exists a unique solution* $u(x,t)\in C_*([0,\infty);\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty})\cap\mathscr{C}_{q(r,p_0)}([0,\infty);L^r(\mathbb{R}^n))$ *to* (1.1) *or* (1.3) *which satisfies*

$$u(x,t)=W(t)\psi+v(x,t)=W(t)\psi+t^{-\frac{\alpha+1}{\rho}}V\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg)=t^{-\frac{\alpha+1}{\rho}}U\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg), \tag{1.15}$$

$$\|v(x,t);L^\infty(\mathbb{R}^+;L^{p_0}(\mathbb{R}^n))\|<\infty, \tag{1.16}$$

$$\sup_{t>0}t^{\frac{1}{q(r,p_0)}}\|u(x,t)\|_r<\infty. \tag{1.17}$$

Moreover, its profile $V(x)$ *satisfies*

$$U(x)=W(1)\psi+V(x)\in L^r(\mathbb{R}^n). \tag{1.18}$$

Corollary 1.2 *Let* $r\ge p>p_0=\frac{n\rho}{2}>1$, $\rho>\max(1,\frac{2}{n})$. *Let* (q,r,p_0) *be any generalized admissible triplet. Let* $\psi(x)\in\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\cap\mathscr{E}_{q(r,p_0),r}(\mathbb{R}^n)$ *with* (1.7). *Then there exists* $\eta(p)>0$ *with the following property*: *If*

$$\Big\|\psi;\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\Big\|+\|\psi;\mathscr{E}_{q(r,p_0),r}(\mathbb{R}^n)\|<\eta(p), \tag{1.19}$$

then there exists a unique self-solution $u(x,t)\in C_*(\mathbb{R}^+;\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty})\cap\mathscr{C}_{q(r,p_0)}(\mathbb{R}^+;L^r(\mathbb{R}^n))$ *to* (1.1) *or* (1.3) *which satisfies* (1.15)–(1.17) *and its profile* $U(x)$ *satisfies* (1.18) *for*

$$p_0\le r<\begin{cases}\dfrac{n(1+\alpha)\rho}{2(1+\alpha)-2\rho}, & \rho<1+\alpha,\\ \infty, & \rho\ge 1+\alpha.\end{cases} \tag{1.20}$$

Remark 1.7 (1) In Theorem 1.1 and Corollary 1.2, for generic data without restriction condition (1.7), one also obtains the global well-posedness of (1.1) or (1.3) on $C_*([0,\infty);\dot B^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty})\cap\mathscr{C}_{q(r,p_0)}([0,\infty);L^r(\mathbb{R}^n))$ but the solutions $u(x,t)$ has no structure of self-similar solutions (1.15) and (1.18).

(2) When $\kappa>0$ is suitably small, the initial data $\psi(x)$ in (1.7) satisfies condition (1.19).

Remark 1.8 (1) If we take $\alpha=0$, Theorem 1.1 and Corollary 1.2 implies the known results in heat equation in [13] and [17].

(2) If we take $\alpha = 1$ in (1.3) and add the initial condition about v_t as φ satisfying

$$\varphi(x) = \lambda^{\frac{2}{\rho}+1}\varphi(\lambda x), \quad \lambda > 0,$$

and substitute

$$\|\psi; \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}\| + \|\psi; \mathscr{E}_{q(r,p_0),r}(\mathbb{R}^n)\| < \eta(p)$$

with a simpler condition

$$\sup_{t>0} t^{\beta}\big\| \cos[(-\triangle)^{\frac{1}{2}}t]\psi + (-\triangle)^{-\frac{1}{2}}\sin[(-\triangle)^{\frac{1}{2}}t]\varphi\big\|_r \le \eta, \quad \beta = \frac{2}{\rho} - \frac{n}{r},$$

where $r \ge \max(2, \frac{n\rho}{2})$, one easily gets the global well-posedness for the wave equation in the space $\mathscr{C}_{\frac{1}{\beta}}(\mathbb{R}; L^r(\mathbb{R}^n))$ by using the method in this paper. For more results on wave equations see [27].

Remark 1.9 The self-similar solution in Theorem 1.1 is driving the large-scale behavior of some global solution to (1.1) or (1.3). If $u(x,t)$ is a global solution to (1.1) or (1.3) with (1.19) and

$$\lim_{t\to\infty} t^{\frac{\alpha+1}{\rho}}\|u(t^{\frac{\alpha+1}{2}}x, t) - V(x)\|_r = 0, \quad V(x) \in L^r(\mathbb{R}^n),$$

then one easily concludes that $v(x,t) = t^{-\frac{\alpha+1}{\rho}}V(t^{-\frac{\alpha+1}{2}}x)$ is a self-similar solution to (1.1) and (1.3). On the other hand, we have

$$u(x,t) = t^{-\frac{\alpha+1}{\rho}}V(t^{-\frac{\alpha+1}{2}}x) + t^{-\frac{\alpha+1}{\rho}}R(t^{-\frac{\alpha+1}{2}}x, t),$$

where $V(x) \in L^r(\mathbb{R}^n)$ and

$$\lim_{t\to\infty}\|R(\cdot,t)\|_r = 0.$$

For the large-scale behavior of the solutions to other evolution equations, see [10], [13] and [11].

Remark 1.10 For the admissible triplet (q, r, p_0) with (1.9), one easily conclude that $u(x,t) \in L^q(\mathbb{R}^+; L^r(\mathbb{R}^n))$ by the space-time estimates which were established in [8].

Before concluding this section, we like to give some notations which we shall use in this paper. $\mathscr{F}v$ and $\mathscr{F}^{-1}v$ denote the Fourier and Fourier inverse transforms of v in $\mathbb{R}^n$ respectively. We also use the more abbreviated notations $\hat{v}$ and $\check{v}$. $\Gamma(\cdot)$ and $B(\cdot,\cdot)$ denote the usual Gamma and Beta functions respectively. Different positive constants might be denoted by the same letter C.

Our plan in the present paper is as follows. In Section 2 we shall recall some well known basic facts and some space-time estimates. Section 3 is devoted to the proof of our main results.

2 Preliminary and Some Lemmas

It is well known that

$$\begin{aligned} v(x,t) =& \big(\widetilde{E}_{1+\alpha} * \psi\big)(x,t) + \int_0^t (t-\tau)^{\alpha}\big(\widetilde{E}_{1+\alpha,1+\alpha}(\cdot, t-\tau) * f(\cdot,\tau)\big)(x)d\tau \\ \triangleq& W(t)\psi + \mathscr{G}f, \end{aligned} \tag{2.1}$$

satisfies the following Cauchy problem for linear integro-differential equation with nonhomogeneous term of the form

$$\begin{cases} \dfrac{\partial v(t)}{\partial t} = \displaystyle\int_0^t R_\alpha(t-s) \triangle u(s)ds + \int_0^t R_\alpha(t-s)f(\cdot,s)ds, \\ v(0) = \psi, \quad \psi = \psi(x), \end{cases} \tag{2.2}$$

where $\widetilde{E}_{1+\alpha}(x,t) = \mathscr{F}^{-1}\big(E_{1+\alpha}(-|\cdot|^2 t^{1+\alpha})\big)(x)$ and

$$\widetilde{E}_{1+\alpha,1+\alpha}(x,t) = \mathscr{F}^{-1}\big(E_{1+\alpha,1+\alpha}(-|\cdot|^2 t^{1+\alpha})\big)(x).$$

Remark 2.1 (i) Integro-differential equation (2.2) corresponds to a fractional order partial differential equation

$$\begin{cases} \dfrac{\partial^{1+\alpha} v}{\partial t^{1+\alpha}} = \triangle\, v + f(\cdot, t), \quad v = v(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ v(0) = \psi, \quad \psi = \psi(x), \quad\quad\quad\quad\quad\quad x \in \mathbb{R}^n, \end{cases} \tag{2.3}$$

formally. If we take $\alpha = 0$, (2.3) is just as follows:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \triangle\, v + f(\cdot, t), \quad v = v(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ v(0) = \psi, \quad \psi = \psi(x), \quad\quad\quad\quad\quad\quad x \in \mathbb{R}^n. \end{cases} \tag{2.4}$$

In view of (2.2) and $E_1(-|\xi|^2 t) = e^{-|\xi|^2 t}$, we can easily see that the solution of (2.4) is

$$v(x,t) = (G(\cdot,t) * \psi)(x) + \int_0^t \big(G(\cdot, t-s) * f(\cdot, s)\big)(x) ds,$$

where $G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.

(ii) If we take $\alpha = 1$ in (2.3) and add the initial condition about v_t as φ, the equation is just as follows:

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \triangle\, v + f(\cdot, t), \quad v = v(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ v(0) = \psi, \quad\quad\quad\quad \psi = \psi(x), \quad x \in \mathbb{R}^n, \\ v_t(0) = \varphi, \quad\quad\quad\quad \varphi = \varphi(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.5}$$

In this case, the representation formula of the solution u changes as follows.

$$v(x,t) = \big(\widetilde{E}_2 * \psi\big)(x,t) + \int_0^t \big(\widetilde{E}_2 * \varphi\big)(x,s) ds + \int_0^t (t-\tau)^\alpha \big(\widetilde{E}_{2,2}(\cdot, t-\tau) * f(\cdot,\tau)\big)(x) d\tau.$$

Using the formulas $E_2(-|\xi|^2 t^2) = \cos(|\xi| t)$ and

$$E_{2,2}(-|\xi|^2 t^2) = \frac{1}{|\xi| t} \sum_{k=0}^{\infty} \frac{(-1)^k (|\xi| t)^{2k+1}}{\Gamma(2k+2)} = \frac{1}{|\xi| t} \sin(|\xi| t),$$

one easily finds that the solution of (2.5) is

$$\begin{aligned} v(x,t) = & \big(\mathscr{F}^{-1}(\cos(|\cdot| t)) * \psi\big)(x) + \left(\mathscr{F}^{-1}\left(\frac{\sin(|\cdot| t)}{|\cdot|}\right) * \varphi\right)(x) \\ & + \int_0^t \left(\mathscr{F}^{-1}\left(\frac{\sin(|\cdot|(t-\tau))}{|\cdot|}\right) * f(\cdot,\tau)\right)(x) d\tau. \end{aligned}$$

(iii) Let $\psi \in \mathscr{S}(\mathbb{R}^n)$, $f \in L^1([0,\infty); \mathscr{S}(\mathbb{R}^n))$ and $0 < \alpha < 1$. Then, (2.2) admits a unique solution $v \in C^{1+\alpha}([0,\infty); \mathscr{S}(\mathbb{R}^n))$. For the proof see [8].

By Mihlin–Hörmander's theorem, Marcinkiewicz's interpolation theorem and other harmonic analysis tool [28–32], one can verify the following propositions; for the detailed proof, see [8].

Proposition 2.1 (L^p-L^r estimates) *Let $1 < r < \infty$ and $n/2 < p \le r$. Then, for any $\psi \in L^p(\mathbb{R}^n)$, we have*

$$\|W(t)\psi\|_r = \|\widetilde{E}_{1+\alpha}(\cdot,t) * \psi\|_r \le C\|\psi\|_r, \quad \forall\, t \in \mathbb{R}^+, \tag{2.7}$$

and for any $\psi \in L^p(\mathbb{R}^n)$, we have

$$\|W(t)\psi\|_r \le C t^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{p} - \frac{1}{r}\right)} \|\psi\|_p, \tag{2.8}$$

$$\|\widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t) * \psi\|_r \le C t^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{p} - \frac{1}{r}\right)} \|\psi\|_p, \tag{2.9}$$

where C is a constant independent of t and φ.

Proposition 2.2 (1) *Let (q,r,p) be any admissible triplet such that $p > n/2$ and $\psi \in L^p(\mathbb{R}^n)$. Then*

$$W(t)\psi = \widetilde{E}_{1+\alpha} * \psi \in C_b(I; L^p) \cap \mathscr{C}_{q(r,p)}(I; L^r), \tag{2.10}$$

and

$$\|W(t)\psi\|_{\mathscr{C}_{q(r,p)}(I;L^r)} \le C\|\psi\|_p, \tag{2.11}$$

where $I=[0,T)$ or $I=[0,\infty)$ and C is a constant independent of I.

(2) *Let (q,r,p) be any admissible triplet such that $p>n/2$ and $\psi\in L^p(\mathbb{R}^n)$. Then*

$$W(t)\psi = \widetilde{E}_{1+\alpha} * \psi \in C_b(I;L^p)\cap L^q(I;L^r), \tag{2.12}$$

and

$$\|W(t)\psi\|_{L^q(I;L^r)} \le C\|\varphi\|_p, \tag{2.13}$$

where $I=[0,T)$ or $I=[0,\infty)$ and C is a constant independent of I.

One easily sees that $v(x,t)=W(t)\psi=\widetilde{E}_{1+\alpha}*\psi$ satisfies (2.1) with $f(x,t)=0$ or the Cauchy Problem to the fractional evolution equation

$$\frac{\partial^{\alpha+1}v}{\partial t^{\alpha+1}} - \triangle v = 0, \quad v(0)=\psi(x). \tag{2.14}$$

Proposition 2.3 *For any initial data ψ satisfying* (1.7)*, then $v(x,t)=W(t)\psi$ is a self-similar solution to* (2.14) *and its profile $V(x)=(W(1)\psi)(x)$.*

Proof In fact,

$$\begin{aligned} v(x,t) = W(t)\psi = W(t)\psi_\rho &\cong \int_{\mathbb{R}^n} e^{ix\cdot\xi}E_{\alpha+1}(-|\xi|^2t^{\alpha+1})\lambda^{\frac{2}{\rho}}\widehat{\psi(\lambda x)}(\xi)d\xi \\ &\cong \lambda^{\frac{2}{\rho}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}E_{\alpha+1}(-|\xi|^2t^{\alpha+1})\lambda^{-n}\hat{\psi}\Big(\frac{\xi}{\lambda}\Big)d\xi \quad \Big(\eta=\frac{\xi}{\lambda}\Big) \\ &\cong \lambda^{\frac{2}{\rho}}\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}E_{\alpha+1}(-|\eta|^2(\lambda^{\frac{2}{\alpha+1}}t)^{\alpha+1})\hat{\psi}(\eta)d\eta \\ &= \lambda^{\frac{2}{\rho}}(W(\lambda^{\frac{2}{\alpha+1}}t)\psi)(\lambda x) = v_\lambda(x,t), \end{aligned} \tag{2.15}$$

where $\cong$ denotes the presence of a constant after $=$. Hence we get

$$v(x,t) = \lambda^{\frac{2}{\rho}}(W(\lambda^{\frac{2}{\alpha+1}}t)\psi)(\lambda x) = \lambda^{\frac{2}{\rho}}v(\lambda x, \lambda^{\frac{2}{\alpha+1}}t). \tag{2.16}$$

Let $\lambda = t^{-\frac{\alpha+1}{2}}$ in (2.16). It follows that

$$t^{-\frac{\alpha+1}{\rho}}(W(1)\psi)\Big(\frac{x}{t^{\frac{\alpha+1}{2}}}\Big) = t^{-\frac{\alpha+1}{\rho}}v\Big(\frac{x}{t^{\frac{\alpha+1}{2}}},1\Big) = t^{-\frac{\alpha+1}{\rho}}V\Big(\frac{x}{t^{\frac{\alpha+1}{2}}}\Big). \tag{2.17}$$

So we get $V(x)=W(1)\psi$.

Remark 2.2 (1) From (2.15) we have

$$\begin{aligned} \triangle v_\lambda(x,t) &= \lambda^{\frac{2}{\rho}}\triangle v(\lambda x, \lambda^{\frac{2}{\alpha+1}}t) \\ &= \lambda^{\frac{2}{\rho}}\triangle\Big\{\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}E_{\alpha+1}(-|\eta|^2(\lambda^{\frac{2}{\alpha+1}}t)^{\alpha+1})\hat{\psi}(\eta)d\eta\Big\} \\ &= \lambda^{\frac{2}{\rho}}\Big\{\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}\lambda^2(-|\eta|^2)E_{\alpha+1}(-|\eta|^2(\lambda^{\frac{2}{\alpha+1}}t)^{\alpha+1})\hat{\psi}(\eta)d\eta\Big\} \\ &= \lambda^2\lambda^{\frac{2}{\rho}}(\triangle v)(\lambda x, \lambda^{\frac{2}{\alpha+1}}t) = \lambda^2(\triangle v)_\lambda. \end{aligned} \tag{2.18}$$

On the other hand, for any $f(t)\in C^2(\mathbb{R}^+)$, a simple computation yields

$$\begin{aligned} \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}f(\lambda^{\frac{2}{\alpha+1}}t) &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\cdot\frac{\partial^2 f(\lambda^{\frac{2}{\alpha+1}}s)}{\partial s^2}ds \\ &= \lambda^{\frac{2}{\alpha+1}}\int_0^{\lambda^{\frac{2}{\alpha+1}}t}\frac{(t-\lambda^{-\frac{2}{1+\alpha}}\tau)^{-\alpha}}{\Gamma(1-\alpha)}\cdot\frac{\partial^2 f(\tau)}{\partial\tau^2}d\tau \\ &= \lambda^{\frac{2}{\alpha+1}+\frac{2\alpha}{\alpha+1}}\int_0^{\lambda^{\frac{2}{\alpha+1}}t}\frac{(\lambda^{\frac{2}{1+\alpha}}t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\cdot\frac{\partial^2 f(\tau)}{\partial\tau^2}d\tau \end{aligned}$$

$$= \lambda^2 \left(\frac{\partial^{\alpha+1} f}{\partial t^{\alpha+1}} \right) (\lambda^{\frac{2}{\alpha+1}} t). \tag{2.19}$$

This implies

$$\lambda^{\frac{2}{\rho}} \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} v(\lambda x, \lambda^{\frac{2}{\alpha+1}} t) = \lambda^2 \lambda^{\frac{2}{\rho}} \left(\frac{\partial^{\alpha+1} v}{\partial t^{\alpha+1}} \right) (\lambda^{\frac{2}{\alpha+1}} t). \tag{2.20}$$

So $v_\lambda(x,t)$ satisfies (1.14) with data ψ_λ by (2.19) and (2.20).

(2) Notice that

$$\begin{aligned}
\frac{\partial}{\partial t} E_{\alpha+1}(-|\xi|^2 t^{\alpha+1}) &= \sum_{k=1}^{\infty} \frac{(1+\alpha)k(-1)^k |\xi|^{2k} t^{(1+\alpha)k-1}}{\Gamma((1+\alpha)k+1)} \\
&= -|\xi|^2 \sum_{k=0}^{\infty} \frac{(1+\alpha)(k+1)(-1)^k |\xi|^{2k} t^{(1+\alpha)k+\alpha}}{\Gamma((1+\alpha)(k+1)+1)} \\
&= -|\xi|^2 \sum_{k=0}^{\infty} \frac{(-1)^k |\xi|^{2k} t^{(1+\alpha)k+\alpha}}{\Gamma((1+\alpha)k+\alpha+1)} \\
&= (-1)^{1-\frac{\alpha}{\alpha+1}} |\xi|^{2-\frac{2\alpha}{\alpha+1}} \sum_{k=0}^{\infty} \frac{(-1)^{k+\frac{\alpha}{\alpha+1}} |\xi|^{2(k+\frac{\alpha}{\alpha+1})} t^{(1+\alpha)(k+\frac{\alpha}{\alpha+1})}}{\Gamma((1+\alpha)(k+\frac{\alpha}{\alpha+1})+1)} \\
&= (-1)^{\frac{1}{\alpha+1}} |\xi|^{\frac{2}{\alpha+1}} \sum_{\tilde{k}=\frac{\alpha}{\alpha+1}}^{\infty+\frac{\alpha}{\alpha+1}} \frac{(-1)^{\tilde{k}} |\xi|^{2\tilde{k}} t^{(1+\alpha)\tilde{k}}}{\Gamma((1+\alpha)\tilde{k}+1)} \\
&\cong (-|\xi|^2)^{\frac{1}{\alpha+1}} E_{\alpha+1}(-|\xi|^2 t^{\alpha+1}).
\end{aligned} \tag{2.21}$$

One can verify that

$$\partial_t W(t)\psi = (-\triangle)^{\frac{1}{\alpha+1}} W(t)\psi. \tag{2.22}$$

Therefore we give another equivalent norm to Besov space by using Mittag–Leffler's functions or $W(t)$.

By simple computations, one can prove the following results:

Proposition 2.4 (1) *Let $r \geq p > p_0 = \frac{n\rho}{2}$, $\rho > \max(1, \frac{2}{n})$. Let (q, r, p_0) be any generalized admissible triplet. Let $\psi(x) \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}) \cap \mathscr{E}_{(q(r,p_0),r)}$ with* (1.7)*. Then*

$$v(x,t) = W(t)\psi = t^{-\frac{\alpha+1}{\rho}} V\left(\frac{x}{t^{\frac{\alpha+1}{2}}} \right) = \widetilde{E}_{\alpha+1}(-|\xi|^2 t^{\alpha+1}) * \psi, \tag{2.23}$$

satisfies

$$v(x,t) \in C_*\Big([0,\infty); \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}\Big) \cap \mathscr{C}_{q(r,p_0)}([0,\infty); L^r(\mathbb{R}^n)). \tag{2.24}$$

Moreover, its profile $V(x) \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$.

(2) *Let $\psi(x) \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}$ satisfy* (1.7)*, (q,r,p) be any admissible triplet with $r > p$. Then*

$$v(x,t) \in \mathscr{L}^q(\mathbb{R}; \dot{B}_{r,\infty}^{s_\rho}) = \dot{B}_{L_t^q L_x^r}^{s_\rho,\infty}, \quad s_\rho = \frac{n}{p} - \frac{2}{\rho} \tag{2.25}$$

and its profile $V(x) \in \dot{B}_{r,q}^{\frac{n}{p}-\frac{2}{\rho}}$.

Proof One easily checks that

$$\sup_{t\in I} t^{\frac{1}{q(r,p_0)}} \|W(t)\psi\|_r \lesssim \|\psi\|_{\mathscr{E}_{(q(r,p_0),r)}}, \tag{2.26}$$

$$\Big\| W(t)\psi; C_*\Big(\mathbb{R}^+, \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}}\Big) \Big\| \leq \Big\| \psi; \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\rho}} \Big\|. \tag{2.27}$$

As for the proof of other results, see Proposition 2.5.

Proposition 2.5 *Let $r \geq p > p_0 = \frac{n\rho}{2}$, $\rho > \max(1, \frac{2}{n})$. Let $\psi(x) \in \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty} \cap \mathscr{E}_{(q(r,p_0),r)}$ with (1.7), and $u(x,t) = t^{-\frac{\alpha+1}{\rho}} U\big(\frac{x}{t^{\frac{\alpha+1}{2}}}\big)$ be the self-similar solution to (1.1) or (1.3). Then we have the following results*:

(1) *Let (q, r, p_0) be any generalized admissible triplet. Then*

$$u(x,t) = t^{-\frac{\alpha+1}{\rho}} U\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg) \in C_*\Big([0,\infty); \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\Big) \cap \mathscr{C}_{q(r,p_0)}([0,\infty); L^r(\mathbb{R}^n)) \tag{2.28}$$

iff its profile $U(x) \in \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$.

(2) *Let (q, r, p) be any admissible triplet with $r > p$. Then*

$$u(x,t) = t^{-\frac{\alpha+1}{\rho}} U\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg) \in \mathscr{L}^q(\mathbb{R}; \dot{B}^{s_\alpha}_{r,\infty}) = \dot{B}^{s_\alpha,\infty}_{L^q_t L^r_x} \tag{2.29}$$

iff its profile $U(x) \in \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{r,q}$, where $\dot{B}^{s,\infty}_X$ is the generalized form of Besov space $\dot{B}^s_{p,\infty}$ in which L^p is replaced by Banach space X.

Proof The proof of (1) comes from

$$\frac{\alpha+1}{\rho} + \frac{\alpha+1}{2}\Big(\frac{n}{p} - \frac{2}{\rho}\Big) - \frac{(\alpha+1)n}{2p} = 0,$$

and

$$-\frac{\alpha+1}{\rho} + \frac{1}{q(r,p_0)} + \frac{(\alpha+1)n}{2r} = 0.$$

Now we only need to prove (2.29). Let $\hat{\psi}_0(\xi) \in C_c^\infty(\mathbb{R}^n)$ with

$$\begin{cases} \hat{\psi}_0(\xi) = 1, & \dfrac{1}{2} \leq |\xi| \leq 2, \\ \hat{\psi}_0(\xi) = 0, & |\xi| \leq \dfrac{1}{4} \quad \text{or} \quad |\xi| \geq 4. \end{cases} \tag{2.30}$$

Define Littlewood–Paley operator as

$$\triangle_\mu f = \mathscr{F}^{-1}(\hat{\psi}_\mu(\xi)\hat{f}) = \psi_\mu * f, \quad \hat{\psi}_\mu(\xi) = \hat{\psi}_0(\mu^{-1}\xi), \quad \mu \in \mathbb{R}^+. \tag{2.31}$$

One easily checks that [17, 20]

$$\|f\|_{\dot{B}^s_{p,q}} \cong \bigg(\int_0^\infty \Big(\mu^s \|\triangle_\mu f\|_p\Big)^q \frac{d\mu}{\mu}\bigg)^{\frac{1}{q}}. \tag{2.32}$$

By Fourier transforms one easily verifies

$$\begin{aligned} \triangle_\mu u &\cong \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\psi}_\mu(\xi)\hat{u}(\xi,t)d\xi \cong t^{-\frac{\alpha+1}{\rho}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\psi}_\mu(\xi) \widehat{U\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg)}(\xi)d\xi \\ &\cong t^{-\frac{\alpha+1}{\rho}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\psi}_\mu(\xi)\widehat{U}(t^{\frac{\alpha+1}{2}}\xi)dt^{\frac{\alpha+1}{2}}\xi \\ &= t^{-\frac{\alpha+1}{\rho}} \Big(\triangle_{t^{\frac{\alpha+1}{2}}\mu} U\Big)\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg), \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \|\triangle_\mu u\|_{L^q_t L^r_x} &= \bigg(\int_0^\infty \bigg(t^{-\frac{\alpha+1}{\rho}} \|(\triangle_{t^{\frac{\alpha+1}{\rho}}\mu} U)\bigg(\frac{x}{t^{\frac{\alpha+1}{2}}}\bigg)\|_r\bigg)^q dt\bigg)^{\frac{1}{q}} \\ &= \bigg(\int_0^\infty \Big(t^{\frac{n(\alpha+1)}{2r} - \frac{\alpha+1}{\rho}} \|\triangle_{t^{\frac{\alpha+1}{2}}\mu} U\|_r\Big)^q dt\bigg)^{\frac{1}{q}} \\ &= \mu^{\frac{2}{\rho} - \frac{n}{r} - \frac{2}{(\alpha+1)q}} \bigg(\int_0^\infty \Big((t^{\frac{\alpha+1}{2}}\mu)^{\frac{n}{r} - \frac{2}{\rho}} \|\triangle_{t^{\frac{\alpha+1}{2}}\mu} U\|_r\Big)^q d(t\mu^{\frac{2}{\alpha+1}})\bigg)^{\frac{1}{q}} \\ &= \Big(\frac{2}{\alpha+1}\Big)^{\frac{1}{q}} \mu^{\frac{2}{\rho} - \frac{n}{r} - \frac{2}{(\alpha+1)q}} \bigg(\int_0^\infty \Big(\eta^{\frac{2}{\rho} - \frac{n}{r}} \|\triangle_\eta U\|_r\Big)^q \eta^{\frac{2}{\alpha+1} - 1} d\eta\bigg)^{\frac{1}{q}} \end{aligned}$$

$$= \left(\frac{2}{\alpha+1}\right)^{\frac{1}{q}} \mu^{\frac{2}{\rho}-\frac{n}{r}-\frac{2}{(\alpha+1)q}} \left(\int_0^\infty \left(\eta^{\frac{2}{\rho}-\frac{n}{r}+\frac{2}{(\alpha+1)q}} \|\triangle_\eta U\|_r\right)^q \frac{d\eta}{\eta}\right)^{\frac{1}{q}}. \tag{2.34}$$

Noting that s_ρ satisfies

$$s_\rho = \frac{n}{p} - \frac{2}{\rho} = \frac{n}{r} + \frac{2}{(\alpha+1)q} - \frac{2}{\rho}, \tag{2.35}$$

we have

$$\sup_{\mu>0} \mu^{s_\rho} \|\triangle_\mu u\|_{L_t^q L_x^r} = \left(\frac{2}{\alpha+1}\right)^{\frac{1}{q}} \sup_{\mu>0} \left(\int_0^\infty \left(\eta^{s_\rho} \|\triangle_\eta U\|_r\right)^q d\eta\right)^{\frac{1}{q}} \cong \|U\|_{\dot{B}^{s_\rho}_{r,q}}. \tag{2.36}$$

One easily sees that for any initial data $f(x)$ with (1.7)

$$\|f; \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\| = \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\rho})} t^{-\frac{1}{\rho}} \left\|H(1) f\left(\frac{x}{\sqrt{t}}\right)\right\|_p = \|H(1)f\|_p, \quad H(t) = e^{t\triangle}, \tag{2.37}$$

$$\|f; \mathscr{E}_{(q(r,p_0),r)}\| = \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q(r,p_0)}} t^{-\frac{\alpha+1}{\rho}} \left\|(W(1)f)\left(\frac{x}{t^{\frac{\alpha+1}{2}}}\right)\right\|_r = \|W(1)f\|_r. \tag{2.38}$$

We now discuss the relation between $\|H(1)f\|_p$ and $\|\triangle_0 f\|_p$, $\|W(1)f\|_r$ and $\|\triangle_0 f\|_r$, for $1 < p, r < \infty$. Now we take $\hat{h}(\xi) = \hat{h}(|\xi|) \in C_c^\infty$ with

$$\begin{cases} \operatorname{supp} \hat{h}(\xi) \subset \{\xi,\ 2^{-2} \le |\xi| \le 2^2\}, \\ \hat{h}(\xi) = 1, \quad 2^{-1} < |\xi| < 2. \end{cases} \tag{2.39}$$

Therefore we easily see that

$$\begin{aligned} \|\triangle_0 f\|_p &= \|\mathscr{F}^{-1}(\hat{\psi}_0(\xi)\hat{f}(\xi))\|_p = \|\mathscr{F}^{-1}(\hat{\psi}_0(\xi)h(\xi)\hat{f}(\xi))\|_p \\ &\le \|\mathscr{F}^{-1}(e^{|\xi|^2} h(\xi)\hat{\psi}_0(\xi)) * \mathscr{F}^{-1}(e^{-|\xi|^2}\hat{f}(\xi))\|_p \\ &\le \|\mathscr{F}^{-1}(e^{|\xi|^2} h(\xi)\hat{\psi}_0(\xi))\|_1 \|\mathscr{F}^{-1}(e^{-|\xi|^2}\hat{f}(\xi))\|_p \\ &\lesssim \|H(1)f\|_p, \end{aligned} \tag{2.40}$$

or

$$\begin{aligned} \|\triangle_0 f\|_r &= \|\mathscr{F}^{-1}(\hat{\psi}_0(\xi)\hat{f}(\xi))\|_r = \|\mathscr{F}^{-1}(\hat{\psi}_0(\xi)h(\xi)\hat{f}(\xi))\|_r \\ &\le \|\mathscr{F}^{-1}(E_{\alpha+1}(|\xi|^2) h(\xi)\hat{\psi}_0(\xi)) * \mathscr{F}^{-1}(E_{\alpha+1}(-|\xi|^2)\hat{f}(\xi))\|_r \\ &\le \|\mathscr{F}^{-1}(E_{\alpha+1}(|\xi|^2) h(\xi)\hat{\psi}(\xi))\|_1 \|\mathscr{F}^{-1}(E_{\alpha+1}(-|\xi|^2)\hat{f}(\xi))\|_r \\ &\lesssim \|W(1)f\|_r, \end{aligned} \tag{2.41}$$

where $\lesssim$ denotes the presence of a constant after $\le$.

On the other hand, we decompose $F = H(1)f$ or $F = W(1)f$ as

$$\begin{aligned} F = H(1)f &= H(1)((1-\varphi) * f) + H(1)\varphi * f \\ &= \mathscr{F}^{-1}(e^{-|\xi|^2}(1-\hat{\varphi})\hat{f}) + \mathscr{F}^{-1}(e^{-|\xi|^2}\hat{\varphi}\hat{f}) \\ &\triangleq F_1 + F_2, \end{aligned} \tag{2.42}$$

or

$$\begin{aligned} F = W(1)f &= W(1)((1-\varphi) * f) + W(1)\varphi * f \\ &= \mathscr{F}^{-1}(E_{\alpha+1}(-|\xi|^2)(1-\hat{\varphi})\hat{f}) + \mathscr{F}^{-1}(E_{\alpha+1}(-|\xi|^2)\hat{\varphi}\hat{f}) \\ &\triangleq F_1 + F_2, \end{aligned} \tag{2.43}$$

where $\hat{\varphi} = 1 - \sum_{k=1}^\infty \hat{\psi}_k(\xi)$. One easily verifies that

$$\hat{F}_1 \subset \{\xi;\ |\xi| \ge 1\}, \quad \triangle_j(F_1) = 0, \quad j \le -1, \tag{2.44}$$

$$\hat{F}_2 \subset \{\xi;\ |\xi| \le 2\}, \quad \triangle_j(F_2) = 0, \quad j \ge 2. \tag{2.45}$$

Since $f(x)$ satisfies the scaling condition (1.7), it follows that

$$\begin{aligned}\triangle_j(f) &= \mathscr{F}^{-1}\Big(\hat{\psi}_0\Big(\frac{\xi}{2^j}\Big)\hat{f}\Big) \cong \int_{\mathbb{R}^n} e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)2^{jn}\hat{f}(2^j\xi)d\xi \\ &\cong \int_{\mathbb{R}^n} e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)\widehat{f\Big(\frac{x}{2^j}\Big)}(\xi)d\xi \\ &\cong \int_{\mathbb{R}^n} e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)2^{j\frac{2}{\rho}}\hat{f}(\xi)d\xi \\ &\cong 2^{j\frac{2}{\rho}}\triangle_0(f)(2^jx). \end{aligned} \tag{2.46}$$

We first estimate $\|\triangle_j(F_2)\|_p$ or $\|\triangle_j(F_2)\|_r$. Since

$$\triangle_j(F_2) = \mathscr{F}^{-1}(\hat{\psi}_j(\xi)\hat{F}_2) = \mathscr{F}^{-1}(\hat{\psi}_j(\xi)\hat{\tilde{\psi}}_j(\xi)\hat{F}_2) = \triangle_j(f) * \tilde{\triangle}_j(\tilde{\varphi}), \quad \hat{\tilde{\psi}}_j = \sum_{\ell=-2}^{2}\hat{\psi}_{j+\ell}, \tag{2.47}$$

where

$$\hat{\tilde{\varphi}}(\xi) = e^{-|\xi|^2}\hat{\varphi}, \quad \text{or} \quad \hat{\tilde{\varphi}}(\xi) = E_{\alpha+1}(-|\xi|^2)\hat{\varphi}, \tag{2.48}$$

we obtain

$$\|\triangle_jF_2\|_\ell \lesssim \|\tilde{\triangle}_j(\tilde{\varphi})\|_1\|\triangle_j(f)\|_\ell \lesssim 2^{j(\frac{2}{\rho}-\frac{n}{\ell})}\|\triangle_0f\|_\ell, \quad j\le 1, \ell=p,r \tag{2.49}$$

where we have used $\|\tilde{\triangle}_j(\tilde{\varphi})\|_1 \le C\|\tilde{\varphi}\|_1$. For $j\ge 0$, we have

$$\begin{aligned}\|\triangle_j(F_1)\|_\ell &= \|H(1)\triangle_j(f_1)\|_\ell \qquad (\text{or} \quad \|W(1)\triangle_j(f_1)\|_\ell) \\ &\lesssim \|\triangle_j(f_1)\|_1 \lesssim \|\triangle_j(f)\|_1 \lesssim 2^{(\frac{2}{\rho}-n)j}\|\triangle_0 f\|_1, \\ \hat{f}_1 &= (1-\hat{\varphi})\hat{f}, \quad \ell = p,r \end{aligned} \tag{2.50}$$

by (2.46). So we have the following proposition:

Proposition 2.6 (1) *Let $f(x)\in L^p(\mathbb{R}^n)$. Then $\|\triangle_0 f\|_p \le C\|H(1)f\|_p$.*

(2) *Let $f(x)\in L^r(\mathbb{R}^n)$. Then $\|\triangle_0 f\|_r \le C\|W(1)f\|_r$.*

(3) *Let $f(x)\in \dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}$ satisfy (1.7) and $\triangle_0 f(x)\in L^1\cap L^p$. Then $H(1)f\in L^p$ such that*

$$\|f;\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}(\mathbb{R}^n)\| = \|H(1)f(x)\|_p \lesssim \|\triangle_0 f\|_1 + \|\triangle_0 f\|_p. \tag{2.51}$$

(4) *Let $f(x)\in \mathscr{E}_{(q(r,p_0),r)}(\mathbb{R}^n)$ satisfy (1.7) and $\triangle_0 f(x)\in L^1\cap L^r$. Then $W(1)f\in L^r$ such that*

$$\|f;\mathscr{E}_{(q(r,p_0),r)}(\mathbb{R}^n)\| = \|W(1)f(x)\|_r \lesssim \|\triangle_0 f\|_1 + \|\triangle_0 f\|_r. \tag{2.52}$$

3 Proof of Theorem 1.1 and Corollary 1.2

It is well known that (1.1) is equivalent to the following integral equation:

$$\begin{aligned}u(\cdot,t) &= \widetilde{E}_{1+\alpha}(\cdot,t)*\psi + \int_0^t \widetilde{E}_{1+\alpha}(\cdot,t-s)*\int_0^s R_\alpha(s-\tau)f(u(\tau))d\tau ds \\ &= \widetilde{E}_{1+\alpha}(\cdot,t)*\psi + \int_0^t (t-\tau)^\alpha \widetilde{E}_{1+\alpha,1+\alpha}(\cdot,t-\tau)*f(u(\tau))d\tau \\ &\stackrel{\triangle}{=} W(t)\psi + \mathscr{G}f(u). \end{aligned} \tag{3.1}$$

Proof of Theorem 1.1 According to the equivalent norm of Besov spaces and Proposition 2.4, we have

$$\begin{aligned}&\|W(t)\psi;\mathscr{C}_{q(r,p_0)}([0,\infty);L^r)\| + \Big\|W(\cdot)\psi; C_*\Big([0,\infty);\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\Big)\Big\| \\ &\le C\Big[\Big\|\psi;\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\Big\| + \|\psi;\mathscr{E}_{(q(r,p_0),r)}(\mathbb{R}^n)\|\Big] \end{aligned} \tag{3.2}$$

for any generalized admissible triplet (q,r,p_0), where $C>0$ is independent of ψ, and (q,r,p_0).

Denote by Λ the set of all generalized admissible triplets (q,r,p_0) satisfying (1.14). Define

$$X := \Big\{u\in C_*\Big(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\Big)\cap \mathscr{C}_{q(r,p_0)}(\mathbb{R}^+;L^r),\ (q,r,p_0)\in\Lambda\Big\} \tag{3.3}$$

with the norm

$$\|u;X\| := \sup_{(q,r,p_0)\in\Lambda}\sup_{t\in\mathbb{R}^+} t^{\frac{1}{q(r,p_0)}}\|u\|_r + \sup_{t\in\mathbb{R}^+}\left\|u;\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\right\|. \tag{3.4}$$

Let us introduce the complete metric space

$$\mathscr{X} = \left\{u\in X,\ \|u;X\|\le M(p),\ M(p)=2C\left[\left\|\psi;\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\right\| + \|\psi;\mathscr{E}_{(q(r,p_0),r)}(\mathbb{R}^n)\|\right]\right\}, \tag{3.5}$$

with the metric

$$d(u,v) = \|u-v;X\|, \tag{3.6}$$

and consider, in the metric space $\mathscr{X}$, the operator $\mathscr{T}$ defined by the right side of (3.1), i.e.

$$\mathscr{T}u := W(t)\psi + \mathscr{G}f(u), \quad f(u)=\mu u^{\rho+1}, \quad u\in\mathscr{X}. \tag{3.7}$$

By L^p-L^r estimate in Proposition 2.1–Proposition 2.2, it follows that

$$\begin{aligned}\|\mathscr{G}f(u);\mathscr{C}_{q(r,p_0)(I;L^r)}\| &\le \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}}\int_0^t |t-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{1+\rho}{r}-\frac{1}{r})}\|u\|_r^{\rho+1}d\tau\\ &\le \int_0^1 |1-\tau|^{\alpha-\frac{n\rho(\alpha+1)}{2r}}\tau^{-\frac{\rho+1}{q}}d\tau\cdot\|u;\mathscr{C}_{q(r,p_0)}([0,\infty);L^r)\|^{\rho+1}\\ &\le C\|u;\mathscr{C}_{q(r,p_0)}(I;L^r)\|^{\rho+1}, \end{aligned} \tag{3.8}$$

where use has been made of the fact that $n\rho/(2r)<1$ and $q>1+\rho$. Moreover, noticing that $L^{p_0}\subset\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}$, we also obtain by Proposition 2.1–Proposition 2.2 that

$$\begin{aligned}&\left\|\mathscr{G}f(u);C_*\left([0,\infty);\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty}\right)\right\| \le C\|\mathscr{G}f(u);C_*([0,\infty);L^{p_0})\|\\ &\le \sup_{t\in\mathbb{R}^+}\int_0^t |t-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{1+\rho}{r}-\frac{1}{p_0})}\|u\|_r^{\rho+1}d\tau\\ &\le C\sup_{t\in\mathbb{R}^+}\int_0^t |t-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{\rho+1}{r}-\frac{1}{p_0})}\tau^{-\frac{\rho+1}{q}}d\tau\|u;\mathscr{C}_{q(r,p_0)}(I;L^r)\|^{\rho+1}\\ &\le C\int_0^1 |1-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{\rho+1}{r}-\frac{1}{p_0})}\tau^{-\frac{\rho+1}{q}}d\tau\|u;\mathscr{C}_{q(r,p_0)}(I;L^r)\|^{\rho+1}\\ &\le C\|u;\mathscr{C}_{q(r,p_0)}(I;L^r)\|^{\rho+1}, \end{aligned} \tag{3.9}$$

for any $(q,r,p_0)\in\Lambda$ and $u\in\mathscr{X}$.

Combining (3.2), (3.8) and (3.9), we obtain that for $u\in\mathscr{X}$,

$$\|\mathscr{T}u;X\| \le C\|\psi;X\| + 2C\|u;X\|^{\rho+1}. \tag{3.10}$$

Similarly to what we do in deriving (3.8) and (3.9) we can easily obtain that for $u,v\in\mathscr{X}$,

$$d(\mathscr{T}u,\mathscr{T}v) \le C[\|u;X\|^\rho + \|v;X\|^\rho]d(u,v). \tag{3.11}$$

Then (3.8)–(3.11) imply that there exists $\eta(p)>0$ with the following properties: If $\|\psi\|_X<\eta(p)$, then there exists a unique solution $u(x,t)\in C_*([0,\infty);\dot{B}^{\frac{n}{p}-\frac{2}{\rho}}_{p,\infty})\cap\mathscr{C}_{q(r,p_0)}([0,\infty);L^r(\mathbb{R}^n))$ to (1.1) which satisfies estimates (1.15)–(1.18) by Proposition 2.3–Proposition 2.5.

Proof of Corollary 1.2 Consider now the case $r\le 1+\rho$. By interpolation between $C_*(\mathbb{R}^+;L^{p_0})$ and any space $\mathscr{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})$ with $(\tilde{q},\tilde{r},p_0)\in\Lambda$, the solution u obtained in Theorem 1.1 with $(\tilde{q},\tilde{r},p_0)\in\Lambda$ satisfies that $u\in\mathscr{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$. It suffices to show that $u\in\mathscr{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$ for any generalized admissible triplet (q,r,p_0) with $r\ge p_0(\rho+1)$. To this end, let $\tilde{r}=p_0(1+\rho)-\epsilon$, where $\epsilon>0$ is so small that

$$\frac{n(\alpha+1)}{2}\left(\frac{\rho+1}{\tilde{r}}-\frac{1}{r}\right)<1, \tag{3.12}$$

which is guaranteed by the fact that $r<np_0(\alpha+1)/(n(\alpha+1)-2p_0)$ if $n(\alpha+1)>2p_0$ and $r<\infty$ if $n(\alpha+1)\le 2p_0$. Let $\frac{1}{\tilde{q}}=\frac{n}{2}(\frac{1}{p_0}-\frac{1}{\tilde{r}})$. Then $(\tilde{q},\tilde{r},p_0)\in\Lambda$ and a simple calculation yields that

$$\|\mathscr{G}f(u)\|_{\mathscr{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)} \le \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}} \int_0^t |t-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{\rho+1}{\tilde{r}}-\frac{1}{r})}\|u\|_{\tilde{r}}^{\rho+1}d\tau$$
$$\le \int_0^1 |1-\tau|^{\alpha-\frac{n(\alpha+1)}{2}(\frac{\rho+1}{\tilde{r}}-\frac{1}{r})}\tau^{-\frac{1+\rho}{\tilde{q}}}d\tau\cdot\|u;\mathscr{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\rho+1}$$
$$\le C\|u;\mathscr{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\rho+1}, \tag{3.13}$$

where use has been made of (3.12) and the fact that $\tilde{q} > 1+\rho$.

Remark 3.1 (1) Let $\Omega \in C^k(\Sigma_{n-1})$, $k>0$ and let

$$\psi(x) = \kappa\frac{\Omega(x')}{|x|^d}, \quad x' = \frac{x}{|x|}, \quad 0<d<n. \tag{3.14}$$

Lemma 4 in Ribaud and Youssfi [RY] assures that

$$|\triangle_0(\psi)(x)| \le C\kappa\|\Omega\|_{C^k}(1+|x|)^{-k-d}. \tag{3.15}$$

Therefore, when $k \ge n$, we have $\psi(x) \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}} \cap \mathscr{E}_{(q(r,p_0),r)}(\mathbb{R}^n)$, in Theorem 1.1 and Corollary 1.2.

(2) According to [15] and [16], one easily sees that the solution $u(x,t) \in L^q(\mathbb{R}^+;L^r(\mathbb{R}^n)$ which is obtained in Theorem 1.1 for any admissible triplet (q,r,p_0).

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