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On Mañé's Proof of the C^1 Stability Conjecture

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Abstract It seems that in Mañé's proof of the C^1 Ω-stability conjecture containing in the famous paper which published in I. H. E. S. (1988), there exists a deficiency in the main lemma which says that for $f \in \mathscr{F}^1(M)$ there exists a dominated splitting $TM_{|\bar{P}_i(f)} = \tilde{E}_i^s \oplus \tilde{F}_i^u$ $(0 < i < \dim M)$ such that if $\tilde{E}_i^s$ is contracting, then $\tilde{F}_i^u$ is expanding. In the first part of the paper, we give a proof to fill up this deficiency. In the last part of the paper, we, under a weak assumption, prove a result that seems to be useful in the study of dynamics in some other stability context.

Keywords The C^1 stability conjecture, Dominated splitting, Shadowing property, Axiom A, No-cycle condition

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1 Introduction

Let M be a smooth closed manifold and let $\mathrm{Diff}^1(M)$ be the set of diffeomorphisms with C^1 topology. In [1], Mañé proved the C^1 stability conjecture for diffeomorphisms. Recall that a C^1 diffeomorphism f belongs to $\mathscr{F}^1(M)$ if and only if there exists a C^1 neighborhood $\mathscr{U}(f)$ of f such that for any $g \in \mathscr{U}(f)$ any periodic orbit P of g is hyperbolic. Let $\overline{P}_i(f)$ be as in [2] for $0 \le i \le \dim M$. Then, by Theorem 1.3 [1] p. 165 or [2], there exists an i-dominated splitting $T_{|\bar{P}_i(f)}M = \tilde{E}_i^s \oplus \tilde{F}_i^u$ for $0 < i < \dim M$. The following theorem is one of the main steps to solve the C^1 stability conjecture.

Theorem 1.4 ([1, p. 166]) *If $f \in \mathscr{F}^1(M)$, $0 < i < \dim M$ and $\tilde{E}_i^s$ is contracting, then $\tilde{F}_i^u$ is expanding.*

We found a deficiency in Mañé's proof of the above theorem. Because this theorem has been cited in several papers (see [3–8]), its importance is more and more increasing.

2 A Deficiency in the Proof

Recall that Theorem 1.4 is a corollary of the following:

Theorem 2.1 *Let Λ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$ such that $\Omega(f|_\Lambda) = \Lambda$, let $T|_\Lambda M = E \oplus F$ be a homogeneous dominated splitting such that E is contracting and suppose*

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that there exists $c > 0$ such that the inequality

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{n} \log \|(Df^{-1})|_{F(f^j(x))}\| \le -c$$

holds for a dense set of points $x \in \Lambda$. Then either F is expanding (therefore Λ is hyperbolic) or for every admissible neighborhood V of Λ and every $\gamma \in (0,1)$, there exists a periodic point $x \in M(f,V)$ with arbitrarily large period N and satisfying

$$\gamma^N \le \prod_{j=1}^{N} \|(Df^{-1})|_{\hat{F}(f^j(x))}\| < 1,$$

where $M(f,V) = \cap_{n\in Z} f^n(V)$ and $\hat{F}$ is given by the unique homogeneous dominated splitting $T|_{M(f,V)}M = \hat{E} \oplus \hat{F}$ that extends $T|_\Lambda M = E \oplus F$.

Mañé proved Theorem 2.1 through the lemmas from 2.2 to 2.7 in [1]. There is no deficiency in the proofs of 2.2 to 2.5. Let us first restate Mañé's Lemma 2.6:

Lemma 2.6 *For every $\epsilon > 0$, for any $\{\gamma_2, \bar{\gamma_2}, \gamma_3\}$, $0 < \gamma_0 < \gamma_2 < \bar{\gamma_2} < \gamma_3 < 1$, there exists a positive integer $N = N(\epsilon; \gamma_2, \bar{\gamma_2}, \gamma_3)$ such that for all $x \in \Lambda$ either $J(x,\Lambda)$ is an (N,γ_3)-set or there exists $\underline{y \in J(x,\Lambda)}$ such that $(y, f^n(y))$ is an $(N(\bar{\gamma_2},\gamma_3),\gamma_3)$-obstruction for all $n \ge N(\bar{\gamma_2},\gamma_3)$. Moreover y satisfies one of the following properties:*

a) $d(x,y) \le \epsilon$;

b) *There exists $z_o \in \Lambda$ arbitrarily near to x with $m \ge 1$, such that $d(f^m(z_0),y) < \epsilon$, and $(z_0, f^m(z_0))$ is a uniform γ_3-string.*

The definition of the terminology and symbols in above lemma will be given in the next section. In the proof of 2.6, there is a deficiency, more precisely, there is no argument to show the assertion $\underline{y \in J(x,\Lambda)}$. The following is what is actually proved in [1].

Lemma 2.6′ *For every $\epsilon > 0$, for any $\{\gamma_2, \bar{\gamma_2}, \gamma_3\}$, $0 < \gamma_0 < \gamma_2 < \bar{\gamma_2} < \gamma_3 < 1$, there exists a positive integer $N = N(\epsilon; \gamma_2, \bar{\gamma_2}, \gamma_3)$ such that for all $x \in \Lambda$ either $J(x,\Lambda)$ is an (N,γ_3)-set or there exists $\underline{y \in \Lambda}$ such that $(y, f^n(y))$ is an $(N(\bar{\gamma_2},\gamma_3),\gamma_3)$-obstruction for all $n \ge N(\bar{\gamma_2},\gamma_3)$. Moreover y satisfies one of the following properties:*

a) $d(x,y) \le \epsilon$;

b) *There exists $z_o \in \Lambda$ arbitrarily near to x with $m \ge 1$, such that $d(f^m(z_0),y) < \epsilon$, and $(z_0, f^m(z_0))$ is a uniform γ_3-string.*

We remark that the Mañé's original proof also used Lemma 2.7 which depends on Lemma 2.6 heavily. We observe that even by using Lemma 2.6′ only, we can prove Theorem 2.1.

3 A New Proof of the Theorem of Mañé

In this section, we first present some notions and lemmas from [1]. One can find complete proofs of these results in [1]. We recall them here just to familiarize the reader with the notations and symbols.

Let M be a smooth closed manifold, d be the distance on M induced from some Riemannian metric on TM, $f \in \mathrm{Diff}^1(M)$, Λ be a compact invariant set of f with a homogenous dominated splitting $T|_\Lambda M = E \oplus F$. We say that a compact neighborhood U of Λ is an *admissible neighborhood* if $T|_{M(f,U)}M$ has one and exactly one homogeneous dominated splitting $T|_{M(f,U)}M = \hat{E} \oplus \hat{F}$ extending the splitting $T|_\Lambda M = E \oplus F$. It is well known that if $T|_\Lambda M$ has a homogeneous dominated splitting, then Λ has an admissible neighborhood U. Here $M(f,U)$ is the maximum invariant set in U, that is, $M(f,U) = \cap_{n\in Z} f^n(U)$. In the following, for the simplicity of notation, we denote the dominated splitting on some admissible neighborhood by $E \oplus F$ also.

Under the above condition, we state Mañé's theorem as follows.

Theorem 3.1 *Let Λ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$ such that $\Omega(f|_\Lambda) = \Lambda$, let $T|_\Lambda M = E \oplus F$ be a homogeneous dominated splitting such that E is contracting and suppose that there exists $c > 0$ such that the inequality*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Df^{-1})|_{F(f^j(x))}\| \le -c$$

holds for points x in a dense subset Λ_0 of Λ. Then either F is expanding (therefore Λ is hyperbolic) or for every admissible neighborhood V of Λ and every $\gamma \in (0,1)$, there exists a periodic point $x \in M(f,V)$ with arbitrarily large period N and satisfying

$$\gamma^N \le \prod_{j=1}^{N} \|(Df^{-1})|_{F(f^j(x))}\| < 1. \qquad (*)$$

Definition 3.1 *For $\gamma \in (0,1)$, $(x, f^n(x)) \subset \Lambda$ $(n \ge 1)$ is called a γ-string if*

$$\prod_{j=1}^{n} \|(Df^{-1})|_{F(f^j(x))}\| \le \gamma^n;$$

$(x, f^n(x)) := \{x, f(x), \ldots, f^n(x)\} \subset \Lambda$ $(n \ge 1)$ is called a uniform γ-string if for all $0 \le k < n$, $(f^k(x), f^n(x))$ is a γ-string.

Definition 3.2 *For $\gamma \in (0,1)$, $\epsilon > 0$, a sequence of uniform γ-strings $\{x_i, f^{n_i}(x_i)\}_{i=1}^k$ is called a periodic (ϵ,γ)-quasi-hyperbolic pseudoorbit if $d(f^{n_i}(x_i), x_{i+1}) < \epsilon$ for all $1 \le i \le k$, where $x_{k+1} = x_1$. For $\delta > 0$, we say a periodic point x is δ-shadowing a sequence of uniform γ-string $\{x_i, f^{n_i}(x_i)\}_{i=1}^k$ if $N = n_1 + \cdots + n_k$ is a period of x and $d(f^n(x), f^n(x_1)) < \delta$ for $1 \le n \le n_1$, setting $N_i = n_1 + \cdots + n_i$, $d(f^{N_i+n}(x), f^n(x_{i+1})) < \delta$ for $1 \le n \le n_{i+1}$, $1 \le i < k$.*

Lemma 3.2 (Generalized Shadowing Lemma) *For all $\gamma \in (0,1)$, $\delta > 0$, there exists $\epsilon = \epsilon(\gamma,\delta)$ such that for any periodic (ϵ,γ)-quasi-hyperbolic pseudoorbit $\{x_i, f^{n_i}(x_i)\}_{i=1}^k$, it can be δ-shadowed by a periodic point x.*

Remark This Generalized Shadowing Lemma was first proved by Liao in [9, 10] (under a more general assumption). Later, Gan [11] gave it another shorter proof.

Lemma 3.3 (Pliss' lemma [12]) *For all $0 < \gamma_1 < \gamma_2 < 1$, there exist a positive integer $N(\gamma_1,\gamma_2)$ and a real number $1 > c(\gamma_1,\gamma_2) > 0$ such that if $(x, f^n(x))$ is a γ_1-string with $n \ge N(\gamma_1,\gamma_2)$, then there exist positive integers $0 < n_1 < \cdots < n_k \le n$, $k \ge 1$, $k \ge nc(\gamma_1,\gamma_2)$ and $(x, f^{n_i}(x))$ is a uniform γ_2-string for all $1 \le i \le k$.*

Definition 3.3 *Let N be a positive integer and $\gamma \in (0,1)$. We say $(x, f^n(x))$ is an (N,γ)-obstruction if $n \ge N$ and $(x, f^m(x))$ is not a γ-string for all $N \le m \le n$, that is, for all $N \le m \le n$,*

$$\prod_{j=1}^{m} \|(Df^{-1})|_{F(f^j(x))}\| > \gamma^m.$$

Lemma 3.4 *Let $0 < \gamma_0 < \gamma_2 < \gamma_3 < 1$, $(x, f^n(x))$ be a γ_0-string. Let $0 < n_1 < \cdots < n_k \le n$ be the set of integers such that $(x, f^{n_i}(x))$ is a uniform γ_3-string. Then for all $1 \le i < k$, either $n_{i+1} - n_i \le N(\gamma_2,\gamma_3)$ or $(f^{n_i}(x), f^{n_{i+1}}(x))$ is an $(N(\gamma_2,\gamma_3),\gamma_2)$-obstruction.*

Lemma 3.5 *Let $0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < 1$, N, l, n be positive integers. Let $(x, f^n(x))$ be a γ_0-string, $(x, f^l(x))$ be an (N,γ_2)-obstruction. They satisfy:*

a) $n \geq N(\gamma_0, \gamma_3), nc(\gamma_0, \gamma_3) > l$;

b) $l \geq N(\gamma_1, \gamma_2), lc(\gamma_1, \gamma_2) > N$.

Then, there exists a uniform γ_3-string $(x, f^m(x))$, with $l \leq m < n$, such that $(x, f^m(x))$ is not a γ_1-string.

Remark In more informal language, these conditions require $n > l > N$.

Definition 3.4 *Let $x \in \Lambda$. Denote by $J(x, \Lambda)$ the set of points $y \in \Lambda$ such that there exist a sequence $\{x_n\} \subset \Lambda$ satisfying $\lim_{n\to+\infty} x_n = x$ and positive integers $m_n \to_n +\infty$ satisfying*

$$y = \lim_{n\to+\infty} f^{m_n}(x_n).$$

Remark By $\Omega(f|_\Lambda) = \Lambda$, we have $x \in J(x, \Lambda)$. The sequence $\{x_n\}$ in the definition can be chosen in the dense subset Λ_0 of Λ. It is easy to see that $J(x, \Lambda)$ is a compact invariant set.

Definition 3.5 *For positive integer t and $\gamma \in (0, 1)$, a point $x \in \Lambda$ is called a (t, γ)-point if there exists $-t < m < t$, such that $(f^{m-n}(x), f^m(x))$ is a γ-string for all $n \geq 1$. Denote by $\Sigma(t, \gamma)$ the set of (t, γ)-points and call it the maximum (t, γ)-set. It is easy to see that $\Sigma(t, \gamma)$ is a compact but not necessarily invariant subset of Λ. A compact invariant subset of $\Sigma(t, \gamma)$ is called a (t, γ)-set. It is easy to see that if for some positive integer t and $\gamma \in (0, 1)$, $\Lambda = \Sigma(t, \gamma)$, then F is expanding (hence Λ is hyperbolic).*

Now we begin the proof of Theorem 3.1.

Proof Let $c > 0$, $\gamma \in (0, 1)$ be as in Theorem 3.1. Fix $\gamma_0 \in (0, 1)$, such that $\max\{\gamma, \exp(-c)\} < \gamma_0$. By the assumption of Theorem 3.1, we have for all $x \in \Lambda_0$, there are infinitely many values of positive integers n satisfying

$$\prod_{j=1}^{n} \|(Df^{-1})|_{F(f^j(x))}\| < \gamma_0^n.$$

To prove the theorem, we need the following lemma.

Lemma 3.6′ *For every $\epsilon > 0$, for any $\{\gamma_2, \bar{\gamma_2}, \gamma_3\}$, $0 < \gamma_0 < \gamma_2 < \bar{\gamma_2} < \gamma_3 < 1$, there exists a positive integer $N = N(\epsilon; \gamma_2, \bar{\gamma_2}, \gamma_3)$, such that for all $x \in \Lambda$, either $J(x, \Lambda)$ is an (N, γ_3)-set or there exists $y \in \Lambda$ such that $(y, f^n(y))$ is an $(N(\bar{\gamma_2}, \gamma_3), \gamma_3)$-obstruction for all $n \geq N(\bar{\gamma_2}, \gamma_3)$. Moreover y satisfies one of the following properties:*

a) $d(x, y) \leq \epsilon$;

b) *There exists $z_o \in \Lambda$ arbitrarily near to x with $m \geq 1$, such that $d(f^m(z_0), y) < \epsilon$, and $(z_0, f^m(z_0))$ is a uniform γ_3-string.*

Now we continue the proof of the theorem. Recall γ_0 to be such that $\max\{\gamma, \exp(-c)\} < \gamma_0 < 1$. Let $\bar{\gamma} < \hat{\gamma}$ be such that $\gamma_0 < \bar{\gamma} < \hat{\gamma} < 1$. Choose $k_0 \in (0, 1)$ satisfying

$$\gamma < k_0^2 \bar{\gamma}; \tag{1}$$

$$k_0^{-1} \hat{\gamma} < 1. \tag{2}$$

Take $\delta > 0$ satisfying that if $a, b \in M(f, V)$ and $d(a, b) < \delta$ then

$$\|(Df^{-1})|_{F(a)}\| \geq k_0 \|(Df^{-1})|_{F(b)}\|. \tag{3}$$

Let $\epsilon = \epsilon(\delta, \hat{\gamma})$ be the constant given by "Generalized Shadowing Lemma". Let the positive integer $s = s(\epsilon/4)$ satisfy that for any given sequence $\{x_i\}_{i=1}^s$ in M there exist $i \neq j$ such that $d(x_i, x_j) < \epsilon/4$. Let the $4(s+1)$ numbers be such that $0 < \gamma_{3s+1} < \gamma_{3s+2} < \gamma_{\bar{3s+1}} < \gamma_{3s+3} < \cdots < \gamma_1 < \gamma_2 < \bar{\gamma_2} < \gamma_3 < 1$ with $\gamma_{3s+1} = \bar{\gamma}$, $\gamma_3 = \hat{\gamma}$.

Now we assume that F is not expanding.

Let $N_i = N(\epsilon/4; \gamma_{3i-1}, \gamma_{\bar{3i-1}}, \gamma_{3i})$ be constants given by Lemma 3.6′, for every $1 \leq i \leq s+1$. Let $\Sigma_i = \Sigma(N_i, \gamma_{3i})$ be the maximum (N_i, γ_{3i})-set for every $1 \leq i \leq s$. Because F is not expanding, it is easy to see that $\cup_{i=1}^s \Sigma_i \neq \Lambda$. We claim that for every $1 \leq i \leq s$, for any point $\tilde{x}_{2i-1} \in \Lambda - \Sigma_i$, there exist $x_{2i-1}, x_{2i} \in \Lambda$, with x_{2i-1} arbitrarily close to $\tilde{x}_{2i-1}$ and two positive integers m_{2i-1}, m_{2i} satisfying:

1) $(x_{2i-1}, f^{m_{2i-1}}(x_{2i-1}))$, $(x_{2i}, f^{m_{2i}}(x_{2i}))$ are uniform γ_{3i}-strings;
2) $(x_{2i}, f^{m_{2i}}(x_{2i}))$ is not a γ_{3i-2}-string;
3) $d(f^{m_{2i-1}}(x_{2i-1}), x_{2i}) < \epsilon/2$;
4) $f^{m2i}(x_{2i}) \in \Lambda - \Sigma_{i+1}$;
5) $\gamma_{3i-2}^{m_{2i}} K^{m_{2i-1}} \geq (k_0\gamma_{3i-2})^{m_{2i-1}+m_{2i}}$,

where $K = \min\{\|(Df^{-1})|_{F(x)}\| : x \in \Lambda\}$.

We will prove the claim later. Now we continue the proof of Theorem 3.1 by this claim and the previous lemmas. First we take a point $\tilde{x}_1 \in \Lambda - \Sigma_1$. By the claim and induction, we can get $2s$ points $\{x_{2i-1}, x_{2i}\}_{i=1}^s$ and $2s$ positive integers $\{m_{2i-1}, m_{2i}\}_{i=1}^s$, satisfying 1), 2), 3), 4), 5) of the claim, for all $1 \leq i \leq s$. By the definition of $s = s(\epsilon/4)$, for sequence $\{x_{2i-1}\}_{i=1}^s$, there exists $k < l$, such that $d(x_{2k-1}, x_{2l-1}) < \epsilon/4$. It is easy to see that

$$\{(x_{2i-1}, f^{m_{2i-1}}(x_{2i-1})), (x_{2i}, f^{m_{2i}}(x_{2i}))\}_{i=k}^{l-1}$$

is a periodic $(\hat{\gamma}, \epsilon)$-quasi-hyperbolic orbit arc. By Generalized Shadowing Lemma, there exists a periodic point $x \in M(f, U)$ which is δ-shadowing that quasi-hyperbolic orbit arc. x is the periodic point satisfying $(*)$ of Theorem 3.1.

First, for all $k \leq i \leq l-1$, by 2), 5) of the claim, we have

$$\prod_{j=1}^{m_{2i-1}} \|(Df^{-1})|_{F(f^j(x_{2i-1}))}\| \cdot \prod_{j=1}^{m_{2i}} \|(Df^{-1})|_{F(f^j(x_{2i}))}\|$$
$$\geq K^{m_{2i-1}} \gamma_{3i-2}^{m_{2i}} \geq (k_0\gamma_{3i-2})^{m_{2i-1}+m_{2i}} \geq (k_0\bar{\gamma})^{m_{2i-1}+m_{2i}}.$$

By 1) of the claim, we have

$$\prod_{j=1}^{m_{2i-1}} \|(Df^{-1})|_{F(f^j(x_{2i-1}))}\| \cdot \prod_{j=1}^{m_{2i}} \|(Df^{-1})|_{F(f^j(x_{2i}))}\| \leq (\gamma_{3i})^{m_{2i-1}+m_{2i}} \leq \hat{\gamma}^{m_{2i-1}+m_{2i}}.$$

Multiplying the above two inequalities for all $k \leq i \leq l-1$, we have

$$(k_0\bar{\gamma})^N \leq \prod_{i=2k-1}^{2(l-1)} \prod_{j=1}^{m_i} \|(Df^{-1})|_{F(f^j(x))}\| \leq \hat{\gamma}^N,$$

where $N = \Sigma_{i=2k-1}^{2(l-1)} m_i$. Because x is δ-shadowing this quasi-hyperbolic orbit arc, by (3), it is easy to see that

$$(K_0^2\bar{\gamma})^N \leq \prod_{j=1}^{N} \|(Df^{-1})|_{F(f^j(x))}\| \leq (k_0^{-1}\hat{\gamma})^N.$$

Since $k_0^2\bar{\gamma} > \gamma$ by (1), we have proved

$$\prod_{j=1}^{N} \|(Df^{-1})|_{F(f^j(x))}\| > \gamma^N.$$

Since $k_0^{-1}\hat{\gamma} < 1$ by (2), we have proved

$$\prod_{j=1}^{N} \|(Df^{-1})|_{F(f^j(x))}\| < 1.$$

This completes the proof of Theorem 3.1.

Now we prove the claim. For $1 \leq i \leq s$, let $\tilde{x}_{2i-1} \in \Lambda - \Sigma_i$. Obviously $J(\tilde{x}_{2i-1}, \Lambda)$ is not an (N_i, γ_{3i})-set. By Lemma 3.6′, there exists x_{2i-1} arbitrarily near $\tilde{x}_{2i-1}$. In particular we can assume that $d(x_{2i-1}, \tilde{x}_{2i-1}) < \epsilon/4$, and there exists $n_0 \geq 0$ such that $f^{n_0}(x_{2i-1})$ is $\epsilon/4$-near a point $\tilde{x}_{2i}$ such that $(\tilde{x}_{2i}, f^n(\tilde{x}_{2i}))$ is an $(N(\bar{\gamma}_{3i-1}, \gamma_{3i}), \gamma_{3i-1})$-obstruction for all $n > N_i$. Moreover, if $n_0 > 0$, then $(x_{2i-1}, f^{n_0}(x_{2i-1}))$ is a uniform γ_{3i}-string. Because Λ_0 is a dense subset of Λ, then when $x_{2i} \in \Lambda_0$ is arbitrarily near $\tilde{x}_{2i}$, there exists l large with respect to $N(\bar{\gamma}_{3i-1}, \gamma_{3i})$, such that $(x_{2i}, f^l(x_{2i}))$ is an $(N(\bar{\gamma}_{3i-1}, \gamma_{3i}), \gamma_{3i-1})$-obstruction. Because $x_{2i} \in \Lambda$, there exist infinitely many integers n such that

$$\prod_{j=1}^{n} \|(Df^{-1})|_{F(f^j(x_{2i}))}\| < \gamma_0^n.$$

Applying Lemma 3.5 to $\gamma_1 = \gamma_{3i-2}$, $\gamma_2 = \gamma_{3i-1}$, $\gamma_3 = \gamma_{3i}$, because n can be chosen large with respect to l, l large with respect to $N(\bar{\gamma}_{3i-1}, \gamma_{3i})$, there exists $l \leq m \leq n$, such that $(x_{2i}, f^m(x_{2i}))$ is a uniform γ_{3i}-string, but not a γ_{3i-2}-string. Because $\gamma_{3i-2} > \gamma_{3(i+1)}$, when l is large enough (so is m), $f^m(x_{2i})$ could not be an $(N_{i+1}, \gamma_{3(i+1)})$-point, that is, $f^m(x_{2i}) \in \Lambda - \Sigma_{i+1}$. Then x_{2i-1}, x_{2i}, $m_{2i-1} = m_0$ and $m_{2i} = m$ satisfy conditions 1), 2), 3), 4) of the claim. Condition 5) holds if m is large with respect to m_{2i-1}. Then in order to satisfy it, take l in the previous construction so large that $\gamma_{3i-2}^j K^{m_{2i-1}} \geq (k_0\gamma_{3i-2})^{j+m_{2i-1}}$ for all $j \geq l$. This will hold for $m_{2i} = m$. Thus we complete the proof of the claim.

4 Another Theorem

In this section, we will prove a theorem similar to the theorem of Mañé. But the condition in this theorem is somewhat different from that of Mañé, and so it may be useful in some other stability context.

Theorem 4.1 *Let $f \in \mathrm{Diff}^1(M)$. Let W be a compact invariant set of f with a homogeneous dominated splitting $T_W M = E \oplus F$. Suppose that $a \in W$. Let Λ be $\omega(a)$, the set of omega-limit points of a. Suppose E is contracting on Λ, that is, there exists $\tau \in (0, 1)$, such that for all $x \in \Lambda$, $\|Df|_{E(x)}\| \leq \tau$. Suppose also that there exists $c > 0$, such that*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Df^{-1})|_{F(f^j(a))}\| \leq -c.$$

Then, either F is expanding on Λ (therefore Λ is hyperbolic), or for all $\gamma \in (0, 1)$, there exists a periodic point x arbitrarily near Λ and with arbitrarily large periodic N, such that

$$\gamma^N \leq \prod_{j=1}^{N} \|(Df^{-1})|_{F(f^j(x))}\| < 1.$$

Proof Suppose that F is not expanding on Λ. Let $c > 0$, $\gamma \in (0, 1)$ be as in the theorem. Fix $\gamma_0 \in (0, 1)$ such that $\max\{\exp(-c), \gamma\} < \gamma_0$. By the assumption of the theorem, we have, for all $x \in \mathrm{orb}(a)$, there are infinitely many positive integers n satisfying

$$\prod_{j=1}^{n} \|(Df^{-1})|_{F(f^j(x))}\| < \gamma_0^n.$$

Now we have numbers $\{\gamma_1, \gamma_2, \bar{\gamma}_2, \gamma_3\}$, such that $0 < \gamma_0 < \gamma_1 < \gamma_2 < \bar{\gamma}_2 < \gamma_3 < 1$.

To prove the theorem, we need a lemma similar to Lemma 3.6$'$ of the previous section.

Lemma 4.1 *Fix $\epsilon > 0$. For all $x \in \Lambda$, there exists $y \in \Lambda$, such that $(y, f^n(y))$ is an $N((\bar{\gamma}_2, \gamma_3), \gamma_2)$-obstruction for all $n \geq N(\bar{\gamma}_2, \gamma_3)$. Moreover, y satisfies one of the following properties:*

a) $d(x, y) \leq \epsilon$;

b) *There exists $z_0 \in \mathrm{orb}(a)$ arbitrarily near x with $m \geq 1$, such that $d(f^m(z_0), y) < \epsilon$ and $(z_0, f^m(z_0))$ is a uniform γ_3-string.*

Proof Denote by $\Lambda(N(\bar{\gamma}_2,\gamma_3))$ the set of points $y\in\Lambda$ such that $(y,f^n(y))$ is an $(N(\bar{\gamma}_2,\gamma_3),\gamma_2)$-obstruction for all $n\ge N(\bar{\gamma}_2,\gamma_3)$. It is easy to check that given $\epsilon>0$, there exist $N_1(\epsilon)>N(\bar{\gamma}_2,\gamma_3)$ and $N_2(\epsilon)\ge 1$ such that when $(f^{n_2}(a),f^{n_1+n_2}(a))$ is an $(N(\bar{\gamma}_2,\gamma_3),\bar{\gamma}_2)$-obstruction and $n_1>N_1(\epsilon)$, $n_2>N_2(\epsilon)$, then $d((f^{n_2}(a),\Lambda(N(\bar{\gamma}_2,\gamma_3))))<\epsilon$ (here we use that $\bar{\gamma}_2>\gamma_2$).

For, otherwise, there exist a sequence of points $\{f^{m_n}(a)\}$ in Λ with $m_n\to_n+\infty$ and $d(f^{m_n}(a),\Lambda(N(\bar{\gamma}_2,\gamma_3)))\ge\epsilon$, and a sequence of positive integers $k_n\to_n+\infty$ such that $(f^{m_n}(a), f^{m_n+k_n}(a))$ is an $(N(\bar{\gamma}_2,\gamma_3),\bar{\gamma}_2)$-obstruction. By choosing subsequence if necessary, without loss of generality, we can assume that there exists $x\in\Lambda=\omega(a)$ such that $f^{m_n}(a)\to_n x$. Obviously $x\in\Lambda(N(\bar{\gamma}_2),\gamma_3)$. On the other hand, because $f^{m_n}(a)\to_n x$, we have $d(x,\Lambda(N(\bar{\gamma}_2,\gamma_3)))\ge\epsilon$, which is a contradiction.

Now we fix an arbitrary point $x\in\Lambda$. Given any point $z\in\Lambda$, there exists a sequence $\{x_n=f^{k_n}(a)|n\ge0\}$ (with $k_n\to_n+\infty$) converging to x and satisfying $z=\lim_{n\to+\infty}f^{m_n}(x_n)$ and $m_n\to_n+\infty$. For $n\ge0$ define $\mathscr{S}(n)=\{m>0;(x_n,f^m(x_n))$ is a uniform γ_3-string$\}\cup\{0\}$. By Pliss' lemma, it is easy to see that $\mathscr{S}(n)$ is unbounded (since $\gamma_0<\gamma_3$ and $x_n\in\Lambda_0$). Set $k_n^+=\min\mathscr{S}(n)\cap[m_n,+\infty)$ and $k_n^-=\max\mathscr{S}(n)\cap[0,m_n)$. Suppose that $\liminf(k_n^+-k_n^-)\le N_1(\epsilon)$. Then there exists $0\le m\le N_1(\epsilon)$ such that $f^m(z)$ is the limit of a subsequence of $\{f^{k_n^+}(x_n)|n\ge0\}$. Hence, if $r>0$, $(f^{m-r}(z),f^m(z))$ is a γ_3-string because it is the limit of a sequence of γ_3-strings $(f^{k_n^+-r}(x_n),f^{k_n^+}(x_n))$ (Indeed these are γ_3-strings for $r\le k_n^+$ because $(x_n,f^{k_n^+}(x_n))$ is a uniform γ_3-string for all n). Therefore, for some $0\le m\le N_1(\epsilon)$, $(f^{m-r}(z),f^m(z))$ is a γ_3-string for all $r>0$. If this holds for all $z\in\Lambda$ then Λ is an $(N_1(\epsilon),\gamma_3)$-set which implies F is expanding. This contradicts our assumption that F is not expanding. So there exists some point $z\in\Lambda$ such that for infinitely many n, $k_n^+-k_n^->N_1(\epsilon)$. Hence $k_n^+-k_n^->N_1(\epsilon)>N(\bar{\gamma}_2,\gamma_3)$ because $N_1(\epsilon)>N(\bar{\gamma}_2,\gamma_3)$. Then, by Lemma 3.4, $(f^{k_n^-}(x_n),f^{k_n^+}(x_n))$ is an $(N(\bar{\gamma}_2,\gamma_3),\bar{\gamma}_2)$-obstruction. Therefore, $d(f^{k_n^-}(x_n),\Lambda(N(\bar{\gamma}_2,\gamma_3)))<\epsilon$ for infinitely many n. If for an unbounded set of these we have $k_n^->0$, we can take $y\in\Lambda(N(\bar{\gamma}_2,\gamma_3))$ such that $d(f^{m_n^-}(x_n),y)<\epsilon$ and then this point y, the point $z_0=x_n$ and $m=k_n^-$ satisfy conclusion b) of the lemma. If $k_n^-=0$ for all sufficiently large n that satisfy $d(f^{k_n^-}(x_n),\Lambda(N(\bar{\gamma}_2,\gamma_3)))<\epsilon$, then $d(x_n,\Lambda(N(\bar{\gamma}_2,\gamma_3)))$, and since $x_n\to_n x$ we obtain that $d(x,\Lambda(N(\bar{\gamma}_2,\gamma_3)))\le\epsilon$. Taking $y\in\Lambda(N(\bar{\gamma}_2,\gamma_3))$ such that $d(x,y)\le\epsilon$, it follows that y satisfies conclusion a) of the lemma. This completes the proof the lemma.

Using this lemma, the process to find the periodic orbit through the Generalized Shadowing Lemma is exactly the Mañé's proof Theorem 3.1 or our proof in the previous section. We omit the details and leave them to the reader.

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